Communications in Mathematics and Applications
Vol. 15, No. 1, pp. 33–42, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i1.2333

http://www.rgnpublications.com

Research Article

Ring in Which Every Element is Sum of Two 5-Potent Elements

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Received: July 5, 2023 **Accepted:** December 23, 2023

Abstract. Every element of a ring R is a sum of two commuting 5-potents if and only if $R \cong R_1 \times R_2 \times R_3 \times R_4$, where $R_1/J(R_1)$ is Boolean and $U(R_1)$ is a group of exponent 4, R_2 is a subdirect product of Z_3's, R_3 is a subdirect product of Z_5's and R_4 is a subdirect product of Z_{13}'s. Also, if in a ring R every element is a sum of two 5-potents and a nilpotent that commute with one another then $R \cong R_1 \times R_2 \times R_3 \times R_4$ where $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil, $R_2 \cong R_a \times R_b \times R_c$ where $R_a = 0$, $R_c = 0$ and $R_b/J(R_b)$ is a subdirect product of rings isomorphic to Z_3, $M_2(Z_3)$ or F_9 with $J(R_b)$ is nil, $R_3/J(R_3)$ is a subdirect product of Z_5's and $J(R_3)$ is nil, $R_4/J(R_4)$ is a subdirect product of Z_{13}'s and $J(R_4)$ is nil.

Keywords. 5-Potents, Chinese Remainder Theorem, Jacobson radical

Mathematics Subject Classification (2020). 16A30, 16A50, 16E50, 16D30

1. Introduction

In the year 1988, Hirano and Tominaga [5] discussed about the properties of a ring R in which every element is sum of two commuting idempotents. They showed R has the identity $x^3 = x$. Then after a long break in 2016, Ying *et al.* [8] discussed about the ring R in which every element is sum of two commuting tripotents. They showed if every element of R is sum of two commuting tripotents if and only if $R \cong R_1 \times R_2 \times R_3$, where $R_1/J(R_1)$ is Boolean with $U(R_1)$ is a group of exponent 2, R_2 is subdirect product of Z_3's, and R_3 is a subdirect product of Z_5's. They questioned about rings in which every element is a sum of two commuting p-potents. Inspiring from these authors work, we discuss about the ring in which every element is sum of two commuting 5-potents in this paper. Then, we discuss about the ring in which every element is sum of 5-potents and a nilpotent all commute each other.

All ring consider here is associative with unity. The Jacobson radical, the group of units, the set of nilpotent elements are denoted by $J(R)$, $U(R)$ and $\mathrm{Nil}(R)$, respectively. Again, the Chinese Remainder Theorem states that for a ring R with I, J are ideals of R such that $I+J=R$ then there exists a ring isomorphism $R/(I\cap J)=R/I\times R/J$. For our work, we take the generalized version which states if I_i, $1\le i\le n$ are ideals of a ring R with $\sum_{i=1}^{n} I_i=R$ and $\cap_{i=1}^{n} I_i=0$ then $R\cong\left(\frac{R}{I_1}\right)\times\left(\frac{R}{I_2}\right)\times\ldots\times\left(\frac{R}{I_n}\right)$.

2. Results and Discussion

Lemma 2.1 ([6]). *Let p be a prime. The following are equivalent for a ring R:*

(i) *$p\in\mathrm{Nil}(R)$ and a^p-a is nilpotent for all $a\in R$.*

(ii) *$J(R)$ is nil and $R/J(R)$ is a subdirect product of Z_p's.*

Lemma 2.2 ([8]). *Let $a\in R$. If a^2-a is nilpotent, then there exists a monic polynomial $\theta(t)\in Z(t)$ such that $\theta(a)^2=\theta(a)$ and $a-\theta(a)$ is nilpotent.*

Lemma 2.3. *$\binom{2^k}{a}$ where $1\ge a\ge 2^k-1$ is always even.*

Proof. We have $\binom{2^k}{a}=\frac{(2^k)!}{a!(2^k-a)!}$.

Now power of 2 in $(2^k)!$ is $\left[\frac{2^k}{2}\right]+\left[\frac{2^k}{2^2}\right]+\left[\frac{2^k}{2^3}\right]+\ldots=2^{k-1}+2^{k-2}+\ldots+2^2+2+1=2^k-1$.

