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Research Article

Basins of Attraction of an Iterative Scheme and Their Applications

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Abstract. This work presents a seventh-order iterative scheme with the help of a variational iteration technique for finding the root of the nonlinear equations. The convergence analysis of the method is discussed and shows that it has seventh-order convergence with four functional evaluations per iteration. Therefore, the efficiency index is 1.6265. The computational performance of the suggested scheme is compared with some well-existing methods of the same order and is tested on various nonlinear equations, including real-world problems. Furthermore, we analyzed the dynamics of the proposed method using basins of attraction in the complex domain by taking some polynomial functions and compared our results with other known methods.

Keywords. Nonlinear equations, Iterative method, Efficiency index, Convergence order, Basins of attraction

Mathematics Subject Classification (2020). 41A25, 65H04, 65H05, 65H20, 65K05

1. Introduction

Finding the solution of nonlinear equations of the following form:

$$g(s) = 0. \tag{1.1}$$

These are the most common problems in mathematics, physics, and engineering. In most cases, the roots of these equations cannot be obtained directly. Hence, we must use an iterative method to approximate the roots of these equations. For these issues, *Newton Raphson Method* (NRM) (Traub [13]) is one of the most fundamental and well-known iterative methods and is represented by

$$s_{n+1} = s_n - \frac{g(s_n)}{g'(s_n)}, \quad g'(s_n) \neq 0, \tag{1.2}$$

which is a single-step method with quadratically convergent and requires two functional evaluations. Numerous numerical iterative algorithms have been developed recently to solve these problems. These methods are created by combining a variety of different techniques. In this paper, we proposed a new three-step iterative method for finding zeros of nonlinear equations. It requires four functional evaluations, and the order of convergence is seven. Therefore, the efficiency index is 1.6265. By evaluating the performance of the proposed method on many real-world applications, we can conclude that the new method outperforms many other known techniques in the same order in terms of the number of iterations and errors. Furthermore, we additionally examine and evaluate the suggested method's stability by applying complex dynamics techniques. Using these tools makes it possible to compare numerous algorithms in terms of their basins of attraction and the iterative method's dynamic behavior on the complex plane.

Some of the following existing seventh-order iterative methods:

Al-Subaihi and Al-Qarni [1] suggested the following seventh-order iterative process (SBM)

$$\left.\begin{aligned} y_n &= s_n - \frac{g(s_n)}{g'(s_n)}, \\ z_n &= y_n + \frac{g(y_n)}{g'(s_n)} - 2\frac{g(s_n)g(y_n)}{g'(s_n)(g(s_n)-g(y_n))}, \\ s_{n+1} &= z_n - \frac{g(z_n)}{g[z_n,y_n]+h[z_n,s_n,s_n](z_n-y_n)}. \end{aligned}\right\} \tag{1.3}$$

Mohamed [7] presented an iterative method (HBM) and is given by

$$\left.\begin{aligned} y_n &= s_n - \frac{g(s_n)}{g'(s_n)}, \\ z_n &= y_n - \frac{g(y_n)(1+\mu/2)}{g'(y_n)}, \\ s_{n+1} &= z_n + \frac{g(z_n)g(y_n)(1+\mu/2)}{(g(z_n)-g(y_n))g'(y_n)}, \end{aligned}\right\} \tag{1.4}$$

where $\mu = \frac{g(y_n)(g'(s_n)-g'(y_n))}{g(s_n)g'(y_n)}$.

Srisarakham and Thongmoon [12] suggested a new iterative scheme (NSM) of order seven is given by

$$\left.\begin{aligned} y_n &= s_n - \frac{g(s_n)}{g'(s_n)}, \\ z_n &= y_n - \frac{g(y_n)}{g'(y_n)} - \frac{g(y_n)^2 Q(s_n,y_n)}{2(g'(y_n))^3}, \\ s_{n+1} &= z_n - \frac{(s_n-z_n)g(z_n)}{g(s_n)-2g(z_n)}, \end{aligned}\right\} \tag{1.5}$$

where $Q(s_n,y_n) = \frac{2}{y_n-s_n}\left[2g'(y_n)+g'(s_n)-3\frac{g(y_n)-g(s_n)}{y_n-s_n}\right]$.

