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Research Article

Numerical Approximation of Stochastic Volterra-Fredholm Integral Equation using Walsh Function

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Abstract. In this paper, a computational method is developed to find an approximate solution to the stochastic Volterra-Fredholm integral equation using the Walsh function approximation and its operational matrix. Moreover, convergence and error analysis of the method is carried out to strengthen its validity. Furthermore, the method is numerically compared to the block pulse function method and the Haar wavelet method for some non-trivial examples.

Keywords. Stochastic Volterra-Fredholm integral equation, Brownian motion, Itô integral, Walsh approximation, Lipschitz condition

Mathematics Subject Classification (2020). 60H05, 60H35, 65C30

1. Introduction

Stochastic Differential Equations (SDE) are widely used in a variety of fields, including the physical sciences, biological sciences, agricultural sciences, and financial mathematics, which includes option pricing, where the *stochastic Volterra-Fredholm integral equation* (SVFIE) plays a crucial role [7, 10]. As with other differential equations, it is practically impossible to find the solution to many SDEs, and the problem becomes more complex in the case of SVFIE. Therefore, the numerical approximation method becomes crucial for finding solutions to the problems. Numerous SVFIEs are determined approximately using a variety of numerical techniques. In recent decades, orthogonal functions such as *block pulse function* (BPF), Haar wavelet, Legendre polynomials, Laguerre polynomials, and Chebyshev's polynomials have been utilised to approximate the solution of SVFIE.

The Walsh functions form an orthonormal system that only accepts the values 1 and -1. As a result, a number of mathematicians consider the Walsh system to be an artificial orthonormal system, which was introduced in 1923 [13] and has numerous applications in digital technology. Walsh functions have a significant advantage over traditional trigonometric functions because a computer can determine the exact value of any Walsh function at any given time with high accuracy. Chen and Hsiao used the Walsh function to solve the variational problem in 1997, as cited in [1]. They used the same concept to solve the integral equation [5] in 1979. The technique's key property is that it converts the problem into a system of algebraic equations, which are then solved to yield an approximation of the solution. In this paper, we use the Walsh function to approximate the solution $x(t)$ of the following linear SVFIE

$$x(t) = f(t) + \int_\alpha^\beta k(s,t)x(s)ds + \int_0^t k_1(s,t)x(s)ds + \int_0^t k_2(s,t)x(s)dB(s), \tag{1.1}$$

where $x(t)$, $f(t)$, $k(s,t)$, $k_1(s,t)$ and $k_2(s,t)$ for $s,t \in [0,T)$, represent the stochastic processes primarily based on the identical probability space (Ω, F, P) and $x(t)$ is unknown. In addition, $B(t)$ represents Brownian motion [7, 10], and $\int_0^t k_2(s,t)x(s)dB(s)$ represents the Itô integral.

In the majority of previous works, the evaluation is predicated on the assumption that the derivatives $f'(t)$, $\frac{\partial^2 k}{\partial s \partial t}$, $\frac{\partial^2 k_i}{\partial s \partial t}$ for $i = 1,2$, exist and are bounded. By converting BPF approximation to Walsh function approximation in this paper, we expect only the Lipschitz continuity of the functions $f(t)$, $k(s,t)$, $k_1(s,t)$ and $k_2(s,t)$ to have the same rate of convergence, which allows us to consider the general form of SVFIE to be integrated. In the final portion, the method is compared to similar techniques [6, 9] that approximate the solution of the SVFIE using the block pulse function and the Haar wavelet.

2. Walsh Function and Its Properties

Definition 2.1 (Rademacher Function)**.** Rademacher function $r_i(t)$, $i = 1,2,\dots$, for $t \in [0,1)$ is defined by [13]

$$r_i(t) = \begin{cases} 1 & i = 0, \\ \operatorname{sgn}(\sin(2^i \pi t)) & \text{otherwise,} \end{cases}$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

Definition 2.2 (Walsh Function)**.** The nth Walsh function for $n = 0,1,2,\dots$, denoted by $w_n(t)$, $t \in [0,1)$ is defined [13] as

$$w_n(t) = (r_q(t))^{b_q} . (r_{q-1}(t))^{b_{q-1}} . (r_{q-2}(t))^{b_{q-2}} \dots (r_1(t))^{b_1},$$

where $n = b_q 2^{q-1} + b_{q-1} 2^{q-2} + b_{q-2} 2^{q-3} + \ldots + b_1 2^0$ is the binary expression of n. Therefore, q, the number of digits present in the binary expression of n is calculated by $q = [\log_2 n] + 1$ in which $[\cdot]$ is the greatest integer less than or equal's to '$\cdot$'.

