Communications in Mathematics and Applications
Vol. 5, No. 3, pp. 91–100, 2014
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

A Characterisation of Strong Integer Additive Set-Indexers of Graphs

Research Article

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Abstract. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathscr{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer (IASI) of a given graph G is an injective function $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathscr{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective, where $f(u) + f(v)$ is the sum set of $f(u)$ and $f(v)$. If $f^+(uv) = k\ \forall\ uv \in E(G)$, then f is said to be a k-uniform IASI. An IASI f is said to be a strong IASI if $|f^+(uv)| = |f(u)||f(v)|\ \forall\ uv \in E(G)$. In this paper, we study the characteristics of certain graph classes, graph operations and graph products that admit strong integer additive set-indexers.

Keywords. Set-indexer; Integer additive set-indexer; Strong integer additive set-indexer; Difference set; Nourishing number of a graph

MSC. 05C78

Received: July 27, 2014 **Accepted:** December 29, 2014

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [4], [9], [15] and [20]. For graph products, we further refer to [14] and [16]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The *sumset* of two non-empty sets A and B, denoted by $A + B$, is the set defined by $A + B = \{a + b : a \in A, b \in B\}$. If either A or B is countably infinite, then $A + B$ will also be countably infinite. Hence, all the sets in present discussion will be non-empty finite sets. We denote the cardinality of a set A by $|A|$.

Invoking the concepts of sumsets of finite sets, the notion of an integer additive set-indexer of a graph G is introduced as follows.

Definition 1.1 ([11])**.** An *integer additive set-indexer* (IASI) of a given graph G is defined as an injective function $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathscr{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. A graph G which admits an integer additive set-indexer is called an *integer additive set-indexed graph* (IASI-graph).

The cardinality of the set-label of an element (a vertex or an edge) of an IASI-graph is called the *set-indexing number* of that element.

The following theorem establishes the bounds for the cardinality of the sumset of two non-empty sets of integers.

Theorem 1.2 ([17])**.** *Let A and B be two non-empty sets of integers. Then, $|A|+|B|-1 \le |A+B| \le |A|\,|B|$.*

The IASIs of a given graph G with respect to which all the edges of G have the highest possible set-indexing number are of special interest. Hence, we have introduced the following notion.

Definition 1.3 ([19])**.** If a graph G has a set-indexer f such that $|f^+(uv)| = |f(u)+f(v)| = |f(u)|\,|f(v)|$ for all vertices u and v of G, then f is said to be a *strong IASI* of G. A graph which admits a strong IASI is called a *strong IASI-graph*.

Definition 1.4 ([19])**.** If G is a graph which admits a k-uniform IASI and $V(G)$ is l-uniformly set-indexed, then G is said to have a *(k,l)-completely uniform IASI* or simply a *completely uniform IASI*.

We use the notation $A < B$ in the sense that $A \cap B = \emptyset$. We notice that the relation $<$ is symmetric, but need not be reflexive and transitive. By the sequence $A_1 < A_2 < A_3 < \ldots < A_n$, we mean that the given sets are pairwise disjoint.

The *difference set* of a non-empty set A, denoted by D_A, is the set defined by $D_A = \{|a-b| : a,b \in A\}$. The following lemma provides a necessary and sufficient condition for the sumset of two sets to have the highest possible cardinality.

Lemma 1.5 ([19])**.** *Let A, B be two non-empty subsets of $\mathbb{N}_0$. Then, $|A+B| = |A|\,|B|$ if and only if their difference sets, denoted by D_A and D_B respectively, follow the relation $D_A < D_B$. In other words, $|A+B| = |A|\,|B|$ if and only if $D_A \cap D_B = \emptyset$.*

A necessary and sufficient condition for a given complete graph to admit a strong IASI is given below.

