Communications in Mathematics and Applications
Vol. 15, No. 1, pp. 145–160, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i1.2368

http://www.rgnpublications.com

Research Article

Some Properties of Kenmotsu Manifolds Admitting a New Type of Semi-Symmetric Non-Metric Connection

Abhishek Singh[1], Lovejoy S. Das[2], Rajendra Prasad[3] and Lalit Kumar*[1]

[1] *Department of Mathematics and Statistics, Dr. Rammanohar Lohia Avadh University, Ayodhya 224001, Uttar Pradesh, India*
[2] *Department of Mathematics, Kent State University, Ohio, USA*
[3] *Department of Mathematics and Astronomy, University of Lucknow, Lucknow 226007, Uttar Pradesh, India*
*__Corresponding author:__ lalitkumargautamjrf@gmail.com

Received: August 11, 2023 **Accepted:** January 27, 2024

Abstract. In this paper, we study some properties of Kenmotsu manifolds admitting a semi-symmetric non-metric connection. Some curvature's properties of Kenmotsu manifolds that admits a semi-symmetric non-metric connection are obtained. Semi-symmetric, Ricci semi-symmetric and locally ϕ-symmetric conditions for Kenmotsu manifolds with respect to semi-symmetric non-metric connection are also studied. It is proved that the manifold endowed with a semi-symmetric non-metric connection is regular. We obtain some conditions for semi-symmetric and Ricci semi-symmetric Kenmotsu manifolds endowed with semi-symmetric non-metric connection $\tilde{\nabla}$. It is further observed that the Ricci soliton of data (g, ξ, Θ) are expanding and shrinking respectively for semi-symmetric and Ricci semi-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection.

Keywords. Kenmotsu manifold, Semi-symmetric non-metric connection, Semi-symmetric manifold, Ricci semi-symmetric manifold, Locally ϕ-symmetric Kenmotsu manifold, Curvature tensor, Ricci tensor, Einstein manifold

Mathematics Subject Classification (2020). 53C15, 53C25, 53C05, 53D10

1. Introduction

The investigations of a differentiable manifold with contact and almost contact metric structures had been initiated by Boothby and Wang [5]. A class of almost contact metric manifold and named as Kenmotsu manifold has been initiated by Kenmotsu [16]. Further, the characteristics of Kenmotsu manifolds have been investigated by many authors such as Sinha and Srivastava [23], Chaubey and Yildiz [6], Chaubey and Ojha [8], Chaubey and Yadav [9], Özgür and De [19] and many others. The main invariants of an affine connection are its torsion and curvature (Friedmann and Schouten [14]). A torsion tensor of a connection is defined as under

$$\mathcal{T}(\mathcal{K}_1,\mathcal{K}_2) = \nabla_{\mathcal{K}_1}\mathcal{K}_2 - \nabla_{\mathcal{K}_2}\mathcal{K}_1 - [\mathcal{K}_1,\mathcal{K}_2], \quad \text{for all } \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M}), \tag{1.1}$$

where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$. A connection ∇ is symmetric and non-symmetric according as $\mathcal{T}(\mathcal{K}_1,\mathcal{K}_2) = 0$, and $\mathcal{T}(\mathcal{K}_1,\mathcal{K}_2) \neq 0$, respectively. The idea of semi-symmetric connection on a differentiable manifold has been proposed by Friedmann and Schouten [14]. A linear connection $\widetilde{\nabla}$ on $\mathfrak{M}^n$ is called semi-symmetric if

$$\widetilde{\mathcal{T}}(\mathcal{K}_1,\mathcal{K}_2) = \eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2, \quad \text{for all } \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M}), \tag{1.2}$$

where η is a 1-form associated with the vector field ξ and satisfies

$$\eta(\mathcal{K}_1) = g(\mathcal{K}_1,\xi), \quad \text{for all } \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M}). \tag{1.3}$$

Further, the idea of metric connection with torsion on a Riemannian manifold has been initiated by Hayden. A connection ∇ is metric and non-metric on $\mathfrak{M}$ according as $\nabla g = 0$ and $\nabla g \neq 0$, respectively, where g is a Reimannian metric in $\mathfrak{M}$. The idea of semi-symmetric non-metric connection has been initiated by Agashe and Chafle [1]. The quarter-symmetric connection in a differentiable manifold with affine connection has been investigated by Golab [15]. Further, characteristics of quarter-symmetric metric connection have been investigated by several geometers like Rastogi [22], Mishra and Pandey [18], Yano and Imai [28], Kumar *et al*. [17], and many others. The semi-symmetric non-metric connection in a Kenmotsu manifold has been investigated by Tripathi and Nakkar [26]. In line with this, Chaubey and Yildiz [6] initiated another semi-symmetric non-metric connection. Later on some other authors, like De and Pathak [10], Pankaj *et al*. [20,21] studied several connections. Tripathi [25] has justified the presence of a new connection and showed that in special cases.

We have decided by above investigations to study characteristics of Kenmotsu manifolds admitting a semi-symmetric non-metric connection. Section 1 is introductory. Section 2 is concerned with some basic results of Kenmotsu manifolds. The necessary results of a semi-symmetric non-metric connection are given in Section 3. Basic characteristics of Riemannian curvature tensor with respect to a semi-symmetric non-metric connection have been investigated in Section 4. Semi-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection have been investigated in Section 5. Ricci semi-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection have been investigated in Section 6. Locally ϕ-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection have been investigated in Section 7. In Section 8, we give an example.

2. Preliminaries

Suppose $\mathfrak{M}$ be an $(2n+1)$-dimensional almost contact metric manifolds with an almost contact metric quartet (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and the Riemannian metric g on $\mathfrak{M}$ satisfying [3, 27]:

$$\eta(\xi)=1, \quad \phi(\xi)=0, \quad \eta(\phi(\mathcal{K}_1))=0, \quad g(\mathcal{K}_1,\xi)=\eta(\mathcal{K}_1), \tag{2.1}$$

$$\phi^2(\mathcal{K}_1)=-\mathcal{K}_1+\eta(\mathcal{K}_1)\xi, \quad g(\mathcal{K}_1,\phi\mathcal{K}_2)=-g(\phi\mathcal{K}_1,\mathcal{K}_2), \tag{2.2}$$

$$g(\phi\mathcal{K}_1,\phi\mathcal{K}_2)=g(\mathcal{K}_1,\mathcal{K}_2)-\eta(\mathcal{K}_1)\eta(\mathcal{K}_2). \tag{2.3}$$

An almost contact metric quartet (ϕ,ξ,η,g) is a Kenmotsu manifolds [16] iff

$$(\nabla_{\mathcal{K}_1}\phi)(\mathcal{K}_2)=g(\phi\mathcal{K}_1,\mathcal{K}_2)\xi-\eta(\mathcal{K}_2)\phi\mathcal{K}_1. \tag{2.4}$$

It is also defined by above investigations.