Power of 2 in $a!$ is $\left[\frac{a}{2}\right]+\left[\frac{a}{2^2}\right]+\left[\frac{a}{2^3}\right]+\ldots$.

Power of 2 in $(2^k-a)!$ is $\left[\frac{2^k-a}{2}\right]+\left[\frac{2^k-a}{2^2}\right]+\left[\frac{2^k-a}{2^3}\right]+\ldots$.

For any $a,b\in R$ we have $[a+b]\ge[a]+[b]$.

So $[2^k/2^l]\ge[a/2^l]+[(2^k-a)/2^l]$ for $0\le a\le 2^k$. Now $1=[2^k/2^k]>[a/2^k]+[(2^k-a)/2^k]=0$ for $1\le a\le 2^k-1$. So power of 2 in $(2^k)!$ is atleast one greater than the combine power of 2 in $a!$ and $(2^k-a)!$. So $\binom{2^k}{a}$ is always even for $1\le a\le 2^k-1$.

For example $\binom{8}{1},\binom{8}{2},\binom{8}{3},\binom{8}{4}$ are all even. □

Lemma 2.4 ([7, Theorem 2.7]). *A ring R is strongly nil-clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil.*

Lemma 2.5. *The $R=\prod R_\alpha$ be direct product of rings. then every element of R is a sum of two commuting n-potents if and only if, for each α, every element of R_α is a sum of two commuting n-potents.*

Lemma 2.6 ([6, Corollary 3.10]). *The following are equivalent for a ring R.*

(i) *a^9-a is nilpotent for all $a\in R$.*

(ii) *$R=R_1\times R_2\times R_3$, where R_1 is zero or $R_1/J(R_1)$ is Boolean with $J(R_1)$ is nil, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of rings isomorphic to Z_3, $M_2(Z_3)$ or F_9 with $J(R_2)$ is nil, and R_3 is zero or $R_3/J(R_3)$ is subdirect product of Z_5's with $J(R_3)$ is nil.*

Theorem 2.1. *The following conditions are equivalent.*

(1) *Let R be a ring in which every element is sum of two commuting five potent elements.*

(2) *R has the following properties:*

(a) *For every $k \in R$, we have*

$$(k-2)(k-1)k(k+1)(k+2)(k^2+1)(k^2+4)(k^2+2k+2)(k^2-2k+2)=0.$$

(b) *$R \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$, where*

(i) *R_1 is zero or a ring with $2^4=0$. R_1 has the identity $k^{64}=k^{32}$ for every $k \in R_1$. For every $n \in \mathrm{Nil}(R)$ we have $n^{16}=0$, $8n^4=0$. $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil. $U(R_1)$ is group of exponent 4.*

(ii) *R_2 is zero or R_2 is a subdirect product of Z_3's.*

(iii) *R_3 is zero or a is a subdirect product of Z_5's.*

(iv) *R_4 is zero or $R_4$is a subdirect product of Z_{13}'s.*

Proof. (a)⇒(b): Let $k \in R$ then there exists $e, f \in R$ with $e^5=e$, $f^5=f$, $ef=fe$ such that $k=e+f$. Now,

$$k^5 = e^5+f^5+5(e^4f+ef^4)+10(e^3f^2+e^2f^3)$$

$$\Rightarrow \quad k^5-k = 5(e^4f+ef^4)+10(e^3f^2+e^2f^3). \tag{2.1}$$

Now,

$$k^5-k=(k^4-1)(e+f)$$

$$\Rightarrow \quad (k^5-k)e^4f^4=(k^4-1)(ef^4+e^4f).$$

Again,

$$\begin{aligned}(k^5-k)e^4f^4 &= 5(e^8f^5+e^5f^8)+10(e^7f^6+e^6f^7)\\ &= 5(e^4f+ef^4)+10(e^3f^2+e^2f^3)\\ &= k^5-k.\end{aligned}$$