The first m Walsh functions for $m \in N$ can be written as an m-vector by

$$W(t) = \begin{bmatrix} w_0(t) & w_1(t) & w_2(t) & \ldots & w_{m-1}(t) \end{bmatrix}^T, \quad t \in [0,1).$$

The Walsh functions satisfy the following properties.

Orthonormality

The set of Walsh functions is orthonormal, i.e.,

$$\int_0^1 w_i(t) w_j(t) dt = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Completeness

For every $f \in L^2[0,1)$

$$\int_0^1 f^2(t) dt = \sum_{i=0}^{\infty} f_i^2 \|w_i(t)\|^2,$$

where $f_i = \int_0^1 f(t) w_i(t) dt$.

Walsh Function Approximation

Any real-valued function $f(t) \in L^2[0,1)$ can be approximated as

$$f_m(t) \approx \sum_{i=0}^{m-1} c_i w_i(t),$$

where $c_i = \int_0^1 f(t) w_i(t) dt$.

The matrix form is given by

$$f(t) \approx F^T T_W W(t), \tag{2.1}$$

where $F = \begin{bmatrix} f_0 & f_1 & f_2 & \ldots & f_{m-1} \end{bmatrix}^T$, $f_i = \int_{ih}^{(i+1)h} f(s) ds$.

Here, $T_W = [w_i(\eta_j)]$ is called as the Walsh operational matrix where $\eta_j \in [jh, (j+1)h)$.

Similarly, function $k(s,t) \in L^2([0,1) \times [0,1))$ can be approximated by

$$k_m(s,t) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij} w_i(s) w_j(t),$$

where, $c_{ij} = \int_0^1 \int_0^1 k(s,t) w_i(s) w_j(t) dt ds$ with the matrix form represented by

$$k(s,t) \approx W^T(s) T_W K T_W W(t) = W^T(t) T_W K^T T_W W(s), \tag{2.2}$$

where $K = [k_{ij}]_{m \times m}, k_{ij} = \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} k(s,t) dt ds$.

3. Relationship Between Walsh Function and Block Pulse Functions (BPFs)

Definition 3.1 (Block Pulse Functions)**.** For a fixed positive integer m, an m-set of BPFs $\phi_i(t), t \in [0,1)$ for $i = 0,1,\dots,m-1$ is defined as

$$\phi_i(t) = \begin{cases} 1 & \text{if } \frac{i}{m} \le t < \frac{(i+1)}{m}, \\ 0 & \text{otherwise,} \end{cases}$$

ϕ_i is known as the ith BPF.

The set of all m BPFs can be written concisely as an m-vector,

$$\Phi(t) = \begin{bmatrix} \phi_0(t) & \phi_1(t) & \phi_2(t) & \dots & \phi_{m-1}(t) \end{bmatrix}^T, \quad t \in [0,1).$$

The BPFs are disjoint, complete, and orthogonal [12].

The BPFs in vector form satisfy

$$\Phi(t)\Phi(t)^T X = \widetilde{X}\Phi(t) \text{ and } \Phi^T(t)A\Phi(t) = \widehat{A}\Phi(t),$$

where $X \in R^{m\times 1}$, $\widetilde{X}$ is the $m \times m$ diagonal matrix with $\widetilde{X}(i,i) = X(i)$ for $i = 1,2,3,\dots,m$, $A \in R^{m\times m}$ and $\widehat{A} = \begin{bmatrix} a_{11} & a_{22} & \dots & a_{mm} \end{bmatrix}^T$ is the m-vector with elements equal to the diagonal entries of A. The integration of BPF vector $\Phi(t)$, $t \in [0,1)$ can be performed by [4]

$$\int_0^t \Phi(\tau)d\tau = P\Phi(t), \quad t \in [0,1). \tag{3.1}$$

Hence, the integral of every function $f(t) \in L^2[0,1)$ can be approximated as

$$\int_0^t f(s)ds = F^T P\Phi(t).$$

The Itô integral of BPF vector $\Phi(t)$, $t \in [0,1)$ can be performed by [8]

$$\int_0^t \Phi(\tau)dB(\tau) = P_S\Phi(t), \quad t \in [0,1). \tag{3.2}$$

Hence, the Itô integral of every function $f(t) \in L^2[0,1)$ can be approximated as

$$\int_0^t f(s)dB(s) = F^T P_S\Phi(t).$$

The following theorem describes a relationship between the Walsh function and the block pulse function.