$$\nabla_{\mathcal{K}_1}\xi=\mathcal{K}_1-\eta(\mathcal{K}_1)\xi, \tag{2.5}$$

$$(\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2)=g(\mathcal{K}_1,\mathcal{K}_2)-\eta(\mathcal{K}_1)\eta(\mathcal{K}_2)=g(\phi\mathcal{K}_1,\phi\mathcal{K}_1), \tag{2.6}$$

$$\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi=\eta(\mathcal{K}_1)\mathcal{K}_2-\eta(\mathcal{K}_2)\mathcal{K}_1, \tag{2.7}$$

$$\mathfrak{R}(\xi,\mathcal{K}_1)\mathcal{K}_2=\eta(\mathcal{K}_2)\mathcal{K}_1-g(\mathcal{K}_1,\mathcal{K}_2)\xi, \tag{2.8}$$

$$\mathfrak{R}(\xi,\mathcal{K}_1)\xi=\mathcal{K}_1-\eta(\mathcal{K}_1)\xi, \tag{2.9}$$

$$\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)=g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)-g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1), \tag{2.10}$$

$$\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)=\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)+2n\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{2.11}$$

$$\mathcal{S}(\mathcal{K}_1,\xi)=-2n\eta(\mathcal{K}_1), \tag{2.12}$$

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)=g(\mathfrak{Q}\mathcal{K}_1,\mathcal{K}_2), \tag{2.13}$$

$\forall\ \mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\in\mathfrak{X}(\mathfrak{M})$, where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$ and $\mathfrak{R}$, $\mathcal{S}$ and $\mathfrak{Q}$ represent the curvature tensor, Ricci tensor and Ricci operator of the manifold $\mathfrak{M}$, respectively, with respect to the Levi-Civita connection ∇.

Definition 2.1. An almost contact metric manifold $\mathfrak{M}$ is said to be an η-Einstein manifolds if there exists the real valued functions Θ_1, Θ_2 such that

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)=\Theta_1 g(\mathcal{K}_1,\mathcal{K}_2)+\Theta_2\eta(\mathcal{K}_1)\eta(\mathcal{K}_2). \tag{2.14}$$

For $\Theta_2=0$, the manifold $\mathfrak{M}$ is an Einstein manifolds.

Definition 2.2. A Ricci soliton $(g,\mathcal{V},\Theta)$ on a Riemannian manifold is defined by

$$\mathfrak{L}_{\mathcal{V}}g+2\mathcal{S}+2\Theta g=0, \tag{2.15}$$

on $\mathfrak{M}$, where $\mathfrak{L}_{\mathcal{V}}g$ is a Lie-derivative along the vector field $\mathcal{V}$ of metric g and $\Theta\in\mathbb{R}$. The Ricci soliton $(g,\mathcal{V},\Theta)$ is shrinking, steady and expanding whenever, $\Theta<0$, $\Theta=0$ and $\Theta>0$, respectively [2].

Definition 2.3. The Ricci tensor $\mathcal{S}$ of a Kenmotsu manifolds is said to be η-parallel if it satisfies

$$(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\mathcal{K}_2,\phi\mathcal{K}_3)=0. \tag{2.16}$$

The idea of Ricci η-parallelity for Sasakian manifolds was investigated by Yano and Kon [27]. In [11] the authors proved that a 3-dimensional Kenmotsu manifold has η-parallel Ricci tensor iff it is of constant scalar curvature.

3. A Semi-Symmetric Non-Metric Connection

Let us define, a linear connection $\widetilde{\nabla}$ [4,13] as

$$\widetilde{\nabla}_{\mathcal{K}_1}\mathcal{K}_2 = \nabla_{\mathcal{K}_1}\mathcal{K}_2 + \frac{1}{2}[\eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2] \tag{3.1}$$

satisfying

$$\widetilde{\mathfrak{T}}(\mathcal{K}_1,\mathcal{K}_2) = \eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2, \tag{3.2}$$

and

$$(\widetilde{\nabla}_{\mathcal{K}_1}g)(\mathcal{K}_2,\mathcal{K}_3) = \frac{1}{2}[2\eta(\mathcal{K}_1)g(\mathcal{K}_2,\mathcal{K}_3) - \eta(\mathcal{K}_2)g(\mathcal{K}_1,\mathcal{K}_3) - \eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_2)]. \tag{3.3}$$

for arbitrary vector fields $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ is said to be a semi-symmetric non-metric connection. Also, we have

$$(\widetilde{\nabla}_{\mathcal{K}_1}\phi)(\mathcal{K}_2) = \frac{1}{2}[2(\nabla_{\mathcal{K}_1}\phi)(\mathcal{K}_2) - \eta(\mathcal{K}_2)\phi(\mathcal{K}_1)], \tag{3.4}$$

$$(\widetilde{\nabla}_{\mathcal{K}_1}\eta)(\mathcal{K}_2) = (\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2), \tag{3.5}$$

$$(\widetilde{\nabla}_{\mathcal{K}_1}g)(\phi\mathcal{K}_1,\mathcal{K}_3) = \frac{1}{2}[2\eta(\mathcal{K}_1)g(\phi\mathcal{K}_2,\mathcal{K}_3) - \eta(\mathcal{K}_3)g(\mathcal{K}_1,\phi\mathcal{K}_2)]. \tag{3.6}$$

On replacing $\mathcal{K}_2$ by ξ in the equation (3.1), we have

$$\widetilde{\nabla}_{\mathcal{K}_1}\xi = \frac{3}{2}\nabla_{\mathcal{K}_1}\xi. \tag{3.7}$$

On replacing $\mathcal{K}_1$ by ξ in the equation (3.3), we have

$$(\widetilde{\nabla}_{\xi}g)(\mathcal{K}_2,\mathcal{K}_3) = g(\phi\mathcal{K}_2,\phi\mathcal{K}_3) = (\nabla_{\mathcal{K}_2}\eta)(\mathcal{K}_3). \tag{3.8}$$

Hence we have the following propositions:

Proposition 3.1. *The vector field ξ with respect to ∇ and $\widetilde{\nabla}$ is related by equation* (3.7).

Proposition 3.2. *Co-variant differentiation of g with respect to contra-variant vector field ξ is given by the equation* (3.8) *in a contact metric manifold admitting connection $\widetilde{\nabla}$.*

The curvature tensor $\widetilde{\mathfrak{R}}$ of $\widetilde{\nabla}$ defined as follows

$$\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \widetilde{\nabla}_{\mathcal{K}_1}\widetilde{\nabla}_{\mathcal{K}_2}\mathcal{K}_3 - \widetilde{\nabla}_{\mathcal{K}_2}\widetilde{\nabla}_{\mathcal{K}_1}\mathcal{K}_3 - \widetilde{\nabla}_{[\mathcal{K}_1,\mathcal{K}_2]}\mathcal{K}_3, \tag{3.9}$$

where $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3 \in \mathfrak{X}(\mathfrak{M})$.
Using equation (3.1) in (3.9), we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 &+ \frac{1}{2}[(\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_3)\mathcal{K}_2 - (\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2)\mathcal{K}_3 \\ &- (\nabla_{\mathcal{K}_2}\eta)(\mathcal{K}_3)\mathcal{K}_1 + (\nabla_{\mathcal{K}_2}\eta)(\mathcal{K}_1)\mathcal{K}_3] \\ &+ \frac{1}{4}[\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2],\end{aligned} \tag{3.10}$$

where

$$\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \nabla_{\mathcal{K}_1}\nabla_{\mathcal{K}_2}\mathcal{K}_3 - \nabla_{\mathcal{K}_2}\nabla_{\mathcal{K}_1}\mathcal{K}_3 - \nabla_{[\mathcal{K}_1,\mathcal{K}_2]}\mathcal{K}_3 \tag{3.11}$$

is the Riemannian curvature tensor [3] of ∇.