Chicharro *et al*. [4] proposed a family of iterative methods (FCM) of order seven is given by

$$\left.\begin{aligned} y_n &= s_n - \frac{g(s_n)}{g'(s_n)}, \\ z_n &= s_n - G(\eta)\frac{g(s_n)}{g'(s_n)}, \\ w_n &= z_n + \frac{g(z_n)}{g'(s_n)}, \\ s_{n+1} &= z_n - \left(1 - 4\left(\frac{g(z_n)}{g(w_n)}\right) + 8\left(\frac{g(z_n)}{g(w_n)}\right)^2\right)\frac{g(z_n)}{g'(s_n)}, \end{aligned}\right\} \tag{1.6}$$

where $G(\eta) = 1 + \frac{g(y_n)}{g(s_n)} + 2\left(\frac{g(y_n)}{g(s_n)}\right)^2$.

The remaining manuscript is structured as follows: In Section 2, we developed a seventh-order iterative scheme. In Section 3, the theoretical order of convergence is derived. In Section 4, some real-world application problems are solved for numerical comparisons and are made between the newly proposed method and some existing methods of the same order. In Section 5, we take three test functions to display the dynamic behavior of the developed method using basins of attraction. In Section 6, conclusions and references are presented.

2. Seventh-Order Method (MSM)

Consider a straight-line equation (Wartono *et al*. [14]):

$$y(s) = e^{p(s-s_n)}(M(s-s_n)+N) \tag{2.1}$$

and its derivatives

$$y'(s) = pe^{p(s-s_n)}(M(s-s_n)+N) + e^{p(s-s_n)}M, \tag{2.2}$$

$$y'(s) = p^2e^{p(s-s_n)}(M(s-s_n)+N) + 2pe^{p(s-s_n)}M. \tag{2.3}$$

Put $s = s_n$ in (2.1) and (2.2), we obtain $y(s_n) = g(s_n)$ and $y'(s_n) = g'(s_n)$, where

$$M = g'(s_n) - pg(s_n), \quad N = g(s_n). \tag{2.4}$$

Let an equation (2.1) through the x-axis at $s = s_{n+1}$, then $y(s_{n+1}) = 0$, we get

$$s_{n+1} = s_n - \frac{N}{M}, \tag{2.5}$$

$$s_{n+1} = s_n - \frac{g(s_n)}{g'(s_n) - pg(s_n)}, \quad n \geq 0. \tag{2.6}$$

From (2.3), we have

$$p^2g(s_n) - 2pg'(s_n) + g''(s_n) = 0. \tag{2.7}$$

Thus, we obtain

$$p = \frac{g''(y_n)}{g'(y_n)}\left(\frac{1}{1+\sqrt{1-\mu}}\right). \tag{2.8}$$

From (2.6) and (2.8), we get

$$s_{n+1} = s_n - \left[\frac{g(s_n)}{g'(s_n)} \cdot \frac{1}{\sqrt{1-\mu}}\right], \tag{2.9}$$

where

$$\mu = \frac{g(s_n)g''(s_n)}{(g'(s_n))^2}.$$

We develop an algorithm using (1.1) as the first step, (2.9) as the second step, and the Newton variant as the third step.

Algorithm 1. The iterative method to compute s_{n+1} is

(1) $y_n = s_n - \frac{g(s_n)}{g'(s_n)}$.

(2) $z_n = y_n - \left[\frac{g(y_n)}{g'(y_n)} \cdot \frac{1}{\sqrt{1-\mu}}\right]$,
where $\mu = \frac{g(y_n)g''(y_n)}{(g'(y_n))^2}$, $g'(y_n) = 2g[y_n, s_n] - g'(s_n)$, $g''(y_n) = \frac{-2(g'(s_n))^2 g(y_n)}{(g(s_n))^2}$.

(3) $s_{n+1} = z_n - \frac{g(z_n)}{g'(z_n)}$, (2.10)
where $g'(z_n) = g[z_n, y_n] + (z_n - y_n)g[z_n, y_n, s_n]$.

The above algorithm (2.10) is denoted as MSM.