Theorem 3.2 ([11])**.** *Let the m-set of Walsh function and BPF vectors be $W(t)$ and $\Phi(t)$, respectively. Then the BPF vectors $\Phi(t)$ can be used to approximate $W(t)$ as $W(t) = T_W\Phi(t)$, $m = 2^k$, and $k = 0,1,\dots$, where $T_W = [c_{ij}]_{m\times m}$, $c_{ij} = w_i(\eta_j)$, for some $\eta_j = \left(\frac{j}{m}, \frac{j+1}{m}\right)$ and $i,j = 0,1,2,\dots m-1$.*

One can see that [2]

$$T_W T_W^T = mI \text{ and } T_W^T = T_W$$

which implies that $\Phi(t) = \frac{1}{m}T_W W(t)$.

Lemma 3.3 (Integration of Walsh Function)**.** *Suppose that $W(t)$ is a Walsh function vector, then the integral of $W(t)$ with respect to t is given by*

$$\int_0^t W(s)ds = \Lambda W(t),$$

where $\Lambda = \frac{1}{m} T_W P T_W$ *and*

$$P = \frac{1}{h}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Lemma 3.4 (Stochastic Integration of Walsh Function [11])**.** *Suppose that $W(t)$ is a Walsh function vector, then the Itô integral of $W(t)$ is given by*

$$\int_0^t W(s)dB(s) = \Lambda_S W(t),$$

where $\Lambda_S = \frac{1}{m} T_W P_S T_W$ *and*

$$P_S = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & \dots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & \dots & B(2h) - B(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B\left(\frac{(2m-1)h}{2}\right) - B((m-1)h) \end{bmatrix}.$$

4. Numerical Solution of Stochastic Volterra-Fredholm Integral Equation

We consider following *linear stochastic Volterra-Fredholm integral equation* (LSVFIE)

$$x(t) = f(t) + \int_0^1 k(s,t)x(s)ds + \int_0^t k_1(s,t)x(s)ds + \int_0^t k_2(s,t)x(s)dB(s), \tag{4.1}$$

where $x(t)$, $f(t)$, $k(s,t)$, $k_1(s,t)$ and $k_2(s,t)$ for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P) and $x(t)$ is unknown. Also, $B(t)$ is Brownian motion and $\int_0^t k_2(s,t)x(s)dB(s)$ is the Itô integral.

Using equations (2.1) and (2.2) in (4.1) we have

$$\begin{aligned} X^T T_W W(t) &= F^T T_W W(t) + \int_0^1 W^T(t) T_W K^T T_W W(s) W^T(s) T_W X ds \\ &\quad + \int_0^t W^T(t) T_W K_1^T T_W W(s) W^T(s) T_W X ds \\ &\quad + \int_0^t W^T(t) T_W K_2^T T_W W(s) W^T(s) T_W X dB(s) \\ &= F^T T_W W(t) + W^T(t) T_W K^T T_W \int_0^1 W(s) W^T(s) T_W X ds \\ &\quad + W^T(t) T_W K_1^T T_W \int_0^t W(s) W^T(s) T_W X ds \end{aligned}$$

$$+W^T(t)T_WK_2^TT_W\int_0^t W(s)W^T(s)T_WXdB(s). \tag{4.2}$$

Now

$$\begin{aligned}\int_0^t W(s)W^T(s)T_WXds &= \int_0^t T_W\Phi(s)\Phi^T(s)T_WT_WXds\\ &= mT_W\widetilde{X}P\frac{1}{m}T_WW(t).\end{aligned}$$