Proposition 3.3. *The relation between Riemannian curvature tensors $\tilde{\mathfrak{R}}$ and $\mathfrak{R}$ with respect to connections $\tilde{\nabla}$ and ∇, respectively is given by the equation* (3.10).

4. Some Curvature Tensor of Kenmotsu Manifolds With a Semi-Symmetric Non-Metric Connection

Now using equation (2.6) in equation (3.10), we have

$$\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + \frac{1}{2}[g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2 - g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1] + \frac{3}{4}[\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2]. \tag{4.1}$$

Contracting of (4.1) with respect to $\mathcal{K}_1$, we have

$$\tilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_3) = \mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) - ng(\mathcal{K}_2,\mathcal{K}_3) + \frac{3}{2}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \tag{4.2}$$

Using (2.13) in equation (4.2), we have

$$\tilde{\mathfrak{Q}}\mathcal{K}_2 = \mathfrak{Q}\mathcal{K}_2 - n(\mathcal{K}_1) + \frac{3}{2}n\eta(\mathcal{K}_2)\xi. \tag{4.3}$$

Again contracting equation (4.2), we have

$$\tilde{\mathfrak{r}} = \mathfrak{r} - \frac{n}{2}(4n-1), \tag{4.4}$$

where $\tilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_3)$; $\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)$, $\tilde{\mathfrak{Q}}$; $\mathfrak{Q}$ and $\tilde{\mathfrak{r}}$; $\mathfrak{r}$ are the Ricci tensors, Ricci operators and scalar curvatures of $\tilde{\nabla}$ and ∇.

On replacing $\mathcal{K}_1$ by ξ in (4.1) and using (2.1), (2.2), we have

$$\tilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3 = \mathfrak{R}(\xi,\mathcal{K}_2)\mathcal{K}_3 - \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\xi + \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\xi - \frac{1}{4}\eta(\mathcal{K}_3)\mathcal{K}_2. \tag{4.5}$$

In view of (2.8) and (4.5), we have

$$\tilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3 = \frac{3}{4}[-2g(\mathcal{K}_2,\mathcal{K}_3)\xi + \eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\xi + \eta(\mathcal{K}_3)\mathcal{K}_2]. \tag{4.6}$$

Again on replacing $\mathcal{K}_3$ by ξ in (4.1) and using (2.1), (2.7), we have

$$\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\xi = \frac{3}{4}\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi = -\frac{3}{4}\tilde{\mathcal{T}}(\mathcal{K}_1,\mathcal{K}_2) \neq 0. \tag{4.7}$$

Thus, we have the following theorem:

Theorem 4.1. *Every* $(2n+1)$*-dimensional Kenmotsu manifold admitting connection $\tilde{\nabla}$ is regular.*

Now operating η on both sides of equation (4.1) and using equation (2.1), we have

$$\eta(\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = \frac{1}{2}[2g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2) - 2g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1) + g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2) - g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)]. \tag{4.8}$$

On contracting of (4.7) with respect to $\mathcal{K}_1$, we have

$$\widetilde{\mathcal{S}}(\mathcal{K}_2,\xi) = -\frac{3}{2}n\eta(\mathcal{K}_2). \tag{4.9}$$

In view of equations (4.2), (4.3) and (4.4), we have the following lemma:

Lemma 4.1. *In a Kenmotsu manifold Ricci tensor, Ricci operator and scalar curvature with respect to connections* $\widetilde{\nabla}$ *and* ∇ *are related by the equations* (4.2)*,* (4.3) *and* (4.4)*.*

Proof. On taking $\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = 0$ in the equation (4.1), we have

$$\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = -\frac{1}{2}[g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2 - g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1] - \frac{3}{4}[\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2]. \tag{4.10}$$

Thus

$$\begin{aligned}{}'\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4) = &-\frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4)\\ &-\frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4).\end{aligned} \tag{4.11}$$

Contracting of (4.11) with respect to vector field $\mathcal{K}_1$, we have

$$\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) = ng(\mathcal{K}_2,\mathcal{K}_3) - \frac{3}{2}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \tag{4.12}$$

Using equation (2.13) in equation (4.12), we have

$$\mathfrak{Q}\mathcal{K}_2 = n\mathcal{K}_2 - \frac{3}{2}n\eta(\mathcal{K}_2)\xi. \tag{4.13}$$

Again contracting equation (4.12), we have

$$\mathfrak{r} = \frac{n}{2}(4n-1). \tag{4.14}$$

□

By virtue of Definition 2.1 and equation (4.12), we state the theorem:

Theorem 4.2. *If Riemannian curvature tensor with respect to connection* $\widetilde{\nabla}$ *in a Kenmotsu manifold vanishes, then the manifold is an* η*-Einstein manifold.*

5. Semi-Symmetric Kenmotsu Manifolds

A $(2n+1)$-dimensional Kenmotsu manifold $\mathfrak{M}$ with $\widetilde{\nabla}$ is said to be semi-symmetric [20] if

$$(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\widetilde{\mathfrak{R}})(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 = 0,$$

i.e.

$$\begin{aligned}&\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 - \widetilde{\mathfrak{R}}(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5\\ &-\widetilde{\mathfrak{R}}(\mathcal{K}_3,\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\mathcal{K}_5 - \widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_5 = 0.\end{aligned} \tag{5.1}$$

On replacing $\mathcal{K}_1$ by ξ, we have

$$\begin{aligned}&\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 - \widetilde{\mathfrak{R}}(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5\\ &-\widetilde{\mathfrak{R}}(\mathcal{K}_3,\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_4)\mathcal{K}_5 - \widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_5 = 0.\end{aligned} \tag{5.2}$$