3. Convergence Criteria

Theorem 3.1 ([6]). *For an open intervalD, let $s_0 \in D$ be a single zero of a sufficiently differentiable function $g(s)$, if the neighborhood of s^* contains s_0. A seventh-order convergence of the method* (2.10) *is then obtained.*

Proof. Let the single zero of $g(s) = 0$ be s^* and $s^* = s_n + \varepsilon_n$. Thus, $g(s^*) = 0$.
By Taylor's series expansion, writing $g(s_n)$ about s^*,

$$g(s_n) = g'(s^*)(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + \dots), \tag{3.1}$$

$$g'(s_n) = g'(s^*)(1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + \dots). \tag{3.2}$$

From the first step of (2.10), we get

$$y_n = s^* + c_2\varepsilon_n^2 + (2c_3 - 2c_2^2)\varepsilon_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)\varepsilon_n^4 + \dots. \tag{3.3}$$

Now, we have

$$g(y_n) = g'(s^*)(c_2\varepsilon_n^2 + (2c_3 - 2c_2^2)\varepsilon_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\varepsilon_n^4 + \dots), \tag{3.4}$$

$$g'(y_n) = g'(s^*)(1 + (2c_2^2 - c_3)\varepsilon_n^2 + (6c_2c_3 - 4c_2^3 - 2c_4)\varepsilon_n^3 + \dots) \tag{3.5}$$

$$g''(y_n) = -2c_2 - 4c_3\varepsilon_n + (6c_4 + 2c_2c_3)\varepsilon_n^2 + \dots. \tag{3.6}$$

From (3.4), (3.5), and (3.6) we obtain

$$\mu = -2c_2^2\varepsilon_n + (4c_2^3 - 8c_2c_3)\varepsilon_2^2 + \dots. \tag{3.7}$$

From the second step of (2.10), we get

$$z_n = s^* + Z_1\varepsilon_n^4 + Z_2\varepsilon_n^5 + Z_3\varepsilon_n^6 + \dots, \tag{3.8}$$

where $Z_1 = 2c_2^3 - c_2c_3$, $Z_2 = 2c_2^2c_3$, $Z_3 = 3c_2c_4 + c_2^2c_3 + 21c_2^3c_3 - 12c_2c_3^2 - 3c_2^2c_4 - \frac{37}{2}c_2^5$.
Again, we have

$$g(z_n) = g(s^*)(Z_1\varepsilon_n^4 + Z_2\varepsilon_n^5 + Z_3\varepsilon_n^6 + \dots). \tag{3.9}$$

Using (3.3) and (3.8), we have

$$z_n - y_n = \eta_1 e^2 + \eta_2 e^3 + \eta_3 e^4 + \dots. \tag{3.10}$$

Here $\eta_1 = -c_2$, $\eta_2 = 2c_2^2 - 2c_3$, $\eta_3 = 6c_2c_3 - 2c_2^3 - 3c_4 + \dots$.
Then, we get

$$g[z_n, y_n] = g'(s^*)(V_1 + V_2\varepsilon_n + V_3\varepsilon_n^2 + \dots) \tag{3.11}$$

where $V_1 = 1$, $V_2 = 0$, $V_3 = c_2^2, \dots,$

$$g[y_n, s_n] = 1 + \beta_1 e + \beta_2 e^2 + \beta_3 e^3 \dots \tag{3.12}$$

where $\beta_1 = c_2$, $\beta_2 = c_3 + c_2^2$, $\beta_3 = c_4 - 2c_2^3 + 3c_2c_3$.
From (3.10) and (3.11), we get

$$g'(z_n) = g'(s^*)(V_1 + (V_2 + \eta_1)e + (V_3 + \eta_2 + \eta_1\beta_1)e^2 + (V_4 + \eta_3 + \eta_2\beta_1)e^3 \dots). \tag{3.13}$$

From the third step of (2.10), we get

$$\varepsilon_{n+1} = (\eta_1 Z_3 + (V_3 + \eta_2 - \eta_1^2 + \eta_1\beta_1)Z_2 + (V_4 + \eta_3 + \eta_2\beta_1 - 2V_3\eta_1 - 2\eta_1\eta_2 + \eta_1^3 + \eta_1^2\beta_1)Z_1)\varepsilon_n^7 + o(\varepsilon_n^8)$$

which shows that the proposed method has seventh-order convergence, and its efficiency index is $7^{\frac{1}{4}} = 1.6265$. □

4. Numerical Examples

In this section, we take some real-world application problems in the form of nonlinear equations to check the effectiveness of the proposed scheme (MSM). We compare our results with some well-existing seventh-order methods, namely, NRM, SBM, HBM, NSM, and FCM. All the calculations are made with mpmath-PYTHON, and the stopping criterion $|f(x_n)| < \varepsilon$, where the tolerance is set to $\varepsilon = 10^{-199}$ and the required precision is set to 690 decimal places.