Hence

$$\int_0^t W(s)W^T(s)T_WXds = T_W\widetilde{X}PT_WW(t). \tag{4.3}$$

Similarly,

$$\int_0^t W(s)W^T(s)T_WXdB(s) = mT_W\widetilde{X}P_S\frac{1}{m}T_WW(t) = T_W\widetilde{X}P_ST_WW(t). \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.2) and using the condition of orthonormality, we get

$$\begin{aligned}X^TT_WW(t) &= F^TT_WW(t)+mW^T(t)T_WK^TX+mW^T(t)T_WK_1^T\widetilde{X}PT_WW(t)\\ &\quad+mW^T(t)T_WK_2^T\widetilde{X}P_ST_WW(t)\\ &= F^TT_WW(t)+mW^T(t)T_WK^TX+W^T(t)T_WH_1T_WW(t)+W^T(t)T_WH_2T_WW(t)\\ &= F^TT_WW(t)+mW^T(t)T_WK^TX+m\widehat{H_1}^TT_WW(t)+m\widehat{H_2}^TT_WW(t)\end{aligned}$$

which implies that

$$\left((I-mK)X^T-F^T-m\widehat{H_1}^T-m\widehat{H_2}^T\right)T_WW(t)=0, \tag{4.5}$$

where $H_1 = mK_1^T\widetilde{X}P$, $H_2 = mK_2^T\widetilde{X}P_S$ and $\widehat{H_i}$ is the m-vector with elements equal to the diagonal elements of H_i.

Hence

$$\left((I-mK^T)X-F-m\widehat{H_1}-m\widehat{H_2}\right)=[0]_{m\times 1} \tag{4.6}$$

can be solved to obtain a non trivial solution of the given stochastic Volterra-Fredholm integral equation (4.1).

5. Error Analysis

In this section, we analyse the error between the approximate solution and the exact solution of the stochastic Volterra- Fredholm integral equation. Before we start the analysis let us define, $\|X\|_2 = E(|X|^2)^{\frac{1}{2}}$.

Theorem 5.1 ([11])**.** *If $f\in L^2[0,1)$ satisfies the Lipschitz condition with Lipschitz constant C, then $\|e_m(t)\|_2 = O(h)$, where $e_m(t) = \left|f(t)-\sum_{i=0}^{m-1}c_iw_i(t)\right|$ and $c_i = \int_0^1 f(s)w_i(s)ds$.*

Theorem 5.2 ([11])**.** *Suppose $k\in L^2\big([0,1)\times[0,1)\big)$ satisfies the Lipschitz condition with Lipschitz constant L. If $k_m(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}c_{ij}w_i(x)w_j(y)$, $c_{ij} = \int_0^1\int_0^1 k(s,t)w_i(s)w_j(t)dtds$, then $\|e_m(x,y)\|_2 = O(h)$, where $|e_m(x,y)| = |k(x,y)-k_m(x,y)|$.*

Theorem 5.3. *Suppose $x_m(t)$ be the approximate solution of the linear SFVIE* (4.1). *If*

(i) $f \in L^2[0,1)$, $k(s,t)$, $k_1(s,t)$ *and* $k_2(s,t) \in L^2([0,1)\times[0,1))$ *satisfies the Lipschitz condition with Lipschitz constants* C, L, L_1 *and* L_2, *respectively,*

(ii) $|x(t)| \le \sigma$, $|k(s,t)| \le \rho$, $|k_1(s,t)| \le \rho_1$ *and* $|k_2(s,t)| \le \rho_2$,

then

$$\|x(t)-x_m(t)\|_2^2 = O(h^2).$$

Proof. Let (4.1) be the given SVFIE and $x_m(t)$ be the approximation to the solution using the Walsh function.

Then

$$\begin{aligned} x(t)-x_m(t) = f(t)-f_m(t) &+ \int_\alpha^\beta (k(s,t)x(s)-k_m(s,t)x_m(s))ds \\ &+ \int_0^t (k_1(s,t)x(s)-k_{1m}(s,t)x_m(s))ds \\ &+ \int_0^t (k_2(s,t)x(s)-k_{2m}(s,t)x_m(s))dB(s) \end{aligned}$$

that implies,

$$\begin{aligned} |x(t)-x_m(t)| \le |f(t)-f_m(t)| &+ \left|\int_\alpha^\beta (k(s,t)x(s)-k_m(s,t)x_m(s))ds\right| \\ &+ \left|\int_0^t (k_1(s,t)x(s)-k_{1m}(s,t)x_m(s))ds\right| \\ &+ \left|\int_0^t \big(k_2(s,t)x(s)-k_{2m}(s,t)x_m(s)\big)dB(s)\right|. \end{aligned}$$

