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Research Article

Ciric Type Theorem for Class of Contravariant Functions in Bi-polar Metric Space

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Abstract. In the current study, utilizing the Ciric type contraction condition, an attempt has been taken to prove new fixed point result for class of contravariant functions in bipolar metric space. The many previous results from the literature are extended, improved, and modified in this paper.

Keywords. Bi-polar, Bi-sequence, Bi-convergent, Fixed point

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

Theory of fixed point is the crucial area in functional analysis. Banach principle is foundation of the entire metric fixed point theory (Banach [2]). Since then, numerous investigators have operated on it, developing the findings in various ways. Boyd and Wong [3], Gaba *et al*. [5], Mutlu *et al*. [8], Özkan and Gürdal [9], Rao *et al*. [10], and Siva [13] contributed to the creation of the contraction condition. New spaces including G-metric, cone metric, 2-metric, D-metric, M-metric, fuzzy metric, quasi metric and most recently bi-polar metric space are the focus of several academics (see Bajović *et al*. [1], Mutlu and Gürdal [7], Mutlu *et al*. [8], Rao *et al*. [10,11], and Roy and Saha [12]). Bipolar metric space was first established as a category of partial distance by Mutlu and Gürdal [7] in 2016. Also, they provided several extensions of well-known fixed point statements like Banach's and Kannan's as well as the connection between bipolar and metric spaces, particularly in situation for completeness.

The work in the current paper differs from all the studies mentioned previously. The objective of the article is to create a new theorem for class of contravariant maps in bipolar metric space. The Ciric type contraction condition [4] is used to prove the theorem.

Definition 1.1 ([9]). A bipolar metric space is a triplet (S,T,d), where S,T are non-empty sets and $d: S\times T \to R^+ = [0,\infty)$ is a function satisfying following properties:

(i) $d(s,t)=0 \Leftrightarrow s=t$, whenever $(s,t)\in S\times T$,

(ii) $d(s,t)=d(t,s)$, whenever $(s,t)\in S\cap T$,

(iii) $d(s_1,t_2)\le d(s_1,t_1)+d(s_2,t_1)+d(s_2,t_2)$, whenever $(s_1,t_1),(s_2,t_2)\in S\times T$.

The pair (S,T) is called bipolar metric.

Definition 1.2 ([9]). Let (S_1,T_1) and (S_2,T_2) be pairs of sets and f is a function $f:(S_1,T_1)\cup(S_2,T_2)$. If $f(S_1)\subseteq T_2$ and $f(T_1)\subseteq S_2$, we call f is a contravariant map from (S_1,T_1) to (S_2,T_2) and is denoted by $f:(S_1,T_1)\rightleftarrows(S_2,T_2)$.

Definition 1.3 ([9, 13]). Let (S,T,d) be bipolar metric space. Then

(i) S = set of left points; T = set of right points; $S\cap T$ = set of central points,

(ii) a sequence in S and sequence in T are known as left and right sequence correspondingly,

(iii) sequence (a_n) is define as convergent to point a iff a is right point and (a_n) is left sequence, also $\lim_{n\to\infty} d(a_n,a)=0$ or a is left point and (a_n) is a right sequence, also $\lim_{n\to\infty} d(a,a_n)=0$,

(iv) sequence (s_n,t_n) in $S\times T$ is called bi-sequence in (S,T). If both sequences (s_n) and (t_n) converge, then (s_n,t_n) is called convergent. If both sequences, (s_n), (t_n) converge to same point $u\in S\cap T$, then (s_n,t_n) is called bi-convergent,

(v) if $\lim_{n,m\to\infty} d(s_n,t_m)=0$, then bi-sequence (s_n,t_n) is called Cauchy bi-sequence,

(vi) each Cauchy bi-sequence must be convergent and hence bi-convergent in order for a bi-polar metric space to be defined as complete.

2. Main Result

The following theorem is established for family of contravariant functions in bi-polar metric space.