Therefore, we have

$$k^5-k=(k^4-1)(ef^4+e^4f).$$

Using (2.1), we have

$$(k^4-6)(e^4f+ef^4)-10(e^3f^2+e^2f^3)=0. \tag{2.2}$$

Now multiplying (2.2) by e^4f^4, we have

$$(k^4-6)(e^3f^2+e^2f^3)-10(e^4f+ef^4)=0. \tag{2.3}$$

Now using equations (2.2) and (2.3), we have

$$[(k^4-6)^2-10^2](e^4f+ef^4)=0$$

$$\Rightarrow \quad [(k^4-6)^2-10^2](k^4-1)(e^4f+ef^4)=0$$

$$\Rightarrow \quad (k^4-16)(k^4+4)(k^5-k)=0$$

$$\Rightarrow \quad (k-2)(k-1)k(k+1)(k+2)(k^2+1)(k^2+4)(k^2+2k+2)(k^2-2k+2)=0$$

Putting $k=3$, we have

$$2\times 3\times 3\times 4\times 5\times 10\times 13\times 85=0$$

$$\Rightarrow \quad 2^4 \times 3 \times 5^3 \times 13 \times 17 = 0$$

Again putting $k = 6$, we have

$$4 \times 5 \times 6 \times 7 \times 8 \times 37 \times 40 \times 26 \times 50 = 2^{11} \times 3 \times 5^4 \times 7 \times 13 \times 37 = 0.$$

Putting $k = 5$, we have

$$2^6 \times 3^2 \times 5 \times 13 \times 17 \times 29 \times 37 = 0.$$

Taking $\gcd(2^4 \times 3 \times 5^3 \times 13 \times 17,\ 2^{11} \times 3 \times 5^4 \times 7 \times 13 \times 37,\ 2^6 \times 3^2 \times 5 \times 13 \times 17 \times 29 \times 37)$, we get

$$2^4 \times 3 \times 5 \times 13 = 0.$$

As for $k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6$ we see that 13 divides $(k^4+4)(k^4-16)(k^5-k)$ (taking modulo 13). Also, 3 divides $(k-1)k(k+1)$ for $k = 0, \pm1$ (taking modulo 3). Again in $(k-2)(k-1)k(k+1)(k+2)$, 3 consecutive even no are present for any integer k so 16 divides $(k-2)(k-1)k(k+1)(k+2)$ and 5 divides k^5-k for any integer k. Hence ultimately $2^4 \times 3 \times 5 \times 13$ divides $(k^4+4)(k^4-16)(k^5-k)$ for integer value of k (i.e. $k \cdot 1_R$ where 1_R is the multiplicative identity of R. Here, we take $1_R = 1$). As $2^4 \times 3 \times 5 \times 13 = 0$. So, by using Chinese Remainder Theorem we have $R \cong R_1 \times R_2 \times R_3 \times R_4$ where $R_1 \cong R/2^4R$, $R_2 \cong R/3R$, $R_3 \cong R/5^3R$, $R_4 \cong R/13R$.

Assume that $R_1 \neq 0$. Now in R_1 we have $2^4 = 0$. For $k \in R_1$ we can write $k = e + f$ where $e, f \in R$ with $e^5 = e$, $f^5 = f$, $ef = fe$. Now $k^4 = e^4 + f^4 + 2F_1$, therefore

$$k^8 = e^8 + f^8 + 2F_2' = e^4 + f^4 + 2F_2' = k^4 - 2F_1 + 2F_2' = k^4 + 2F_2$$

$$\Rightarrow \quad k^8 = k^4 + 2F_2$$

so $(k^8-k^4)^4 = 0$, $2^3(k^8-k^4) = 0$. Similarly, $k^{16} = k^8 + 4F_3$, $k^{32} = k^{16} + 8F_4$, $k^{64} = k^{32} + 16F_5 \Rightarrow k^{64} = k^{32}$, where F_1, F_2', F_2, F_3, F_4, F_5 are functions of e, f. Now for $n \in \mathrm{Nil}(R_1)$ we have $1 - n^\alpha \in U(R_1)$, where $\alpha \in N$. Now for $n \in \mathrm{Nil}(R_1)$ we have

$$(n^8 - n^4)^4 = 0$$

$$\Rightarrow \quad n^{16}(n^4-1)^4 = 0$$

$$\Rightarrow \quad n^{16} = 0.$$

Also,

$$8(n^8 - n^4) = 0$$

$$\Rightarrow \quad 8n^4 = 0$$

Again

$$(k^2-k)^{32} = k^{64} + k^{32} + 2F(k) = 2(k^{32} + F(k))$$

$$\Rightarrow \quad (k^2-k)^{32\times32} = 0$$

using Lemma 2.3, where $F(k)$ is a function of k. Therefore, $k^2 - k$ is nilpotent, so by using Lemma 2.1 we have $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil. Now as $R_1/J(R_1)$ is Boolean so for $u \in U(R_1)$ we have