We know that, $(a+b+c+d)^2 \le 7a^2+7b^2+7c^2+7d^2$. Hence,

$$\begin{aligned} E(|x(t)-x_m(t)|^2) \le 7E(|f(t)-f_m(t)|^2) &+ 7E\left(\left|\int_\alpha^\beta (k(s,t)x(s)-k_m(s,t)x_m(s))ds\right|^2\right) \\ &+ 7E\left(\left|\int_0^t (k_1(s,t)x(s)-k_{1m}(s,t)x_m(s))ds\right|^2\right) \\ &+ 7E\left(\left|\int_0^t (k_2(s,t)x(s)-k_{2m}(s,t)x_m(s))dB(s)\right|^2\right). \end{aligned} \tag{5.1}$$

Now for $i=1,2$, we have

$$\begin{aligned} |k_i(s,t)x(s)-k_{im}(s,t)x_m(s)| \le |k_i(s,t)||x(s)-x_m(s)| &+ |k_i(s,t)-k_{im}(s,t)||x(s)| \\ &+ |k_i(s,t)-k_{im}(s,t)||x(s)-x_m(s)|. \end{aligned}$$

For $i=1,2$, let $|k_i(s,t)| \le \rho_i$, $|x(s)| \le \sigma$ and using Theorem 5.2, we get

$$|k_i(s,t)x(s)-k_{im}(s,t)x_m(s)| \le \sqrt{2}L_i h\sigma + (\rho_i+\sqrt{2}L_i h)|x(t)-x_m(t)| \tag{5.2}$$

which gives,

$$\begin{aligned} E\left(\left|\int_0^t (k_1(s,t)x(s)-k_{1m}(s,t)x_m(s))ds\right|^2\right) &\le E\left(\left(\int_0^t |k_1(s,t)x(s)-k_{1m}(s,t)x_m(s)|ds\right)^2\right) \\ &\le E\left(\left(\int_0^t (\sqrt{2}L_i h\sigma + (\rho_i+\sqrt{2}L_i h)|x(t)-x_m(t)|)ds\right)^2\right). \end{aligned}$$

By Cauchy-Schwarz inequality, for $t > 0$ and $f \in L^2[0,1)$

$$\left|\int_0^t f(s)ds\right|^2 \le t\int_0^t |f|^2 ds$$

this implies,

$$E\left(\left|\int_0^t (k_1(s,t)x(s) - k_{1m}(s,t)x_m(s))ds\right|^2\right)$$
$$\le 2E\left(\int_0^t ((\sqrt{2}L_1h\sigma)^2 + (\rho_1+\sqrt{2}L_1h)^2|x(t)-x_m(t)|^2)ds\right).$$

Therefore,

$$E\left(\left|\int_0^t (k_1(s,t)x(s) - k_{1m}(s,t)x_m(s))ds\right|^2\right)$$
$$\le 2(\sqrt{2}L_1h\sigma)^2 + 2(\rho_1+\sqrt{2}L_1h)^2 E\left(\int_0^t |x(t)-x_m(t)|^2 ds\right). \tag{5.3}$$

Similarly, for $|k(s,t)| \le \rho$ and using Theorem 5.2, we get

$$E\left(\left|\int_\alpha^\beta \big(k(s,t)x(s) - k_m(s,t)x_m(s)\big)ds\right|^2\right)$$
$$\le 2(\beta-\alpha)(\sqrt{2}Lh\sigma)^2 + 2(\rho+\sqrt{2}Lh)^2 E\left(\int_\alpha^\beta |x(t)-x_m(t)|^2 ds\right). \tag{5.4}$$

Now,

$$E\left(\left|\int_0^t (k_2(s,t)x(s) - k_{2m}(s,t)x_m(s))dB(s)\right|^2\right)$$
$$\le E\left(\int_0^t |k_2(s,t)x(s) - k_{2m}(s,t)x_m(s)|^2 ds\right)$$
$$\le 2E\left(\int_0^t ((\sqrt{2}L_2h\sigma)^2 + (\rho_2+\sqrt{2}L_2h)^2|x(t)-x_m(t)|^2)ds\right).$$