Theorem 2.1. *If (S,T,d) is complete bi-polar metric space, J an indexing set and $\{\mu_i\}_{i\in J}$ be a family of contravariant mappings $\mu_i:(S,T,d)\rightleftarrows(S,T,d)$ which satisfy*

$$d(\mu_i s,\mu_j t)\le \lambda u(s,t), \tag{2.1}$$

$$u\in M\{\mu_i,\mu_j;S,T\}=\lambda\max\left[d(s,t),d(s,\mu_i s),d(t,\mu_j t),\frac{1}{2}\{d(s,\mu_j t)+d(t,\mu_i s)\}\right],$$

where $\lambda=\lambda(i)\in(0,1)$. Then, all $\mu_i: S\cup T\to S\cup T$ have unique fixed point.

Proof. Let $s_0\in S$ and $t_0\in T$. For each $n\in N$, define

$$\mu_i(s_n)=t_n,\mu_j(t_n)=s_{n+1}. \tag{2.2}$$

Then (s_n,t_n) is a bi-sequence on (S,T,d).

$$d(s_n,t_n)=d(\mu_j t_{n-1},\mu_i s_n)=d(\mu_i s_n,\mu_j t_{n-1})\le\lambda u_1, \tag{2.3}$$

where

$$u_1 \in \max\left[d(s_n,t_{n-1}), d(s_n,\mu_i s_n), d(t_{n-1},\mu_j t_{n-1}), \frac{1}{2}\{d(s_n,\mu_j t_{n-1}) + d(t_{n-1},\mu_i s_n)\}\right]$$

$$\Rightarrow \quad u_1 \in \max\left[d(s_n,t_{n-1}), d(s_n,t_n), d(t_{n-1},s_n), \frac{1}{2}\{d(s_n,s_n) + d(t_{n-1},t_{n-1})\}\right]$$

$$\Rightarrow \quad u_1 \in d(s_n,t_{n-1})$$

Thus

$$d(s_n,t_n) \le \lambda d(s_n,t_{n-1}). \tag{2.4}$$

Now

$$d(s_n,t_{n-1}) = d(\mu_j t_{n-1},\mu_i s_{n-1}) = d(\mu_i s_{n-1},\mu_j t_{n-1}) \le \lambda u_2,$$

where

$$u_2 \in \max\left[d(s_{n-1},t_{n-1}), d(s_{n-1},\mu_i s_{n-1}), d(t_{n-1},\mu_j t_{n-1}), \frac{1}{2}\{d(s_{n-1},\mu_j t_{n-1}) + d(t_{n-1},\mu_i s_{n-1})\}\right]$$

$$\Rightarrow \quad u_2 \in \max\left[d(s_{n-1},t_{n-1}), d(s_{n-1},t_{n-1}), d(t_{n-1},s_{n-1}), \frac{1}{2}\{d(s_{n-1},s_n) + d(t_{n-1},t_{n-1})\}\right]$$

$$\Rightarrow \quad u_2 \in d(s_{n-1},t_{n-1})$$

Therefore

$$d(s_n,t_{n-1}) \le \lambda d(s_{n-1},t_{n-1}), \tag{2.5}$$

$$d(s_n,t_n) \le \lambda d(s_n,t_{n-1}) \le \lambda^2 d(s_{n-1},t_{n-1}) \le \ldots \le \lambda^{2n} d(s_0,t_0),$$

$$d(s_n,t_{n-1}) \le \lambda^{2n-1} d(s_0,t_0). \tag{2.6}$$

Case 1: For all positive integers, if $m > n$:

$$d(s_n,t_m) \le d(s_n,t_n) + d(s_{n+1},t_n) + d(s_{n+1},t_m) \le (\lambda^{2n} + \lambda^{2n+1} + \ldots + \lambda^{2m}) d(s_0,t_0).$$

Case 2: For all positive integers, if $m < n$:

$$d(s_n,t_m) \le d(s_{m+1},t_m) + d(s_{m+1},t_{m+1}) + d(s_n,t_{m+1}) \le (\lambda^{2m} + \lambda^{2m+1} + \ldots + \lambda^{2n}) d(s_0,t_0).$$