$$u^2 - u \in J(R_1)$$

$$\Rightarrow \quad u - 1 \in J(R)$$

So $U(R_1) \subseteq 1+J(R_1)$. Again as $J(R)$ is nil so for every $j \in J(R_1)$ we have $1+j \in U(R_1)$. Therefore, $1+J(R_1) \subseteq U(R_1)$. Hence $1+J(R_1) = U(R_1)$. Now for $u \in U(R_1)$ we have

$$(u^4+4)(u^4)(u^5-u) = 0$$
$$\Rightarrow \quad (u^4+4)(u^4-1) = 0$$

as $u \in U(R_1)$ and $16 = 0$. Again as $u^4 \in U(R_1)$ so

$$u^4 = 1+j$$
$$\Rightarrow \quad u^4+4 = 1+(4+j)$$

Now as $2 \in \mathrm{Nil}(R)$ so $4+j \in \mathrm{Nil}(R_1)$. As $n^{16} = 0$ for $n \in \mathrm{Nil}(R_1)$ so $1+n \in U(R_1)$ which imply $u^4+4 \in U(R_1)$. Therefore,

$$u^4-1 = 0$$
$$\Rightarrow \quad u^4 = 1$$

Hence $U(R_1)$ is a group of exponent 4.

Assume that $R_2 \neq 0$. Now in R_2 we have $3 = 0$. Suppose $k^2 = 0$ in R_2. For $k \in R_2$ we can write $k = e+f$ where $e, f \in R$ with $e^5 = e, f^5 = f, ef = fe$. Now

$$k^3 = e^3+f^3+3e^2f+3ef^2 = e^3+f^3$$
$$\Rightarrow \quad k^9 = e^9+f^9 = k$$
$$\Rightarrow \quad k = 0$$

as $e^9 = e^5e^4 = e^5 = e$. Therefore, R_2 is a reduced ring, so R_2 is a subdirect product of domains $\{R_\alpha\}$. Now for $x \in R_\alpha$ with $x^5 - x = 0$, we have

$$x(x-1)(x+1)(x^2+1) = 0$$
$$\Rightarrow \quad x = 0, 1, -1 \ \text{ or } \ x^2+1 = 0$$

But $3 = 0$ in R_α so $x^2+1 \neq 0$ as if $x^2 = -1$ then as $x^2 = 1$ or 0 (as $3 = 0$) which imply $1 = 0$ or $-1 = 0$ which is a contradiction. So, $-1, 0, 1$ are only trivial 5-potents R_α, so we conclude that $R_\alpha = \{-2, -1, 0, 1, 2\}$. But $3 = 0$ in R_α so $2 = -1, -2 = 1$. Thus $R_\alpha = \{0, 1, 2\}$, which is isomorphic to Z_3. Hence R_2 is a subdirect product of Z_3's.

Assume that $R_3 \neq 0$. In R_3 we have $5 = 0$. Suppose $k^2 = 0$ in R_3. For $k \in R_3$ we can write $k = e+f$ where $e, f \in R$ with $e^5 = e$, $f^5 = f$, $ef = fe$. Now

$$0 = k^5 = e^5+f^5+5F_1 = k$$
$$\Rightarrow \quad k = 0$$

Therefore, R_3 is a reduced ring. Hence R_2 is a subdirect product of domains $\{R_\alpha\}$. Now for $x \in R_\alpha$ with $x^5 - x = 0$ we have

$$x(x-1)(x+1)(x^2+1) = 0$$
$$\Rightarrow \quad x = 0, 1, -1 \ \text{ or } \ x^2+1 = 0$$

As $5 = 0$ in R_α so $x^2+1 = 0$ is satisfied by $x = 2, 3$. So $0, 1, 2, 3, -1 = 4$ are 5-potent elements R_α. Hence $R_\alpha = \{0, 1, 2, 3, 4\}$ which is isomorphic to Z_5. So R_3 is a subdirect product of Z_5's.