In view of equations (2.1), (2.2), (4.6), (4.7) and (4.8), we have

$$
\begin{aligned}
'\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5,\mathcal{K}_2) = {} & g(\mathcal{K}_2,\mathcal{K}_3)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_4)\mathcal{K}_5) - \frac{1}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_4)\mathcal{K}_5) \\
& + \frac{1}{2}\eta(\mathcal{K}_3)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_2,\mathcal{K}_4)\mathcal{K}_5) - g(\mathcal{K}_2,\mathcal{K}_4)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_3)\mathcal{K}_5) \\
& + \frac{1}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_3)\mathcal{K}_5) + \frac{1}{2}\eta(\mathcal{K}_4)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_2)\mathcal{K}_5) \\
& + g(\mathcal{K}_2,\mathcal{K}_5)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\xi) - \frac{1}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_5)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\xi) \\
& + \frac{1}{2}\eta(\mathcal{K}_5)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_2).
\end{aligned} \tag{5.3}
$$

By using equations (2.1), (2.2), (4.6), (4.7) and (4.8), we have

$$
\begin{aligned}
'\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5,\mathcal{K}_2) = {} & -\frac{3}{2}g(\mathcal{K}_2,\mathcal{K}_3)g(\mathcal{K}_4,\mathcal{K}_5) + \frac{3}{2}g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_4)\eta(\mathcal{K}_5) - \frac{3}{4}\eta(\mathcal{K}_4)\eta(\mathcal{K}_5) \\
& + \frac{3}{2}g(\mathcal{K}_4,\mathcal{K}_5)g(\mathcal{K}_5,\mathcal{K}_3) - \frac{9}{4}g(\mathcal{K}_4,\mathcal{K}_2)\eta(\mathcal{K}_5)\eta(\mathcal{K}_3).
\end{aligned} \tag{5.4}
$$

Hence, we have

$$
\begin{aligned}
\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 = {} & -\frac{3}{2}g(\mathcal{K}_4,B_5)\mathcal{K}_3 + \frac{3}{2}\eta(\mathcal{K}_4)\eta(\mathcal{K}_5)\mathcal{K}_3 - \frac{3}{4}\eta(\mathcal{K}_4)\eta(\mathcal{K}_5) + \frac{3}{2}g(\mathcal{K}_5,\mathcal{K}_3)\mathcal{K}_4 \\
& - \frac{9}{4}\eta(\mathcal{K}_5)\eta(\mathcal{K}_3)\mathcal{K}_4.
\end{aligned} \tag{5.5}
$$

Contracting equation (5.5) with respect to $\mathcal{K}_3$, we have

$$\widetilde{\mathcal{S}}(\mathcal{K}_4,\mathcal{K}_5) = -3ng(\mathcal{K}_4,\mathcal{K}_5) + \frac{3}{2}(2n-1)\eta(\mathcal{K}_4)\eta(\mathcal{K}_5). \tag{5.6}$$

Using equation (4.2) in above equation, we obtain

$$\mathcal{S}(\mathcal{K}_4,\mathcal{K}_5) = -2ng(\mathcal{K}_4,\mathcal{K}_5) + \frac{3}{2}(n-1)\eta(\mathcal{K}_4)\eta(\mathcal{K}_5). \tag{5.7}$$

Using equation (2.13) in above equation, we have

$$\mathfrak{Q}\mathcal{K}_4 = -2n\mathcal{K}_4 + \frac{3}{2}(n-1)\eta(\mathcal{K}_4)\xi. \tag{5.8}$$

Again contracting equation (5.7), we obtain

$$\mathfrak{r} = -\frac{1}{2}(8n^2+n+3). \tag{5.9}$$

By virtue of Definition 2.1 and equation (5.7), we can state

Theorem 5.1. *A semi-symmetric Kenmotsu manifold admitting connection $\widetilde{\nabla}$ is an η-Einstein manifold.*

The Ricci soliton of data $(g,\mathcal{V},\Theta)$ is defined by (2.15), where g, $\mathcal{V}$, Θ are Riemannian metric, a vector field and a real constant. Here two conditions come out with regard to the $\mathcal{V}: \mathcal{V} \in \operatorname{span}\{\xi\}$ and $\mathcal{V} \perp \operatorname{span}\{\xi\}$. Now taking $\mathcal{V} \in \operatorname{span}\{\xi\}$. The Ricci soliton of data (g,ξ,Θ) on a Kenmotsu manifold admitting connection $\widetilde{\nabla}$ defined as under:

$$(\widetilde{\mathfrak{L}}_\xi g)(\mathcal{K}_1,\mathcal{K}_2) + 2\widetilde{\mathcal{S}}(\mathcal{K}_1,\mathcal{K}_2) + 2\Theta g(\mathcal{K}_1,\mathcal{K}_2) = 0. \tag{5.10}$$

$\forall\ \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M})$. Here $\widetilde{\mathfrak{L}}_\xi g$, the Lie-derivative of g with respect to ξ admitting connection $\widetilde{\nabla}$, is defined as under

$$(\widetilde{\mathfrak{L}}_\xi g)(\mathcal{K}_1,\mathcal{K}_2) = g(\widetilde{\nabla}_{\mathcal{K}_1}\xi,\mathcal{K}_2) + g(\mathcal{K}_1,\widetilde{\nabla}_{\mathcal{K}_2}\xi) - 2g(\phi K_1,\phi K_2). \tag{5.11}$$

Now, using equations (2.1), (2.3), (2.5), (3.7) and (5.11), we have

$$(\widetilde{\mathfrak{L}}_\xi g)(\mathcal{K}_1,\mathcal{K}_2) = g(\phi\mathcal{K}_1,\phi\mathcal{K}_2). \tag{5.12}$$

Using equations (5.6) and (5.12) in the equation (5.10), we have

$$g(\phi\mathcal{K}_1,\phi\mathcal{K}_2) - 6ng(\mathcal{K}_1,\mathcal{K}_2) + 3(2n-1)\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) + 2\Theta g(\mathcal{K}_1,\mathcal{K}_2) = 0. \tag{5.13}$$

On taking $\mathcal{K}_1 = \mathcal{K}_2 = \xi$ and using (2.1) in (5.13), we have

$$\Theta = \frac{3}{2} > 0. \tag{5.14}$$

Thus, we state the theorem:

Theorem 5.2. *A semi-symmetric Kenmotsu manifold admitting connection* $\widetilde{\nabla}$*, the Ricci soliton of data* (g,ξ,Θ) *is always expanding.*

6. Ricci Semi-Symmetric Kenmotsu Manifolds

A $(2n+1)$-dimensional contact metric manifolds $\mathfrak{M}$ with respect to connection $\widetilde{\nabla}$ is said to be Ricci semi-symmetric [20] if

$$(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4) = 0.$$

i.e.

$$\widetilde{\mathcal{S}}(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4) + \widetilde{\mathcal{S}}(\mathcal{K}_3,\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4) = 0. \tag{6.1}$$

On replacing $\mathcal{K}_1$ by ξ and using (4.6) in (6.1), we have

$$\widetilde{\mathcal{S}}(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4) + \widetilde{\mathcal{S}}(\mathcal{K}_3,\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_4) = 0, \tag{6.2}$$

i.e.