Table 1. Comparison of efficiency-index

Method	P	N	*E.I.*
NRM	2	2	1.4142
SBM	7	4	1.6266
HBM	7	5	1.4757
NSM	7	5	1.4757
FCM	7	5	1.4757
MSM	7	4	1.6266

Note: P is the order of convergence and N is the functional evaluations

5. Applications

Application 1 (Azeotropic point of a binary solution, [5, 11])**.** To find the azeotropic point of an equation:

$$g_1(s) = \frac{MN(N(1-s)^2 - Ms^2)}{(s(M-N)+M)^2} + 0.14845.$$

We took, $M = 0.38969$ and $N = 0.55954$ were used. The root of the nonlinear equation is 0.69147373574714144, is displayed in the table below.

Table 2

Method	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$
$g_1(s)$	0.2				0.9			
NRM		10	2.60e-275	2.86e-275		9	1.19e-237	1.31e-237
SBM			Divergent			4	1.89e-288	1.41e-288
HBM		5	4.10e-691	7.18e-691		4	6.88e-202	7.57e-202
NSM		5	6.84e-691	7.18e-691		4	2.39e-303	2.63e-303
FCM		5	1.91e-690	1.57e-690		4	1.06e-264	1.17e-264
MSM		4	9.72e-227	1.07e-226		4	7.39e-358	8.14e-358

Application 2 (Ideal and Non-Ideal Gas Laws, [3])**.** The computation of the molal volume of ideal and non-ideal gas is given by

$$g_2(V) = \left(p + \frac{a}{V^2}\right)(V-b) - RT.$$

We take the values of the parameters as, the universal constant of the gas $R = 0.082054\,\mathrm{L\,atm/(mol\,K)}$ for carbon dioxide, temperature $T = 300\,\mathrm{K}$, pressure $p = 1\,\mathrm{atm}$ and $a = 3.592$, $b = 0.04267$ are constants. Therefore, 24.5125881284415006 is the root of the nonlinear equation $g_2(V)$.

Table 3

Method	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$
$g_2(v)$	4				30			
NRM		9	4.67e-382	4.64e-382		8	3.78e-376	3.76e-376
SBM		4	1.72e-207	1.71e-207		4	3.11e-688	6.13e-689
HBM		5	4.38e-690	3.94e-689		4	2.97e-589	2.95e-589
NSM		4	8.19e-261	8.14e-261		4	0	3.94e-689
FCM		4	9.55e-265	9.49e-265		4	1.22e-688	6.13e-689
MSM		4	6.49e-332	6.45e-332		4	3.51e-689	3.94e-689

Application 3 (Study of Multifactor Effect [5, 10])**.** The equation

$$s(t) = s_0 + (s_0 + eE_0(mw)^{-1}\sin(wt_0+\eta))(t-t_0) + eE_0(mw^2)^{-1}(\cos(wt_0+\eta)+\sin(wt_0+\eta))$$

describes the moment of an electron in the space between two parallel plates. Regarding the specific values, it is reduced in polynomial form as

$$g_3(s) = s - 0.5\cos s + \frac{\pi}{4}.$$

This function has a simple root at $s^* \approx -0.309466139208214$.