Assume that $R_4 \neq 0$. Now in R_4 we have $13 = 0$. Suppose $k^2 = 0$ in R_4. For $k \in R_4$ we can write $k = e+f$ where $e, f \in R$ with $e^5 = e$, $f^5 = f$, $ef = fe$. Now $0 = k^{13} = e^{13}+f^{13}+13F(k) = k$

as $e^{13} = (e^5)^2e^3 = e^5 = e$. Therefore, R_4 is a reduced ring, hence R_4 is a subdirect product of domains $\{R_\alpha\}$. Now for $x \in R_\alpha$ with $x^5 - x = 0$ we have

$$x(x-1)(x+1)(x^2+1) = 0$$
$$\Rightarrow \quad x = 0, 1, -1 \text{ or } x^2+1 = 0$$

As $13 = 0$ in R_α so $x^2+1=0$ is satisfied by $x = 5, 8$. So $0,1,5,8,12$ are only trivial 5-potent of R_α. Therefore, $R_\alpha = \{0,1,2,5,6,9,10,8,16,17,12,20,24\} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ as $13 = 0$ in R_α, which is isomorphic to Z_{13}. So R_4 is a subdirect product of Z_{13}.

(b)⇒(a): Let (b) hold. R_1, R_2, R_3, R_4 are defined as in (b). Now in R_1 we have $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil. So by Lemma 2.4 R_1 is strongly nil clean. So for $a \in R_1$ there exist $e \in R_1$ with $e^2 = e$ and $n \in \text{Nil}(R_1)$ such that

$$a - 1 = e + n$$
$$\Rightarrow \quad a = e + (1+n)$$

where $en = ne$. As $e^2 = e$ so $e^5 = e$ and as $1 + n \in U(R_1)$ so

$$(1+n)^4 = 1$$
$$\Rightarrow \quad (1+n)^5 = (1+n)$$

So R_1 is sum of two commuting 5-potent elements.

Using [8, Proposition 3.9] we have R_2 is subdirect product of Z_3's if and only if R_2 is a strong SIT-ring with $3 = 0$. So every element k of R_2 can be expressed as $k = e + f$ where $e^2 = e$, $f^3 = f$, $ef = fe$. Clearly, $e^5 = e$, $f^5 = f$ so we have the result.

Using converse part of [8, Theorem 5.2] we have R_3 is subdirect product of Z_5's if and only if every element of R_3 is a sum of two commuting tripotents. Consequently, every element of R_3 is sum of two commuting 5-potents.

Finally, we have to show in R_4 every element of R_4 is a sum of two commuting 5-potents. Suppose R is a subdirect product of $\{R_\alpha : \alpha \in \wedge\}$ where $R_\alpha = Z_{13}$ for all $\alpha \in \wedge$. So R_4 is a subring of $\prod_{\alpha\in\wedge} R_\alpha$. Let $x = (x_\alpha) \in R_4$. So $\wedge$ is a disjoint union of $\wedge_0$, $\wedge_1$, $\wedge_2$, $\wedge_3$, $\wedge_4$, $\wedge_5$, $\wedge_6$, $\wedge_7$, $\wedge_8$, $\wedge_9$, $\wedge_{10}$, $\wedge_{11}$, $\wedge_{12}$ such that $x_\alpha = i$ if and only if $\alpha \in \wedge_i$ for $i = 0,1,2,3,4,5,6,7,8,9,10,11,12$. Without loss of generality we can denote $x = (0_{\wedge_0}, 1_{\wedge_1}, 2_{\wedge_2}, 3_{\wedge_3}, 4_{\wedge_4}, 5_{\wedge_5}, 6_{\wedge_6}, 7_{\wedge_7}, 8_{\wedge_8}, 9_{\wedge_9}, 10_{\wedge_{10}}, 11_{\wedge_{11}}, 12_{\wedge_{12}})$. As we know in Z_{13} the 5-potents are $0,1,5,8,12$. So if $u = (0_{\wedge_0}, 1_{\wedge_1}, 1_{\wedge_2}, 8_{\wedge_3}, 5_{\wedge_4}, 5_{\wedge_5}, 1_{\wedge_6}, 8_{\wedge_7}, 8_{\wedge_8}, 8_{\wedge_9}, 5_{\wedge_{10}}, 12_{\wedge_{11}}, 12_{\wedge_{12}})$ and $v = (0_{\wedge_0}, 0_{\wedge_1}, 1_{\wedge_2}, 8_{\wedge_3}, 12_{\wedge_4}, 0_{\wedge_5}, 5_{\wedge_6}, 12_{\wedge_7}, 0_{\wedge_8}, 1_{\wedge_9}, 5_{\wedge_{10}}, 12_{\wedge_{11}}, 0_{\wedge_{12}})$ then $u^5 = u, v^5 = v, uv = vu$ and $x = u + v$ which shows every element of R_4 is sum of two commuting 5-potents. Hence using Lemma 2.5 we have every element of R can be expressed as sum of two 5-potent elements.