Theorem 1.6 ([19])**.** *Let each vertex v_i of the complete graph K_n be labeled by the set $A_i \in \mathscr{P}(\mathbb{N}_0)$. Then K_n admits a strong IASI if and only if there exists a finite sequence of sets $D_1 < D_2 < D_3 < \cdots < D_n$ where each D_i is the set of all differences between any two elements of the set A_i.*

The hereditary nature of the existence of strong IASI of a given graph has been established in the following result.

Theorem 1.7 ([19]). *If a graph G admits a strong IASI then its subgraphs also admit strong IASI.*

Corollary 1.8 ([19]). *A connected graph G (on n vertices) admits a strong IASI if and only if each vertex v_i of G is labeled by a set A_i in $\mathscr{P}(\mathbb{N}_0)$ and there exists a finite sequence of sets $D_1 < D_2 < D_3 < \cdots < D_m$, where $m \le n$ is a positive integer and each D_i is the set of all differences between any two elements of the set A_i.*

2. New Results on Strong IASI Graphs

In this paper, as an extension to the studies on strong IASI-graphs, done in [19], we study certain characteristics and properties of certain graph classes, graph operations and graph products which admit strong IASIs.

2.1 Strong IASI of Certain Graph Classes

Let us denote the difference set of the set-label A_i of a vertex v_i in G by D_i. The relation $<$ is called the *difference relation* between the set-labels of the corresponding vertices in G. A *chain* of difference sets is the sequence of difference sets $D_1 < D_2 < D_3 < \ldots < D_n$ and the *length* of a chain is the number of difference sets in that chain.

For any strong IASI-graph with m vertices and n edges, there are m difference sets, one each for each vertex and n relations $<$, one each corresponding to each edge of G. Note that all these difference sets need not be necessarily pair wise disjoint. But, if G is a strong IASI-graph, then the difference sets of two adjacent vertices u and v in G must be disjoint. That is, there exists a difference relation for the set-labels of any two adjacent vertices in G and these relations forms one or more chain of difference sets. The lengths of such chains of difference sets are noteworthy and hence we have the following notion.

Definition 2.1. The *nourishing number* of a set-labeled graph is the minimum length of the maximal chain of difference sets in G. The nourishing number of a graph G is denoted by $\kappa(G)$.

Remark 2.2. In other words, the nourishing number of a given strong IASI-graph can be defined as the minimum value required for the order of its maximal (connected) subgraph, the difference sets of the set-labels of all whose vertices are pair wise disjoint.

The study about nourishing number of different graphs and graph classes arouses much interest. In this section, we discuss about the nourishing number of various graphs.
In view of the above definition we can rewrite Theorem 1.6, as given below.

Theorem 2.3. *The nourishing number of a complete graph is n. That is, $\kappa(K_n) = n$.*

The following theorem determines the nourishing number of bipartite graphs.

Theorem 2.4. *The nourishing number of a bipartite graph is* 2.

Proof. Let G be a bipartite graph with bipartition (X,Y) which admits a strong IASI. Let $X = \{u_1,u_2,u_3,\dots,u_m\}$ and $Y = \{v_1,v_2,v_3,\dots,v_n\}$. Define $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ such that for two adjacent vertices $u_i \in X$ and $v_j \in Y$, the difference sets of $f(u_i)$ and $f(v_j)$, denoted by D_{u_i} and D_{v_j} respectively, hold the relation $D_{u_i} < D_{v_j}$. Hence, corresponding to each edge in G, there exists a difference relation between the set-labels of its end vertices. Hence, f is a strong IASI and the minimum length of the maximal chain in G is 2. That is, $\kappa(G) = 2$. □

Proposition 2.5. *The nourishing number of a triangle-free graph is* 2.