$$\begin{aligned} -\frac{3}{2}g(\mathcal{K}_2,\mathcal{K}_3)\widetilde{\mathcal{S}}(\xi,\mathcal{K}_4) + \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\widetilde{\mathcal{S}}(\xi,\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_3)\widetilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_4) - \frac{3}{2}g(\mathcal{K}_2,\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3,\xi) \\ + \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3,\xi) - \frac{3}{4}\eta(\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3,\mathcal{K}_2) = 0. \end{aligned} \tag{6.3}$$

In view of equation (4.9), the above equation yields

$$\begin{aligned} \frac{9}{4}ng(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_4) - \frac{9}{8}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_3)\widetilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_4) + \frac{9}{4}ng(\mathcal{K}_2,\mathcal{K}_4)\eta(\mathcal{K}_3) \\ - \frac{9}{8}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3,\mathcal{K}_2) = 0. \end{aligned} \tag{6.4}$$

Again replacing $\mathcal{K}_4$ by ξ and using (4.9) in (6.4), we have

$$\widetilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_3) = 3ng(\mathcal{K}_2,\mathcal{K}_3) + \frac{3}{2}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \tag{6.5}$$

Using (4.2) in (6.5), we have

$$\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) = 4ng(\mathcal{K}_2,\mathcal{K}_3). \tag{6.6}$$

On contracting equation (6.6), we have

$$\mathfrak{r} = 4n(2n+1), \tag{6.7}$$

with the help of equation (6.7), equation (4.4) takes the form

$$\tilde{\mathfrak{r}} = \frac{3n}{2}(4n+3). \tag{6.8}$$

In view of equation (6.6), we can state following:

Theorem 6.1. *A Ricci semi-symmetric Kenmotsu manifold equipped with connection $\tilde{\nabla}$ is an Einstein manifold.*

Using equation (4.1) in the given below equation

$$(\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\tilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4) = -\tilde{\mathcal{S}}(\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4) - \tilde{\mathcal{S}}(\mathcal{K}_3,\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4), \tag{6.9}$$

we have

$$\begin{aligned}(\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\tilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4) = {} & (\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4) - \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4)\\ & + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4)\\ & + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4) - \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)\\ & + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1)\\ & + \frac{3}{4}\eta(B_1)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) - \frac{3}{2}n\cdot\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\eta(\mathcal{K}_4)\\ & - \frac{3}{2}n\cdot\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\eta(\mathcal{K}_3).\end{aligned} \tag{6.10}$$

If we assume $(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4) = (\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\tilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4)$, then from equation (6.10), we have

$$\begin{aligned}& -\frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4)\\ & \quad + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4) - \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1)\\ & \quad - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\eta(\mathcal{K}_4)\\ & \quad - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\eta(\mathcal{K}_3) = 0,\end{aligned} \tag{6.11}$$

where

$$(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4) = -\mathcal{S}(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4) - \mathcal{S}(\mathcal{K}_3,\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4). \tag{6.12}$$

Now, replacing $\mathcal{K}_4$ by ξ in the equation (6.11), we have

$$\begin{aligned}& -\frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\xi) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\xi) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\xi) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\xi)\\ & \quad - \frac{1}{2}g(\mathcal{K}_1,\xi)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) + \frac{1}{2}g(\mathcal{K}_2,\xi)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\xi)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\xi)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)\\ & \quad - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\eta(\xi) - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi)\eta(\mathcal{K}_3) = 0.\end{aligned} \tag{6.13}$$

Now, using equations (2.1), (2.10) and (6.5) in equation (6.13), we have

$$-3ng(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)-\frac{3}{2}ng(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)+3ng(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)+\frac{3}{2}ng(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)=0, \tag{6.14}$$

i.e.

$$\frac{9}{2}n[\eta(\mathcal{K}_1)g(\mathcal{K}_2,\mathcal{K}_3)-\eta(\mathcal{K}_2)g(\mathcal{K}_1,\mathcal{K}_3)]=0, \tag{6.15}$$

which is not possible. Hence we have the following:

Corollary 6.1. *In a Ricci semi-symmetric Kenmotsu manifold admitting connection* $\widetilde{\nabla}$

$$(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4)\neq(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4). \tag{6.16}$$

Using equations (5.12) and (6.5) in the equation (5.10), we have

$$2(3n+\Theta)g(\mathcal{K}_1,\mathcal{K}_2)+g(\phi\mathcal{K}_1,\phi\mathcal{K}_2)+3n\eta(\mathcal{K}_1)\eta(\mathcal{K}_2)=0. \tag{6.17}$$

On taking $\mathcal{K}_1=\mathcal{K}_2=\xi$ and using (2.1) in (6.17), we have

$$\Theta=-\frac{9n}{2}<0. \tag{6.18}$$

Thus, we have the following:

Theorem 6.2. *A Ricci semi-symmetric Kenmotsu manifold admitting connection* $\widetilde{\nabla}$*, the Ricci soliton of data* (g,ξ,Θ) *is always shrinking.*

7. Locally ϕ-Symmetric Kenmotsu Manifolds

Definition 7.1. A Kenmotsu manifolds $\mathfrak{M}$ admitting connection $\widetilde{\nabla}$ is called locally ϕ-symmetric [24] if

$$\phi^2((\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)=0$$

$\forall$ $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4$ are orthogonal to ξ.

Taking covariant differentiation of $\mathfrak{R}$ with respect to $\mathcal{K}_4$, we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3&=\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3-\mathfrak{R}(\widetilde{\nabla}_{\mathcal{K}_4}\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3\\&\quad-\mathfrak{R}(\mathcal{K}_1,\widetilde{\nabla}_{\mathcal{K}_4}\mathcal{K}_2)\mathcal{K}_3-\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)(\widetilde{\nabla}_{\mathcal{K}_4}\mathcal{K}_3).\end{aligned} \tag{7.1}$$

Now using equations (2.10) and (3.1) in equation (7.1), we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3&=(\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\frac{1}{2}[2\eta(\mathcal{K}_4)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3-\eta(\mathcal{K}_1)\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3\\&\quad-\eta(\mathcal{K}_2)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3-\eta(\mathcal{K}_3)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4+g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_4\\&\quad-g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_4].\end{aligned} \tag{7.2}$$

Applying covariant differentiation on (4.1) with respect to $\mathcal{K}_4$, we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3&=(\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\frac{1}{2}[(\widetilde{\nabla}_{\mathcal{K}_4}g)(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2-(\widetilde{\nabla}_{\mathcal{K}_4}g)(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1]\\&\quad+\frac{3}{4}[(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1+(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_1-(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2\\&\quad-(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_2].\end{aligned} \tag{7.3}$$

Using equations (2.6), (3.3), (3.5) and (7.2), we have

$$(\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = (\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + \frac{1}{2}[2\eta(\mathcal{K}_4)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \eta(\mathcal{K}_1)\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3 - \eta(\mathcal{K}_2)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3 - \eta(\mathcal{K}_3)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4 + g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_4 - g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_4] + \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_2 - \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1 - g(\mathcal{K}_4,\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2 + g(\mathcal{K}_4,\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - g(\mathcal{K}_4,\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_2 + g(\mathcal{K}_4,\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_1 - \frac{3}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1 + \frac{3}{2}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1. \tag{7.4}$$