Table 4

Method	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$
$g_3(s)$	−0.7				0.5			
NRM		9	5.09e-241	4.32e-241		10	4.84e-374	4.11e-374
SBM		4	3.07e-291	2.60e-291		4	6.22e-246	5.28e-246
HBM		4	1.26e-204	1.07e-204		5	4.79e-691	5.47e-691
NSM		4	4.17e-338	3.54e-338		4	1.16e-262	9.85e-263
FCM		4	8.59e-288	7.29e-288		5	4.65e-690	1.37e-690
MSM		4	2.94e-361	2.49e-361		4	5.41e-277	4.59e-277

Application 4 (The Vertical Stress, [5]). The vertical stress is one of the basic stresses that describing about underground structures, and it can be written in the form of nonlinear equation as

$$g_4(s) = \frac{s + \cos s \cdot \sin s}{\pi} - \frac{1}{4}.$$

The nonlinear equation $g_4(s) = 0$ has a root of 0.4160444988100767043.

Table 5

Method	s_0	n	$\|s_{n+1} - s_n\|$	$\|g(s_{n+1})\|$	s_0	n	$\|s_{n+1} - s_n\|$	$\|g(s_{n+1})\|$
$g_4(s)$	0.6				0.4			
NRM		9	9.39e-260	4.99e-260		8	1.93e-276	1.03e-276
SBM		4	2.80e-295	1.49e-295		4	6.23e-678	3.31e-678
HBM		4	2.83e-219	1.51e-219		4	3.09e-466	1.64e-466
NSM		4	6.06e-234	3.23e-234		4	3.42e-691	3.42e-691
FCM		4	1.67e-298	8.91e-299		4	3.98e-677	2.12e-677
MSM		4	1.21e-332	6.45e-333		4	6.84e-692	1.37e-691

Application 5 (Volume from van der Waals Equation, [5]). An equation

$$\left(p + \frac{An^2}{V^2}\right)(V - nB) = nRT$$

represents the non-ideal gas in the Van der Waals equation. Regarding particular values, it is converted to nonlinear polynomial function

$$g_5(s) = 40s^3 - 95.26535116s^2 + 35.28s - 5.6998368.$$

It has three roots in which one is real, i.e., 1.9707842194070294.

Table 6

Method	s_0	n	$\|s_{n+1} - s_n\|$	$\|g(s_{n+1})\|$	s_0	n	$\|s_{n+1} - s_n\|$	$\|g(s_{n+1})\|$
$g_5(s)$	2.5				1.8			
NRM		26	1.90e-203	2.39e-201		25	2.97e-201	3.74e-199
SBM		8	1.78e-214	2.24e-212		8	2.51e-217	3.16e-215
HBM		8	9.42e-407	1.18e-404		7	1.15e-292	1.45e-291
NSM		8	3.63e-206	4.57e-204		8	2.03e-213	2.56e-211
FCM		7	4.03e-207	5.07e-205		7	1.32e-215	1.66e-213
MSM		6	2.67e-282	3.36e-280		6	4.39e-385	5.52e-383

Application 6 (Blood Rheology Model, [8]). To evaluate the plug flow of Caisson fluids, we consider the following function as a nonlinear equation:

$$H = 1 - \frac{16}{7}\sqrt{u} + \frac{4}{3}u - \frac{1}{21}u^4,$$

where $H = 0.4$ calculates the flow rate reduction. We get the nonlinear equation

$$g_6(s) = \frac{1}{441}s^8 - \frac{8}{63}s^5 - 0.0571428571s^4 + \frac{16}{9}s^2 - 3.624489796s + 0.3.$$

The root of $g_6(s) = 0$ is 0.0864335580522467.

Table 7

Method	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$
$g_6(s)$	0				0.2			
NRM		9	3.62e-347	1.20e-346		9	2.64e-305	8.75e-305
SBM		4	1.38e-501	4.58e-501		4	2.51e-413	8.34e-413
HBM		4	1.72e-331	5.69e-331		4	5.31e-266	1.76e-265
NSM		4	1.18e-450	3.90e-450		4	7.91e-396	2.62e-395
FCM		4	3.99e-408	1.32e-407		4	9.29e-357	8.08e-356
MSM		4	1.11e-520	3.69e-520		4	1.51e-438	5.01e-432

Application 7 (A reactor of the stirred tank, [9]). Think about the stirred tank's reactor. The reactor receives materials at rates of β and $q-\beta$, respectively. The equipment enhances mixed reaction as follows:

$$H_1+H_2\to H_3;\ \ H_3+H_2\to H_4;\ \ H_4+H_2\to H_5;\ \ H_4+H_2\to H_6\,.$$

In their initial investigation of this intricate control system, and the nonlinear polynomial equation shown below:

$$\frac{2.98*(s+2.25)}{(s+1.45)*(s+2.85)^2*(s+4.35)}=\frac{1}{T_c},$$

where T_c is the proportional controller's gain. By taking $T_c=0$, we have

$$g_7(s)=s^4+11.50s^3+47.49s^2+83.06325s-51.23266875=0\,.$$

The root of the above equation is -1.45.