Proof. Since G is a triangle-free graph, G can not be a complete graph. Since the relation $<$ is not transitive, we can not find a chain consisting of three or more difference sets corresponding to the set-labels of vertices of G. Hence $\kappa(G) = 2$. □

Due to Proposition 2.5, our problem of finding the nourishing number of a graph G is reduced to finding the order of a maximal complete subgraph of G. We know that a clique of graph G is a complete subgraph of G. We recall that the clique number $\omega(G)$ of a graph is the number of vertices in a maximal clique in G. Hence, we have

Proposition 2.6. *The nourishing number of a graph G is equal to the clique number ω of G.*

Proof. Let H be the maximal clique of a given graph G. Then, H is complete graph. Hence, by Theorem 1.6, there exists a chain of difference sets $D_1 < D_2 < D_3 < \dots < D_r$, where $r = |V(H)|$. Moreover, no cliques in $G - H$ can have more vertices in G than H. Therefore, $\kappa(G) = r = \omega(G)$. □

Interesting questions that arise in this context are about the nourishing number of different graph operations. In the following discussions, we address these problems. First, we check the admissibility of strong IASIs by the union of two graphs and its nourishing number in the following results.

2.2 Strong IASIs of Graph Operations

By the term *last vertex* of a subgraph H of G, we mean a vertex $v \in V(H)$ whose adjacent vertices in G are not in $V(H)$.

Theorem 2.7. *The union $G_1 \cup G_2$ of two graphs G_1 and G_2, admits a strong IASI if and only if both G_1 and G_2 admit strong IASIs.*

Proof. Let G_1 and G_2 admit strong IASIs say f_1 and f_2. If G_1 and G_2 are disjoint graphs, then we observe that the IASI f defined by $f(v) = f_i(v)$ if $v \in V(G_i)$, $i = 1,2$ is a strong IASI for $G_1 \cup G_2$.

If G_1 and G_2 are not disjoint graphs, then re-label the vertices in $G_1 \cap G_2$ in such a way that the difference sets of the set-labels of adjacent vertices in $G_1 \cap G_2$ and the last vertices of $G_1 \cap G_2$ and their adjacent vertices in G_1 and G_2 follow the relation '<'. Hence, f is a strong IASI of $G_1 \cup G_2$.

Conversely, assume that $G_1 \cup G_2$ admits a strong IASI. Hence, by Theorem 1.7, being the subgraphs of a strong IASI-graph $G_1 \cup G_2$, G_1 and G_2 admit the (induced) strong IASIs, $f|_{G_1}$ and $f|_{G_2}$, where $f|_{G_i}$ is the restriction of f to the (sub)graph G_i. □

Theorem 2.8. *Let G_1 and G_2 be two strong IASI-graphs. Then, $\kappa(G_1 \cup G_2) \geq \max\{\kappa(G_1), \kappa(G_2)\}$.*

Proof. Let G_1 and G_2 be two strong IASI-graphs. Let H_1 and H_2 be the maximal cliques in G_1 and G_2 respectively. If G_1 and G_2 are disjoint, so are H_1 and H_2. Without loss of generality, let $|V(H_1) \geq |V(H_2)|$. Then, H_1 is the maximal clique in $G_1 \cup G_2$. Hence, $\kappa(G_1 \cup G_2) = \kappa(G_1)$. Therefore, in general, for disjoint graphs G_1 and G_2, $\kappa(G_1 \cup G_2) = \max\{\kappa(G_1), \kappa(G_2)\}$. If G_1 and G_2 are not disjoint, then there may exist a subgraph H'_1, not necessarily complete, in G_1 and a subgraph H'_2, not necessarily complete, in G_2 such that $H'_1 \cup H'_2 = K_l$, where $l \geq |V(H_1)|, |V(H_2)|$. In this case, $\kappa(G_1 \cup G_2) \geq \max\{\kappa(G_1), \kappa(G_2)\}$. This completes the proof. □

Invoking Theorem 2.8, we observe the following theorem.