Since $\lambda \in (0,1)$, therefore, $d(s_n,t_m)$ can be reduced randomly by integer m,n and henceforth (s_n,t_m) is a Cauchy bi-sequence.
Since (S,T,d) is complete, (s_n,t_m) is a bi-convergent.
Let v be the point to which (s_n,t_m) bi-convergence. Then $(s_n) \to v$, $(t_n) \to v$ and $v \in S \cap T$.
Also, $t_n = \mu_i(s_n) \to \mu_i(v)$. Since (t_n) has limit in $S \cap T$, this limit is unique. Hence $\mu_i(v_1) = v_1$ and so μ_i has a unique fixed point. If v_2 is any fixed point of μ_j, then $\mu_j(v_2) = v_2$,

$$d(v_1,v_2) = d(\mu_i v_1,\mu_j v_2) \le \lambda u.$$

Here

$$u \in \max\left[d(v_1,v_2), d(v_1,\mu_i v_1), d(v_2,\mu_j v_2), \frac{1}{2}\{d(v_1,\mu_j v_2) + d(v_2,\mu_i v_1)\}\right] = d(v_1,v_2),$$

$$d(v_1,v_2) \le \lambda d(v_1,v_2) \Rightarrow v_1 = v_2.$$

Hence

$$\mu_i v_1 = \mu_j v_1 = v_1.$$

Hence family of functions $\{\mu_i\}_{i\in J}$ have unique common fixed point. □

Corollary 2.1. *If (S,T,d) is complete bipolar metric space, J an indexing set and $\{\mu_i\}_{i\in J}$ be a family of contravariant mappings $\mu_i:(S,T,d)\rightleftarrows(S,T,d)$ which satisfy*

$$d(\mu_i s,\mu_j t)\le \lambda u(s,t),$$
$$u\in M\{\mu_i,\mu_j;S,T\}=\lambda\max[d(s,t),d(s,\mu_i s),d(t,\mu_j t)],$$

where $\lambda=\lambda(i)\in(0,1)$. Then, all $\mu_i:S\cup T\to S\cup T$ have unique fixed point.

Proof. Proof is in line with Theorem 2.1. □

Corollary 2.2. *If (S,T,d) is complete bipolar metric space, J an indexing set and $\{\mu_i\}_{i\in J}$ be a family of contravariant mappings $\mu_i:(S,T,d)\rightleftarrows(S,T,d)$ which satisfy*

$$d(\mu_i s,\mu_j t)\le \lambda u(s,t),$$
$$u\in M\{\mu;S,T\}=d(s,t),$$

where $\lambda=\lambda(i)\in(0,1)$. Then, all $\mu_i:S\cup T\to S\cup T$ have unique fixed point.

Proof. By setting $M\{\mu;S,T\}=d(s,t)$ in Theorem 2.1, one can obtain the result. □

Corollary 2.3. *If (S,T,d) is complete bipolar metric space, μ_1,μ_2 be contravariant mappings which satisfy*

$$d(\mu_1 s,\mu_2 t)\le \lambda u(s,t),$$
$$u\in M\{\mu_1,\mu_2;S,T\}=\lambda\max\left[d(s,t),d(s,\mu_1 s),d(t,\mu_2 t),\frac{1}{2}\{d(s,\mu_2 t)+d(t,\mu_1 s)\}\right],$$

where $\lambda\in(0,1)$. Then μ_1, μ_2 have unique fixed point.

Proof. In Theorem 2.1, setting $\mu_i=\mu_1$, $\mu_j=\mu_2$, one can obtain the result. □

Corollary 2.4. *If (S,T,d) is complete bipolar metric space, μ be contravariant mappings which satisfy*

$$d(\mu s,\mu t)\le \lambda u(s,t),$$
$$u\in M\{\mu;S,T\}=\lambda\max\left[d(s,y),d(s,\mu s),d(t,\mu t),\frac{1}{2}\{d(s,\mu t)+d(t,\mu s)\}\right],$$

where $\lambda\in(0,1)$. Then μ has unique fixed point.

Proof. In Corollary 2.3, setting $\mu_1=\mu=\mu_2$, one can get the result. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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