Example 2.1. There are many ring in which every element is sum of two commuting 5-potents. Some of which are given below:

(i) Ring R with the identity $x^3 = x$ for every $x \in R$. Ring in which every element is sum or difference of two commuting idempotents that commute one another.

(ii) All SIT rings or a ring R with the identity $x^6 = x^4$ for every $x \in R$ (ring in which every element is a sum of a tripotent and an idempotent that commute each other). Also, the rings in which every element is a difference of a tripotent and an idempotent that commute with one another.

(iii) Ring in which every element is sum of two commuting tripotents.

(iv) All strongly nil clean rings R with $n^2 = 0$, $2n = 0$ or $n^4 = 0$, $2n = 0$ for every $n \in \mathrm{Nil}(R)$. Also, all rings R in which every element is a sum of tripotent and nilpotent that commute each other with $n^2 = 0$, $2n = 0$ or $n^4 = 0$, $2n = 0$ for every $n \in \mathrm{Nil}(R)$.

(v) All strongly clean rings R with $U(R)$ of exponent 2 or 4. Also, the rings in which every element is sum of a tripotent and an unit that commute with each other and $U(R)$ is a group of exponent 2 or 4.

(vi) $Z_2 \times Z_3 \times Z_5 \times Z_{13}$, $Z_5 \times Z_{13}$, $Z_3 \times Z_5$, $Z_5 \times Z_5$ etc. are some ring with the given property.

Theorem 2.2. *If every element of a ring is a sum of two 5-potents and a nilpotent, all commute one another then $R \cong R_1 \times R_2 \times R_3 \times R_4$, where*

(i) *$R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil. R_1 is a strongly nil clean.*

(ii) *$R_2 \cong R_a \times R_b \times R_c$ where $R_a = 0$, $R_c = 0$ and $R_b/J(R_b)$ is a subdirect product of rings isomorphic to Z_3, $M_2(Z_3)$ or F_9 with $J(R_b)$ is nil.*

(iii) *$R_3/J(R_3)$ is a subdirect product of Z_5's and $J(R_3)$ is nil.*

(iv) *$R_4/J(R_4)$ is a subdirect product of Z_{13}'s and $J(R_4)$ is nil.*

Proof. Let $k \in R$ so k can be expressed as $k = e + f + n$ where $e^5 = e$, $f^5 = f$, $n \in \mathrm{Nil}(R)$, $ef = fe$, $ne = en$, $en = nf$. Now $k - n = e + f$ which is sum of two commuting tripotents. So, Theorem 2.1, we have

$$[(k-n)^4 - 16][(k-n)^4 + 4](k-n)^5 - (k-n)] = 0$$
$$\Rightarrow \quad (k^4 - 16)(k^4 + 4)(k^5 - k) = nf(n)$$

where $f(n)$ is a function of n. So $(k^4 - 16)(k^4 + 4)(k^5 - k)$ is a nilpotent element for every $k \in (R)$. Now from Theorem 2.1 we get $2^4 \times 3 \times 5 \times 13$ divides $(k^4 - 16)(k^4 + 4)(k^5 - k)$ for every integer value of k (i.e., $k = k.1_R$, where 1_R is the multiplicative identity of R, here we take $1_R = 1$).

Let m be the least integer such that

$$(2^4 \times 3 \times 5 \times 13)^m = 0$$
$$\Rightarrow \quad 2^{4m} \times 3^m \times 5^m \times 13^m = 0$$

Now by using Chinese Remainder Theorem, we have $R \cong R_1 \times R_2 \times R_3 \times R_4$ where $R_1 \cong \frac{R}{2^{4m}R}$, $R_2 \cong \frac{R}{3^m R}$, $R_3 \cong \frac{R}{5^m R}$ and $R_4 \cong \frac{R}{13^m R}$.