Theorem 2.9. *Let G_1 and G_2 be two strong IASI-graphs. Then, $\kappa(G_1 \cup G_2) = max\{\kappa(G_1), \kappa(G_2)\}$ if $G_1 \cap G_2$ is triangle-free.*

Proof. Every complete graph K_n with more than two vertices contains triangles. Hence, since, $G_1 \cap G_2$ is triangle-free, it does not contain any clique. Therefore, there does not exist a subgraph H'_i in G_i, $i = 1, 2$ such that $H'_1 \cup H'_2 = K_l$, $l > 2$. Hence, $\kappa(G_1 \cup G_2) = max\{\kappa(G_1), \kappa(G_2)\}$. □

The above theorem may not hold for the union of two graphs whose intersection contains triangles. For example, consider two cycles C_m and C_n such that $C_m \cap C_n = K_3$. We know that $\kappa(C_m) = \kappa(C_n) = 2$. But, $\kappa(C_m \cup C_n) = 3$ which is not equal to $\max\{\kappa(C_m), \kappa(C_n)\}$.

Hence, we have the following theorem for the union of two graphs.

Theorem 2.10. *Let G_1 and G_2 be two strong IASI graphs. Then, the nourishing number of G is*

$$\kappa(G_1 \cup G_2) = \begin{cases} \max\{3, \kappa(G_1), \kappa(G_2)\} & \text{if } K_3 \subseteq G_1 \cap G_2 \\ \max\{\kappa(G_1), \kappa(G_2)\} & \text{if } K_3 \not\subseteq G_1 \cap G_2. \end{cases}$$

The following theorem is a necessary and sufficient condition for the admissibility of strong IASI by the join of two strong IASI-graphs.

Theorem 2.11. *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two strong IASI-graphs. Then, their join $G_1 + G_2$ admits a strong IASI if and only if the difference set of the set-label of every vertex in G_1 is disjoint from the difference sets of the set-labels of all vertices of G_2.*

Proof. Let $V(G_1) = \{u_1, u_2, u_3, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, v_3, \ldots, v_n\}$. Now let $E_3 = \{u_iv_j : u_i \in V(G_1), v_j \in V(G_2)\}$. Let G_3 be the subgraph of $G_1 + G_2$ with the edge set E_3. Therefore, $G_1 + G_2 = G_1 \cup G_2 \cup G_3$. Also, let f_1, f_2, f_3 be the IASIs defined on G_1, G_2, G_3 respectively. Given that f_1 and f_2 are strong IASIs. Let f be an IASI defined on $G_1 + G_2$ by $f(v) = f_i(v)$ if $v \in V(G_i)$, $i = 1, 2, 3$. Then, by Theorem 2.7, f is a strong IASI if and only if f_3 is a strong IASI.

First assume that f is a strong IASI. Then, f_3 is also a strong IASI. Hence, $|g_{f_3}(u_iv_j)| = |f_3(u_i)| \cdot |f_3(v_j)|\ \forall\ u_i \in V(G_1),\ v_j \in V(G_2)$. By Lemma 1.5, the difference sets of the set-labels of these vertices follow the relation $D_{u_i} < D_{v_j},\ \forall\ u_i \in V(G_1),\ v_j \in V(G_2)$.

Conversely, assume that $D_{u_i} < D_{v_j},\ \forall\ u_i \in V(G_1),\ v_j \in V(G_2)$. Then, $|g_{f_3}(u_iv_j)| = |f_3(u_i)| \cdot |f_3(v_j)|\ \forall\ u_iv_j \in E_3$. Therefore, f_3 is a strong IASI on G_3 and hence f is a strong IASI on $G_1 + G_2$. This completes the proof. □

The nourishing number of the join of two strong IASI-graphs has been determined in the following theorem.

Theorem 2.12. *Let G_1 and G_2 be two strong IASI-graphs. Then, $\kappa(G_1+G_2)=\kappa(G_1)+\kappa(G_2)$.*