Now applying ϕ^2 on both sides of equation (7.4) and using equation (2.2), we have

$$\phi^2((\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = \phi^2((\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) + \frac{1}{2}[-2\eta(\mathcal{K}_4)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + 2\eta(\mathcal{K}_4)\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\xi + \eta(\mathcal{K}_1)\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3 - \eta(\mathcal{K}_1)\eta(\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3)\xi + \eta(\mathcal{K}_2)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3 - \eta(\mathcal{K}_2)\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3)\xi + \eta(\mathcal{K}_3)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi - \eta(\mathcal{K}_3)\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\xi - \eta(\mathcal{K}_2)g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_4 + 2\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)g(\mathcal{K}_1,\mathcal{K}_3)\xi + \eta(\mathcal{K}_1)g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_4 - 2\eta(\mathcal{K}_1)\eta(\mathcal{K}_4)g(\mathcal{K}_2,\mathcal{K}_3)\xi - \eta(\mathcal{K}_4)g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2 + \eta(\mathcal{K}_4)g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1 + 2\eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_2 - 2\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4)\xi - 2\eta(\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4)\mathcal{K}_1 + 2\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4)\xi + 2\eta(\mathcal{K}_1)g(\mathcal{K}_4,\mathcal{K}_3)\mathcal{K}_2 - 2\eta(\mathcal{K}_2)g(\mathcal{K}_4,\mathcal{K}_3)\mathcal{K}_1 + 3\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1 - 3\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_2]. \tag{7.5}$$

Taking $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3$ and $\mathcal{K}_4$ orthogonal to ξ, then from equation (7.5), we have

$$\phi^2((\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = \phi^2((\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3). \tag{7.6}$$

Theorem 7.1. *The necessary and sufficient condition for a Kenmotsu manifold to be locally ϕ-symmetric with respect to connection $\widetilde{\nabla}$ is that the manifold is also locally ϕ-symmetric with respect to the connection ∇.*

8. Example of a Three-Dimensional Kenmotsu Manifold

Let three-dimensional manifold $\mathfrak{M}^3 = \{(t_1,t_2,t_3) \in \mathbb{R}^3 : t_3 > 0\}$, where (t_1,t_2,t_3) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields [12]

$$\varsigma_1 = t_3\frac{\partial}{\partial t_1}, \quad \varsigma_2 = t_3\frac{\partial}{\partial t_2}, \quad \varsigma_3 = -t_3\frac{\partial}{\partial t_3}$$

are linearly independent at each point of $\mathfrak{M}$. Let g be the Riemannian metric defined by

$$\left.\begin{aligned} g(\varsigma_1,\varsigma_2) = g(\varsigma_2,\varsigma_3) = g(\varsigma_3,\varsigma_1) = 0,\\ g(\varsigma_1,\varsigma_1) = g(\varsigma_2,\varsigma_2) = g(\varsigma_3,\varsigma_3) = 1,\end{aligned}\right\} \tag{8.1}$$

where

$$g = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Let η be the 1-form defined by $\eta(\mathcal{K}_1) = g(\mathcal{K}_1,\varsigma_3)$ for any $\mathcal{K}_1 \in \mathfrak{X}(\mathfrak{M})$. Let ϕ be the $(1,1)$-tensor field defined by

$$(\phi\varsigma_1) = -\varsigma_2, \quad (\phi\varsigma_2) = \varsigma_1, \quad (\phi\varsigma_3) = 0. \tag{8.2}$$

Now for $\mathcal{K}_1 = \mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2 + \mathcal{K}_1^3\varsigma_3$ and $\xi = \varsigma_3$, using linearity of ϕ and g, we have

$$\eta(\varsigma_3) = \eta(\xi) = 1, \quad \phi^2(\mathcal{K}_1) = -\mathcal{K}_1 + \eta(\mathcal{K}_1)\varsigma_3 = -(\mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2) \tag{8.3}$$

where $\mathcal{K}_1^1, \mathcal{K}_1^2, \mathcal{K}_1^3$ are the scalars and $\forall\ \mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M})$. Thus for $\varsigma_3 = \xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on $\mathfrak{M}$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, we have

$$\left.\begin{aligned} &[\varsigma_1,\varsigma_1] = 0, && [\varsigma_1,\varsigma_2] = 0, && [\varsigma_1,\varsigma_3] = \varsigma_1,\\ &[\varsigma_2,\varsigma_1] = 0, && [\varsigma_2,\varsigma_2] = 0, && [\varsigma_2,\varsigma_3] = \varsigma_2,\\ &[\varsigma_3,\varsigma_1] = -\varsigma_1, && [\varsigma_3,\varsigma_2] = -\varsigma_2, && [\varsigma_3,\varsigma_3] = 0.\end{aligned}\right\} \tag{8.4}$$

Now using equation (2.3), we have

$$g(\mathcal{K}_1,\mathcal{K}_2) = \mathcal{K}_1^1\mathcal{K}_2^1 + \mathcal{K}_1^2 B_2^2 + \mathcal{K}_1^3\mathcal{K}_2^3. \tag{8.5}$$

Let us consider ∇, a Levi-Civita connection admitting a Riemannian metric g. Using the Koszul formula

$$\begin{aligned} 2g(\nabla_{\mathcal{K}_1}\mathcal{K}_2,\mathcal{K}_3) &= \mathcal{K}_1 g(\mathcal{K}_2,\mathcal{K}_3) + \mathcal{K}_2 g(\mathcal{K}_3,\mathcal{K}_1) - \mathcal{K}_3 g(\mathcal{K}_1,\mathcal{K}_2)\\ &\quad + g([\mathcal{K}_1,\mathcal{K}_2],\mathcal{K}_3) - g([\mathcal{K}_2,\mathcal{K}_3],\mathcal{K}_1) + g([\mathcal{K}_3,\mathcal{K}_1],\mathcal{K}_2). \end{aligned} \tag{8.6}$$

By virtue of (8.6), we have

$$\left.\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1 = 0, && \nabla_{\varsigma_1}\varsigma_2 = 0, && \nabla_{\varsigma_1}\varsigma_3 = \varsigma_1,\\ &\nabla_{\varsigma_2}\varsigma_1 = 0, && \nabla_{\varsigma_2}\varsigma_2 = -\varsigma_3, && \nabla_{\varsigma_2}\varsigma_3 = \varsigma_2,\\ &\nabla_{\varsigma_3}\varsigma_1 = 0, && \nabla_{\varsigma_3}\varsigma_2 = 0, && \nabla_{\varsigma_3}\varsigma_3 = 0.\end{aligned}\right\} \tag{8.7}$$

Again for $\mathcal{K}_1 = \mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2 + \mathcal{K}_1^3\varsigma_3$ and $\xi = \varsigma_3$, we have

$$\frac{3}{2}\nabla_{\mathcal{K}_1}\xi = \frac{3}{2}[\mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2], \tag{8.8}$$

i.e.