Assume that $R_1 \neq 0$. In R_1 we have $2^{4m} = 0$. Now let $k \in R_1$ so there exist $e, f, n \in R_1$ such that $e^5 = e, f^5 = f$ and $n \in \mathrm{Nil}(R_1)$. As 2 is nilpotent so all odd numbers of R_1 are unit. now we have $k^8 - k^4 = (e+f+n)^8 - (e+f+n)^4 = e^8 + f^8 - e^4 - f^4 + n^8 - n^4 + 2F_1(e,f,n) = e^4 + f^4 - e^4 - f^4 + n^4(n^4 - 1) + 2F_1 = n^4(n^4 - 1) + 2F_1$, where $F_1(e,f,n) = F_1$ is a function of e, f, n. As $n^4(n^4 - 1) + 2F_1$ is nilpotent (as $n, 2 \in \mathrm{Nil}(R)$ and e, f, n, F_1 are commutative), so $k^8 - k^4$ is nilpotent. Therefore, $k^8 - k^4 = n_1$ for some $n_1 \in \mathrm{Nil}(R_1)$. Clearly, k, n_1 commute each other as $n_1 = n^4(n^4 - 1) + 2F_1$. Suppose $n_1^{2^p} = 0$ for some integer p. Now continue with squiring, we get

$$k^8 = k^4 + n_1$$
$$\Rightarrow \quad k^{2^4} = k^{2^3} + 2F_2 + n_1^2$$
$$\Rightarrow \quad k^{2^5} = k^{2^4} + 2F_3 + n_1^{2^2}$$

$$\Rightarrow \quad k^{2^6} = k^{2^5} + 2F_4 + n_1^{2^3}$$

Continuing in this way ultimately we get

$$k^{2^{p+2}} = k^{2^{p+1}} + 2F_p + n_1^{2^p}$$

$$\Rightarrow \quad k^{2^{p+2}} = k^{2^{p+1}} + 2F_p$$

Now again continue squiring we get

$$k^{2^{p+3}} = k^{2^{p+2}} + 2^2 F_{p+1}$$

$$\Rightarrow \quad k^{2^{p+4}} = k^{2^{p+3}} + 2^3 F_{p+2}, \dots$$

$$\Rightarrow \quad k^{2^{p+4m+1}} = k^{2^{p+4m}} + 2^{4m} F_{p+4m-1}$$

$$\Rightarrow \quad k^{2^{p+4m+1}} = k^{2^{p+4m}}$$

as $2^{4m} = 0$. Here F_i's are functions of e, f, n. Now $(k^2 - k)^{2^{p+4m}} = k^{2^{p+4m+1}} + k^{2^{p+4m}} + 2F(e,f,n) = 2k^{p+4m} + 2F(e,f,n) = 2[k^{p+4m} + F(e,f,n)]$ using Lemma 2.3. Now, $(k^2 - k)^{2^{p+4m} \times 4m} = 2^{4m}[k^{p+4m} + F(e,f,n)] = 0$ which imply $k^2 - k$ is nilpotent. As k is arbitrary element of R_1 so for every $k \in R_1$ we have $k^2 - k$ is nilpotent. So by using Lemma 2.1, $R_1/J(R_1)$ is a subdirect product of Z_2's i.e. $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil, using Lemma 2.4 we get R_1 is strongly nil-clean.

Assume that $R_2 \neq 0$. In R_2 we have $3^m = 0$. Let $k \in R_2$ so it can be expressed as $k = e + f + n$, where $e^5 = e$, $f^5 = f$, $n \in \text{Nil}(R_2)$, $ef = fe$, $en = ne$, $fn = nf$. Now, $k^9 - k = (e+f+n)^9 - (e+f+n) = n^9 - n + 3F(e,f,n) = n(n^8-1) + 3F(e,f,n)$ as $e^9 = e^5e^4 = e^5 = e$, $f^9 = f$ where $F(e,f,n)$ is a function of e,f,n. Now $n(n^8-1)+3F(e,f,n)$ is nilpotent as $n, 3 \in \text{Nil}(R)$ and e,f,n are all commutative. So $k^9 - k$ is nilpotent for every $k \in R_2$. Now as 3 is nilpotent so 2, 5 are unit otherwise $1 = 0 \Rightarrow R_2 = 0$ which is a contradiction. Now using Lemma 2.6 we have $R_2 \cong R_a \times R_b \times R_c$ where $R_2 = 0$ as 2 is unit and R_c is zero as 5 is unit, and in R_b we have $R_b/J(R_b)$ is a subdirect product of rings isomorphic to Z_3, $M_2(Z_3)$ or F_9 with $J(R_b)$ is nil.