Hence,

$$E\left(\left|\int_0^t (k_2(s,t)x(s) - k_{2m}(s,t)x_m(s))dB(s)\right|^2\right)$$
$$\le 2(\sqrt{2}L_2h\sigma)^2 + 2(\rho_2+\sqrt{2}L_2h)^2 E\left(\int_0^t |x(t)-x_m(t)|^2 ds\right). \tag{5.5}$$

Using Theorem 5.1, equations (5.3), (5.4) and (5.5) in (5.1), we get

$$\begin{aligned} E\big(|x(t)-x_m(t)|^2\big) &\le 7C^2h^2 + 7\left(2(\beta-\alpha)(\sqrt{2}Lh\sigma)^2 + 2(\rho+\sqrt{2}Lh)^2 E\left(\int_\alpha^\beta |x(t)-x_m(t)|^2 ds\right)\right) \\ &\quad + 7\left(2(\sqrt{2}L_1h\sigma)^2 + 2(\rho_1+\sqrt{2}L_1h)^2 E\left(\int_0^t |x(t)-x_m(t)|^2 ds\right)\right) \\ &\quad + 7\left(2(\sqrt{2}L_2h\sigma)^2 + 2(\rho_2+\sqrt{2}L_2h)^2 E\left(\int_0^t |x(t)-x_m(t)|^2 ds\right)\right) \\ &\le R_1 + R_2\int_0^t E(|x(s)-x_m(s)|^2)ds, \end{aligned} \tag{5.6}$$

where

$$R_1 = 7(C^2h^2 + 2(\beta-\alpha)(\sqrt{2}Lh\sigma)^2 + 2(\sqrt{2}L_1h\sigma)^2 + 2(\sqrt{2}L_2h\sigma)^2) \quad \text{and}$$

$$R_2 = 7(2(\rho+\sqrt{2}Lh)^2 + 2(\rho_1+\sqrt{2}L_1h)^2 + 2(\rho_2+\sqrt{2}L_2h)^2).$$

By using Gronwall's inequality, we have

$$E(|x(t)-x_m(t)|^2) \le R_1 \exp\left(\int_0^t R_2 ds\right), \tag{5.7}$$

which implies that,

$$\|x(t)-x_m(t)\|_2^2 = E(|x(t)-x_m(t)|^2) \le R_1 e^{R_2} = O(h^2). \tag{5.8}$$

□

6. Numerical Examples

To illustrate the method given in the above section, we consider following examples and compute the approximate solution. The computations are done using MATLAB 2013a.

Example 6.1 ([6]). Consider the following linear SVFIE,

$$x(t) = f(t) + \int_0^1 \cos(s+t)x(s)ds + \int_0^t (s+t)x(s)ds + \int_0^t e^{-3(s+t)}x(s)dB(s),$$

where $s,t \in [0,1)$ in which $f(t) = t^2 + \sin(s+t) - 2\cos(1+t) - 2\sin(t) - \frac{7t^4}{12} + \frac{1}{40}B(t)$, $B(t)$ is a Brownian motion, and $x(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P).

Table 1. Numerical result for $m = 32$ and $m = 64$ in Example 6.1

t	$m = 2^5$			$m = 2^6$		
	WFM	BPF [6]	HWM [9]	WFM	BPF [6]	HWM [9]
0.1	0.0114759	0.0199110	0.0189403	0.0085404	0.0155137	0.0184610
0.3	0.0839521	0:1174676	0.1026368	0.0998259	0.0583251	0.1033269
0.5	0.3296197	0.2741207	0.2469981	0.3385104	0.2775350	0.2462734
0.7	0.4891180	0.5144708	0.4624837	0.4933237	0.4886760	0.4644731
0.9	0.7826759	0.7685722	0.7642845	0.8223408	0.8222331	0.7640509