$$\nabla_{\mathcal{K}_1}\xi = \mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2, \tag{8.9}$$

$$\mathcal{K}_1 - \eta(\mathcal{K}_1)\xi = \mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2, \tag{8.10}$$

where $\mathcal{K}_1^1, \mathcal{K}_1^2, \mathcal{K}_1^3$ are scalars. From equations (8.9) and (8.10) it follows that the manifold satisfies equation (2.5) for $\xi = \varsigma_3$. Thus manifold is a Kenmotsu manifold. In reference of equations (2.1), (3.1) and (8.7), we have the following:

$$\left.\begin{array}{lll}\widetilde{\nabla}_{\varsigma_1}\varsigma_1 = 0, & \widetilde{\nabla}_{\varsigma_1}\varsigma_2 = 0, & \widetilde{\nabla}_{\varsigma_1}\varsigma_3 = \frac{3}{2}\varsigma_1 \\ \widetilde{\nabla}_{\varsigma_2}\varsigma_1 = 0, & \widetilde{\nabla}_{\varsigma_2}\varsigma_2 = 0, & \widetilde{\nabla}_{\varsigma_2}\varsigma_3 = \frac{3}{2}\varsigma_2 \\ \widetilde{\nabla}_{\varsigma_3}\varsigma_1 = -\frac{\varsigma_1}{2}, & \widetilde{\nabla}_{\varsigma_3}\varsigma_2 = -\frac{1}{2}\varsigma_2, & \widetilde{\nabla}_{\varsigma_3}\varsigma_3 = 0.\end{array}\right\} \tag{8.11}$$

In equations (3.2) and (3.3), we have

$$\begin{aligned}\widetilde{\mathfrak{T}}(\varsigma_1,\varsigma_3) &= \eta(\varsigma_3)\varsigma_1 - \eta(\varsigma_1)\varsigma_3 \\ &= g(\varsigma_3,\varsigma_3)\varsigma_1 - g(\varsigma_1,\varsigma_3)\varsigma_3 \\ &= \varsigma_1 \neq 0 \end{aligned} \tag{8.12}$$

and

$$\begin{aligned}(\widetilde{\nabla}_{\varsigma_1}g)(\varsigma_1,\varsigma_3) &= \frac{1}{2}\{2\eta(\varsigma_1)g(\varsigma_1,\varsigma_3) - \eta(\varsigma_1)g(\varsigma_1,\varsigma_3) - \eta(\varsigma_3)g(\varsigma_1,\varsigma_1)\} \\ &= -\frac{1}{2} \neq 0.\end{aligned} \tag{8.13}$$

Thus it is clear from (3.1) that $\widetilde{\nabla}$ is a semi-symmetric non-metric connection. Now

$$\begin{aligned}\widetilde{\nabla}_{\mathcal{K}_1}\xi &= \widetilde{\nabla}_{\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2+\mathcal{K}_1^3\varsigma_3}\varsigma_3 \\ &= \mathcal{K}_1^1\widetilde{\nabla}_{\varsigma_1}\varsigma_3 + \mathcal{K}_1^2\widetilde{\nabla}_{\varsigma_2}\varsigma_3 + \mathcal{K}_1^3\widetilde{\nabla}_{\varsigma_3}\varsigma_3 \\ &= \frac{3}{2}(\mathcal{K}_1^1\varsigma_1 + \mathcal{K}_1^2\varsigma_2).\end{aligned} \tag{8.14}$$

By virtue of (8.8) and 8.12, we have verified the equations (3.6) and (3.7). The $\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k$; $i,j,k = 1,2,3$ of connection ∇ can be estimated by using (3.11), (8.4) and (8.7), we have

$$\left.\begin{array}{lll}\mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_1 = 0, & \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_2 = 0, & \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_3 = 0, \\ \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_1 = 0, & \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_2 = 0, & \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_3 = -\varsigma_1, \\ \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_1 = 0, & \mathfrak{R}(\varsigma_2,\varsigma_3)_\varsigma 2 = 0, & \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_3 = -\varsigma_2,\end{array}\right\} \tag{8.15}$$

along with $\mathfrak{R}(\varsigma_i,\varsigma_i)\varsigma_i = 0$; $\forall\ i = 1,2,3$. By above discussions it has been verified equations (2.7), (2.8), (2.10) and (2.12) hold.

Analogously, we can estimate the $\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k$; $i,j,k = 1,2,3$ of connection $\widetilde{\nabla}$ by using equations (3.10), (8.4) and (8.11), we have

$$\left.\begin{array}{lll}\widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_1 = 0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_2 = 0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_3 = 0, \\ \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_1 = 0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_2 = 0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_3 = -\frac{3}{4}\varsigma_1, \\ \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_1 = 0, & \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_2 = 0, & \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_3 = -\frac{3}{4}\varsigma_2,\end{array}\right\} \tag{8.16}$$

along with $\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_i)\varsigma_i = 0$; $\forall\ i = 1,2,3$.

By virtue of (8.15) and (8.16), we have verified equations (4.1), (4.5), (4.6), (4.7) and (4.8). The Ricci tensors $\mathcal{S}(\varsigma_j,\varsigma_k)$; $j,k = 1,2,3$ of connection ∇ can be estimated by using (8.15) as under

$$\mathcal{S}(\varsigma_j,\varsigma_k) = \sum_{i=1}^{3} g(\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It is as under:

$$\left.\begin{aligned} &\mathcal{S}(\varsigma_1,\varsigma_1)=0, \quad \mathcal{S}(\varsigma_2,\varsigma_2)=0, \quad \mathcal{S}(\varsigma_3,\varsigma_3)=-2,\\ &\mathcal{S}(\varsigma_1,\varsigma_2)=0, \quad \mathcal{S}(\varsigma_1,\varsigma_3)=0, \quad \mathcal{S}(\varsigma_2,\varsigma_3)=0. \end{aligned}\right\} \tag{8.17}$$

In view of equation (8.17), we can easily verify equation (2.12).

Also in view of equation (8.17) we have verified the following:

$$\left.\begin{aligned} &(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_1,\phi\varsigma_2)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_2,\phi\varsigma_3)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_1,\phi\varsigma_1)=0,\\ &(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_1,\phi\varsigma_3)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_3,\phi\varsigma_1)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_2,\phi\varsigma_2)=0,\\ &(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_2,\phi\varsigma_1)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_3,\phi\varsigma_2)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_3,\phi\varsigma_3)=0. \end{aligned}\right\} \tag{8.18}$$

Thus we note that

$$(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\mathcal{K}_2,\phi\mathcal{K}_3)=0. \tag{8.19}$$

$\forall\ \mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\in\mathfrak{X}(\mathfrak{M})$. Hence the Ricci tensor is η-parallel. In view of equation (8.18) we can easily verify the equation (2.16).