Assume that $R_3 \neq 0$. In R_3 we have $5^m = 0$. Let $k \in R_3$ so it can be expressed as $k = e + f + n$, where $e^5 = e$, $f^5 = f$, $n \in \text{Nil}(R_2)$, $ef = fe$, $en = ne$, $fn = nf$. Now, $k^5 - k = (e+f+n)^5 - (e+f+n) = e^5 + f^5 + n^5 + 5F(e,f,n) - e - f - n = n(n^4 - 1) + 5F(e,f,n)$, where $F(e,f,n)$ is a function of e,f,n. As $n, 5 \in \text{Nil}(R)$ and e,f,n are commutative so $n(n^4-1) + 5F(e,f,n)$ is nilpotent which imply $k^5 - k$ is nilpotent for every $k \in R_3$. So, by using Lemma 2.1 we have $R_3/J(R_3)$ is a subdirect product of Z_5's and $J(R_2)$ is nil.

Assume that $R_4 \neq 0$. In R_4 we have $13^m = 0$. Let $k \in R_3$ so it can be expressed as $k = e + f + n$, where $e^5 = e$, $f^5 = f$, $n \in \text{Nil}(R_2)$, $ef = fe$, $en = ne$, $fn = nf$. Now, $k^{13} - k = (e + f + n)^{13} - (e + f + n) = e^{13} + f^{13} + n^{13} + 13F(e,f,n) - e - f - n = n(n^{12} - 1) + 13F(e,f,n)$, where $F(e,f,n)$ is a function of e,f,n. As $n, 13 \in \text{Nil}(R)$ and e,f,n are commutative so $n(n^{12} - 1) + 13F(e,f,n)$ is nilpotent which imply $k^{13} - k$ is nilpotent for every $k \in R_4$. So by using Lemma 2.1 we have $R_4/J(R_4)$ is a subdirect product of Z_{13}'s and $J(R_4)$ is nil. □

Now, the question arises: What is the structure of a ring in which every element is sum of three commuting 5-potent or three 5-potent and an nilpotent that commute one another? It is still open while we make little progress in it. We are ending our study by the following proposition:

Proposition 2.1. *Let R be ring. If $k \in R$ can be expressed as $k = e + f + g$ where $e^5 = e$, $f^5 = f$, $g^5 = g$, $ef = fe$, $fg = gf$, $eg = ge$ then we have $(k-2)(k-1)k(k+1)(k+2)(k^2+1)(k^2+2k+2)(k^2-2k+2)(e^4-e)^{13} = 0$. Similar result we get for f and g.*

Proof. First, we prove the following results for $e \in R$ where $e^5 = e$. Then for $k \in R$ with $ke = ek$ and integer a, b, we have

(i) $(k-a-e)(e^4-e) = (k-a)(e^4-e)$,

(ii) $[(k-e+a)^2+b](e^4-e)^2 = [(k-a)^2+b](e^2-e)^2$.

We have $(k-a-e)(e^4-e) = (k-a)(e^4-e)-(e^5-e) = (k-a)(e^4-e)$. Again

$$\begin{aligned}[(k-e+a)^2+b](e^4-e)^2 &= [\{(k+a)(e^4-e)-(e^5-e)\}^2+b(e^4-e)^2] \\ &= [\{(k+a)\}^2(e^4-e)^2+b(e^4-e)^2] \\ &= [(k+a)^2+b](e^4-e)^2.\end{aligned}$$

Now

$$\begin{aligned} & k = e+f+g \\ \Rightarrow\quad & k-e = f+g.\end{aligned}$$

Therefore, $k-e$ can be expressed as sum of two commuting 5-potent. Now by using Theorem 2.1, we have

$$\begin{aligned}&(k-e-2)(k-e-1)(k-e)(k-e+1)(k-e+2) \\ &\cdot\{(k-e)^2+4\}\{k-e)^2+1\}\{(k-e+1)^2+1\}\{(k-e-1)^2+1\} = 0.\end{aligned}$$

Now multiplying it by $(e^2-e)^{13}$ and using above two formulas we get

$$(k-2)(k-1)k(k+1)(k+2)(k^2+1)(k^2+4)(k^2+2k+2)(k^2-2k+2)(e^4-e)^{13} = 0. \qquad \square$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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