Proof. Let H_1 be a maximal clique with order m in G_1 and H_2 be a maximal clique with order n in G_2. Since every vertex of G_1 is joined to every vertex of G_2 in G_1+G_2, it is so in H_1+H_2 also. Since H_1 and H_2 are cliques, they are complete graphs. Hence, every vertex of H_1 is adjacent to all other vertices of H_1 and all vertices of H_2. Similarly, every vertex of H_2 is adjacent to all other vertices of H_2 and all vertices of H_1. Hence, H_1+H_2 is an $(m+n-1)$-regular graph on $m+n$ vertices. Therefore, H_1+H_2 is a complete graph and hence is a clique in G_1+G_2. Since H_1 and H_2 are maximal cliques, H_1+H_2 is maximal in G_1+G_2. Hence, $\kappa(G_1+G_2)=m+n=\kappa(G_1)+\kappa(G_2)$. □

Next, let us discuss the nourishing number of the complement of a strong IASI-graph G. A graph G and its complement $\bar{G}$ have the same set of vertices and hence G and $\bar{G}$ have the same set-labels for their corresponding vertices. We observe that the strong IASIs, except some, defined on G do not induce strong IASI on $\bar{G}$. A set-labeling of $V(G)$ that defines a strong IASI for both the graphs G and its complement $\bar{G}$ may be called a *strongly concurrent set-labeling*. The set-labels of the vertex set of G mentioned in this section are strongly concurrent. Hence, we propose the following results.

Theorem 2.13. *Let G be a strong IASI-graph on n vertices and let $\bar{G}$ be its complement. Then, $\bar{G}$ admits a strong IASI if and only if the length of the chain of difference sets of set-labels of vertices in G or in $\bar{G}$ is n.*

Proof. We have $G\cup\bar{G}=K_n$. Therefore, the length of the chain of difference sets in $G\cup\bar{G}$ is n. Since $\kappa(G\cup\bar{G})\ge max\{\kappa(G),\kappa(\bar{G})\}$, the length of the chain of difference sets of set-labels of vertices in G or in $\bar{G}$ must be n.

Conversely, assume that the length of the chain of difference sets of set-labels of vertices in G or in $\bar{G}$ is n. Then, length of the chain of difference sets in $G\cup\bar{G}$ is n, which is the maximum possible length of a chain of difference sets. Therefore, both G and $\bar{G}$ admit strong IASI under the same set-labels for the vertices of G. This completes the proof. □

Corollary 2.14. *If G is a self-complementary graph on n vertices, which admits a strong IASI, then $\kappa(G)=n$.*

Proof. Since G is self complementary, we have $G\cong\bar{G}$ and hence $\kappa(G)=\kappa(\bar{G})$. Therefore, $n=\kappa(G\cup\bar{G})=max\{\kappa(G),\kappa\bar{G}\}=\kappa(G)$. That is, $\kappa(G)=\kappa(\bar{G})=n$. Therefore, G and $\bar{G}$ admit strong IASI. This completes the proof. □

2.3 Strong IASIs of Graph Products

In this section, we discuss the admissibility of strong IASI by certain products of strong IASI graphs. In graph products, we may have to make certain number of copies of the graphs and to make suitable of attachments between these copies and the given graphs. Therefore, we have to establish a suitable IASI, if exists, to each of these copies. Hence, we make the following remark.

Remark 2.15. Let G_i is a copy of a given graph G, which appears in a graph product. Let $n\cdot A = \{na_i : a_i \in A\}$, for $n \in \mathbb{N}_0$. Note that $n\cdot A \neq nA$. If f is a strong IASI on G, then the i-th copy of G denoted by G_i has the set-label f_i where $f_i(v_i) = r\cdot f(v)$, $r \in \mathbb{N}$, where v_i is the vertex in G_i corresponding to the vertex v in G. We observe that if two sets A and B are disjoint, then $n\cdot A$ and $n\cdot B$ are also disjoint. Hence, if f is a strong IASI of G, then f_i is a strong IASI of G_i.

In this section, we verify the admissibility of strong IASI by the Cartesian product of two graphs. In this section, by the term *product of graphs* we mean the Cartesian product of graphs. we recall the definition the Cartesian product of two graphs as follows.

Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two graphs. Then, the *Cartesian product* or simply *product* of G_1 and G_2, denoted by $G_1\Box G_2$, is the graph with vertex set $V_1\times V_2$ defined as follows. Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two points in $V_1\times V_2$. Then, u and v are adjacent in $G_1\Box G_2$ whenever $[u_1=v_1$ and u_2 is adjacent to $v_2]$ or $[u_2=v_2$ and u_1 is adjacent to $v_1]$. If $|V_i|=p_i$ and $E_i=q_i$ for $i=1,2$, then $|V(G_1\Box G_2)|=p_1p_2$ and $|E(G_1\Box G_2)|=p_1q_2+p_2q_1$.

Remark 2.16. We observe that the product $G_1\Box G_2$ is obtained as follows. Make p_2 copies of G_1. Denote these copies by G_{1_i}, $1\le i\le p_2$, which corresponds to the vertex v_i of G_2. Now, join the corresponding vertices of two copies G_{1_i} and G_{1_j} if the corresponding vertices v_i and v_j are adjacent in G_2. Thus, we view the product $G_1\Box G_2$ as a union of p_2 copies of G_1 and a finite number of edges connecting two copies G_{1_i} and G_{1_j} of G_1 according to the adjacency of the corresponding vertices v_i and v_j in G_2, where $1\le i,\ j\le p_2$, $i\neq j$.

Hence, we make the following inferences on the admissibility of strong IASI by the product of two strong IASI-graphs.

Theorem 2.17. *Let G_1 and G_2 be two strong IASI-graphs. Then, the product $G_1\Box G_2$ admits a strong IASI if and only if the difference sets of the set-labels of corresponding vertices different copies of G_1 which are adjacent in $G_1\Box G_2$ are disjoint.*

Proof. Assume that the graphs G_1 and G_2 admit strong IASIs, say f and g respectively and $G_1\Box G_2$ admits a strong IASI, say F. Since f is a strong IASI on G, then by Remark 2.15, the function f_i defined on the i-th copy G_{1_i} of G_1, by $f_i(v_i)=r\cdot f(v)$, for some positive integer r, is a strong IASI on G_{1_i}. If the corresponding vertices of two copies $G1_i$ and G_{1_j} are adjacent in $G_1\Box G_2$, then $|g_f(v_iv_j)|=|f(v_i)|\cdot|f(v_j)|=|f_i(v_i)|\cdot|f_j(v_j)|$, $\forall\ v_i\in G_{1_i}$, $v_j\in G_{1_j}$. Hence, $D_{v_i}<D_{v_j}$. That is, the difference sets of $f_i(v_i)$ and $f_j(v_j)$ are disjoint.

Conversely, assume that the difference sets of the set-labels of corresponding vertices different copies of G_1 which are adjacent in $G_1\Box G_2$ are disjoint. Also, the function f_i defined on the i-th copy G_{1_i} of G_1, by $f_i(v_i)=r\cdot f(v)$, for some positive integer r, is a strong IASI on G_{1_i}, $1\le i\le|V(G_2)|$. Hence, the difference sets of the set-labels of all the adjacent vertices in $G_1\Box G_2$ are disjoint. Therefore, $G_1\Box G_2$ admits a strong IASI. □

Invoking the above theorem, we determine the nourishing number of the Cartesian product of two strong IASI-graphs in the following theorem.

Theorem 2.18. *Let G_1 and G_2 be two graphs which admit strong IASIs. Then, $\kappa(G_1 \Box G_2) = max\{\kappa(G_1), \kappa(G_2)\}$.*

Proof. Let H_1 and H_2 be the maximal clique in G_1 and G_2 respectively. Without loss of generality, let H_1 be greater than H_2 in terms of the number of vertices in them. For $1 \le i \le |V(G_2)|$, let H_{1_i} be the copy of H_1 in G_{1_i} and is the maximal clique in G_{1_i}. Now, observe that no vertex of another copy G_{1_j} is adjacent to all the vertices of H_{1_i}. Since, all cliques H_{1_i} are isomorphic, for $1 \le i \le |V(G_2)|$, H_{1_i} is the maximal clique in $G_1 \Box G_2$. Hence, $\kappa(G_1 \Box G_2) = |V(H_{1_i})| = V(H_1)|$. Therefore, in general, $\kappa(G_1 \Box G_2) = \max\{\kappa(G_1), \kappa(G_2)\}$. □

Another graph product we consider in this occasion is the corona of two graphs. The *corona* of two graphs G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained by taking $|V(G_1)|$ copies of G_2 and then joining the i-th vertex of G_1 to every vertex of the i-th copy of G_2.