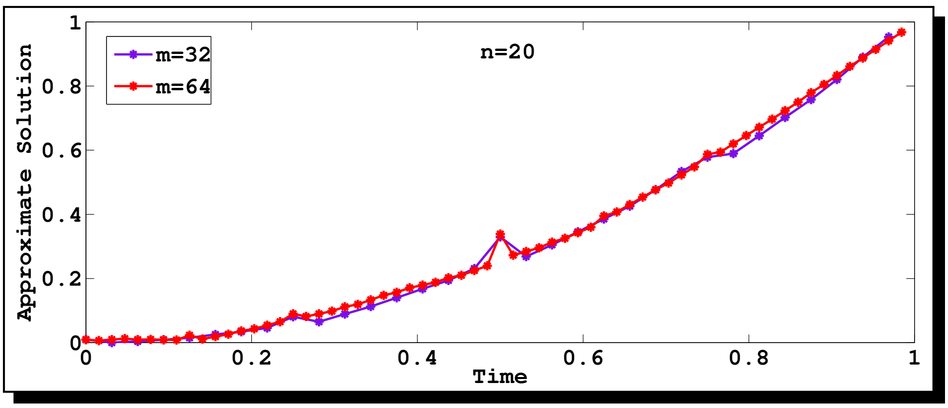

Figure 1. Example 6.1's approximate solution for $m = 32$ and $m = 64$

Example 6.2 ([9]). Consider the following linear SVFIE,

$$x(t) = f(t) + \int_0^1 (s+t)x(s)ds + \int_0^t (s-t)x(s)ds + \frac{1}{125}\int_0^t \sin(s+t)x(s)dB(s),$$

where $s,t \in [0,1)$ in which $f(t) = 2-\cos(1)-(1+t)\sin(1)+\frac{1}{250}\sin(B(t))$, $B(t)$ is a Brownian motion, and $x(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P).

Table 2. Numerical result for $m=32$ and $m=64$ in Example 6.2

t	$m=2^5$			$m=2^6$		
	WFM	BPF [6]	HWM [9]	WFM	BPF [6]	HWM [9]
0.1	0.9976241	0.9983232	0.9526175	0.9912432	0.9958677	0.9535115
0.3	0.9592595	0.9427155	0.9044299	0.9510972	0.9618340	0.9058330
0.5	0.8470106	0.8930925	0.8149461	0.8345253	0.8503839	0.8160360
0.7	0.7669107	0.7695923	0.6922649	0.7610515	0.7566968	0.6943825
0.9	0.6438552	0.6924411	0.5480265	0.6105657	0.6120356	0.5496713

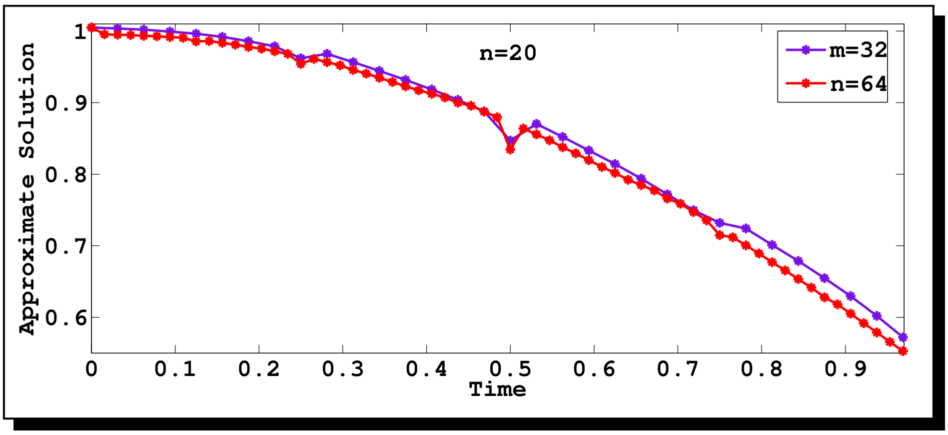

Figure 2. Example 6.2's approximate solution for $m=32$ and $m=64$

7. Conclusion

Since it is challenging to find the exact solution for the majority of the SVFIEs, the numerical technique is crucial in solving these issues. Several numerical solutions have also been developed earlier to determine the approximate solution of SVFIEs. This article also proposes a numerical method to find an approximate solution to SVFIE. It also includes numerical estimates for some SVFIEs. The important part is that error analysis of the approach has been undergone by considering the functions satisfying the Lipschitz condition to confirm the validity of the methodology, which gives an upper hand to consider more general SVFIEs than the previous methods. This method can be further developed to address nonlinear stochastic integral equations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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