The $\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k)$; $j,k=1,2,3$ of $\widetilde{\nabla}$ estimated by using (8.16) as under

$$\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k)=\sum_{i=1}^{3} g(\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It follows as under:

$$\left.\begin{aligned} &\widetilde{\mathcal{S}}(\varsigma_1,\varsigma_1)=0, \quad \widetilde{\mathcal{S}}(\varsigma_2,\varsigma_2)=0, \quad \widetilde{\mathcal{S}}(\varsigma_3,\varsigma_3)=-\frac{3}{2},\\ &\widetilde{\mathcal{S}}(\varsigma_1,\varsigma_2)=0, \quad \widetilde{\mathcal{S}}(\varsigma_1,\varsigma_3)=0, \quad \widetilde{\mathcal{S}}(\varsigma_2,\varsigma_3)=0. \end{aligned}\right\} \tag{8.20}$$

In view of equation (8.20), we can say that the example validate the equations (4.2) and (4.9).

Hence, we can say that given example is suitable for verification.

Acknowledgement

This work is supported by Council of Scientific and Industrial Research (CSIR), India, under Senior Research Fellowship with File No. 09/703(0007)/2020-EMR-I.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] N. S. Agashe and M. R. Chafle, A semi-symmetric non-metric connection on a Riemannian manifold, *Indian Journal of Pure and Applied Mathematics* **23** (1992), 399 – 409.

[2] C. S. Bagewadi and G. Ingalahalli, Ricci solitons in Lorentzian α-Sasakian manifolds, *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis* **28**(1) (2012), 59 – 68, URL: https://www.emis.de/journals/AMAPN/vol28_1/8.html.

[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics (LNM, Vol. **509**), Springer, Berlin – Heidelberg, viii + 148 pages (1976), DOI: 10.1007/BFb0079307.

[4] S. K. Chaubey, On semi-symmetric non-metric connection, *arXiv:1711.01035* [math.DG] (2007), 10 pages, DOI: 10.48550/arXiv.1711.01035.

[5] M. M. Boothby and R. C. Wang, On contact manifolds, *Annals of Mathematics* **68** (1958), 421 – 450.

[6] S. K. Chaubey and A. Yildiz, Riemannian manifolds admitting a new type of semisymmetric nonmetric connection, *Turkish Journal of Mathematics* **43**(4) (2019), Article 7, DOI: 10.3906/mat-1902-2.

[7] S. K. Chaubey and R. H. Ojha, On a semi-symmetric non-metric and quarter symmetric metric connections, *Tensor, N.S.* **70**(2) (2008), 202 – 213.

[8] S. K. Chaubey and R. H. Ojha, On a semi-symmetric non-metric connection, *Filomat* **26**(2) (2012), 269 – 275, URL: https://www.jstor.org/stable/24895727.

[9] S. Chaubey and S. K. Yadav, Study of Kenmotsu manifolds with semi-symmetric metric connection, *Universal Journal of Mathematics and Applications* **1**(2) (2018), 89 – 97, DOI: 10.32323/ujma.650209.

[10] S. K. Chaubey, Pankaj and S. K. Yadav, Projective curvature tensor of Riemannian manifolds admitting a projective semi-symmetric connection, *Universal Journal of Mathematics and Applications* **3**(2) (2020), 78 – 85, DOI: 10.32323/ujma.650209.

[11] U. C. De and G. Pathak, On 3-dimensional Kenmotsu manifolds, *Indian Journal of Pure and Applied Mathematics* **35** (2004), 159 – 165.

[12] A. De, On Kenmotsu manifold, *Bulletin of Mathematical Analysis and Applications* **2**(3) (2010), 1 –6, URL: https://www.emis.de/journals/BMAA/repository/docs/BMAA2-3-1.pdf.

[13] U. C. De., Y. L. Han and P. B. Zhao, A special type of semi-symmetric non-metric connection on a Riemannian manifold, *Facta Universitatis Niš: Series Mathematics and Informatics* **31**(2) (2016), 529 – 541, URL: http://casopisi.junis.ni.ac.rs/index.php/FUMathInf/article/view/1536.

[14] A. Friedmann and A. Schouten, Über die Geometrie der halbsymmetrischen Übertragungen *Mathematische Zeitschrift* **21** (1924), 211 – 223, DOI: 10.12691/tjant-3-4-1.

[15] S. Golab, On semi-symmetric and quarter symmetric linear connections, *Tensor, N.S.* **29** (1975), 249 – 254.

[16] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Mathematical Journal* **24** (1972), 93 – 103, DOI: 10.2748/tmj/1178241594.

[17] K. T. P. Kumar, C. S. Bagewadi and Venkatesha, On projective ϕ-symmetric K-contact manifold admitting quarter-symmetric metric connection, *Differential Geometry-Dynamical Systems* **13** (2011), 128 – 137.

[18] R. S. Mishra and S. N. Pandey, On quarter-symmetric metric F-connections, *Tensor, N.S.* **34** (1980), 1 – 7.

[19] C. Özgür and U.C. De, On the quasi-conformal curvature tensor of a Kenmotsu manifold, *Mathematica Pannonica* **17**(2) (2006), 221 – 228, URL:https://eudml.org/doc/226507.

[20] Pankaj, S.K. Chaubey and R. Prasad, Sasakian manifolds with admitting a non-symmetric non-metric connection, *Palestine Journal of Mathematics* **9**(2) (2020), 698 – 710, URL: https://pjm.ppu.edu/sites/default/files/papers/PJM_May2020_698_to_710.pdf.

[21] Pankaj, S.K. Chaubey and R. Prasad, Trans-Sasakian manifolds with respect to a non-symmetric non-metric connection, *Global Journal of Advanced Research on Classical and Modern Geometries* **7**(1) (2018), 1 – 10.

[22] S. C. Rastogi, On quarter-symmetric metric connection, *Comptes rendus de l'Académie bulgare des Sciences* **31** (1978), 811 – 814.

[23] B.B. Sinha and A.K. Srivastava, Curvatures on Kenmotsu manifold, *Indian Journal of Pure and Applied Mathematics* **22**(1) (1991), 23 – 28.

[24] T. Takahashi, Sasakian ϕ-symmetric spaces, *Tohoku Mathematical Journal* **29** (1977), 91 – 113, DOI: 10.2748/tmj/1178240699.

[25] M.M. Tripathi, A new connection in a Riemannian manifold, *International Electronic Journal of Geometry* **1**(1) (2008), 15 – 24, URL: https://dergipark.org.tr/tr/download/article-file/741700.

[26] M.M. Tripathi and N. Nakkar, On a semi-symmetric non-metric connection in a Kenmotsu manifold, *Bulletin of the Calcutta Mathematical Society* **16**(4) (2001), 323 – 330.

[27] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, Vol. **3**, World Scientific, Singapore (1985), DOI: 10.1142/0067.

[28] K. Yano and T. Imai, Quarter-symmetric metric connections and their curvature tensor, *Tensor, N.S.* **38** (1982), 13 – 18.