The following result establishes the admissibility of a strong IASI by the corona of two strong IASI-graphs.

Theorem 2.19. *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two strong IASI-graphs. Then, their corona $G_1 \odot G_2$ admits a strong IASI if and only if the difference set of the set-label of every vertex in G_1 is disjoint from the difference sets of the set-labels of all vertices of the corresponding copy of G_2.*

Proof. Let f and g be the strong IASI on G_1 and G_2 respectively. Let $g_i = r \cdot g$, $1 \le i \le |V(G_1)|$, r being a positive integer, be an IASI defined on the i-th copy G_i of G. Then, by Remark 2.15, g_i is a strong IASI on G_i for all $i = 1, 2, 3, \ldots, |V(G_1)|$.

First assume that $G_1 \odot G_2$ admits a strong IASI. Also, each copy of G_2 admits strong IASIs. Since $G_1 \odot G_2$ admits a strong IASI, the difference set of the set-label of each vertex u_i of G_1 and the difference set of the set-label of each vertex v_{j_i} of G_{2_i}, where $1 \le i \le |V(G_1)$, $1 \le j \le |V(G_2)$, must be disjoint.

Conversely, assume that the difference set of the set-label of every vertex in G_1 is disjoint from the difference sets of the set-labels of all vertices of the corresponding copy of G_2. Since each copy of G_2 is also strong IASI-graph, for every pair of adjacent vertices u, v in $G_1 \odot G_2$, the difference sets D_u and D_v hold the relation $D_u < D_v$. Hence, $G_1 \odot G_2$ admits a strong IASI. □

In view of the above theorem, the nourishing number of the corona of two strong IASI-graphs, is determined in the theorem given below.

Theorem 2.20. *If G_1 and G_2 are two strong IASI-graphs, then*

$$\kappa(G_1 \odot G_2) = \begin{cases} \kappa(G_1) & \text{if } \kappa(G_1) > \kappa(G_2) \\ \kappa(G_2) + 1 & \text{if } \kappa(G_2) > \kappa(G_1). \end{cases}$$

Proof. Let H_1 and H_2 be the maximal cliques in G_1 and G_2 respectively. Then, H_{2_i} is the copy of H_2 in G_{2_i}, which is maximal in G_{2_i}. Since the vertex u_i of H_1 is adjacent to all vertices of the copy H_{2_i} in $G_1 \odot G_2$, we can find $|V(G_1)|$ cliques in $G_1 \odot G_2$ with clique number $1 + |V(H_{2_i}| = 1 + |V(H_2)| = 1 + \kappa(G_2)$.

If $\kappa(G_1) > \kappa(G_2)$, then clearly, $\kappa(G_1 \odot G_2) = \kappa(G_1)$. If $\kappa(G_1) < \kappa(G_2)$, then the maximal clique in $G_1 \odot G_2$ is $H_{2_i} + \{u_i\}$. Therefore, $\kappa(G_1 \odot G_2) = 1 + \kappa(G_2)$. This completes the proof. □

3. Conclusion

In this paper, we have reviewed the properties and characteristics of certain strong IASI-graphs and studied the admissibility of strong IASIs by certain graph classes, graph operations and graph products and determined the nourishing numbers. The admissibility of strong IASI by various other graph classes, operations and products are yet to be verified and finding their corresponding nourishing numbers are to be estimated.

More properties and characteristics of strong IASIs, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs still remain unsettled. All these facts highlight a great scope for further studies in this area.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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