Table 8

Method	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$
$g_7(s)$	-1.4				-1.5			
NRM		9	6.03e-277	3.42e-276		9	1.59e-263	9.08e-263
SBM		4	1.64e-367	9.31e-367		4	2.61e-328	1.48e-327
HBM		4	1.47e-224	8.39e-224		4	4.04e-215	2.29e-214
NSM		4	1.35e-357	1.73e-357		4	1.32e-339	7.49e-339
FCM		4	1.51e-315	8.61e-315		4	3.52e-309	2.00e-308
MSM		4	2.33e-455	1.32e-454		4	7.58e-352	6.63e-349

Application 8 (Parachutist Problem, [2,5]). The total force for parachutists is calculated as

$$F=mg-sv\,,$$

where m is the mass, g is the acceleration caused by gravity, s is the drag coefficient, v is the parachutist's velocity, and from the aforementioned equation, we obtain the nonlinear equation

$$g_8(s)=\frac{gm}{s}(1-e^{-\frac{s}{m}t})-v\,.$$

We suppose that the parameters will have values of $g=9.8\,\text{m/s}^2$, $m=68\,\text{kg}$, $t=8\,\text{s}$, and $v=41\,\text{m/s}$. Therefore, 12.533522848184467 is the nonlinear equation's root.

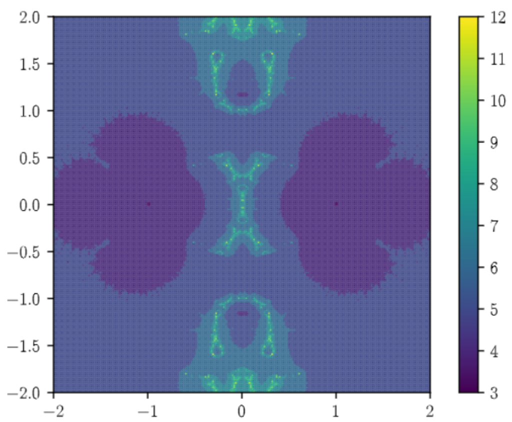
Table 9

Method	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$	s_0	n	$\lvert s_{n+1}-s_n\rvert$	$\lvert g(s_{n+1})\rvert$
$g_8(s)$	13				4.5			
NRM		9	6.03e-277	3.42e-276		10	1.41e-302	2.60e-302
SBM		4	2.44e-584	4.49e-584			Divergent	
HBM		4	7.55e-382	1.39e-381			Divergent	
NSM		4	1.82e-602	3.36e-602		5	3.94e-687	2.63e-689
FCM		4	3.81e-564	7.01e-564		5	1.29e-689	2.63e-689
MSM		4	1.91e-602	3.51e-606		5	1.09e-689	2.63e-689

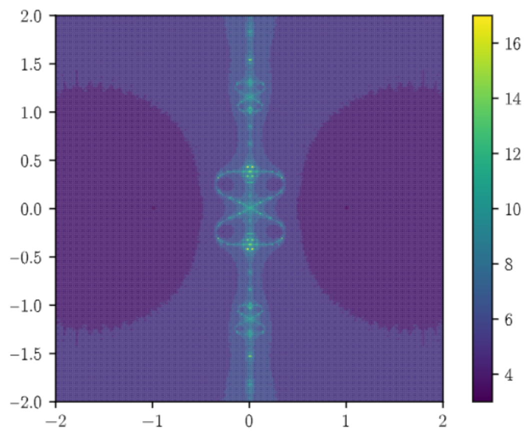
Basins of Attraction

In this section, using computer technology, we compare the proposed method (MSM) with the other existing algorithms, such as SBM, HBM, NSM, and FCM of same order. We consider a square region $[-2,2]\times[-2,2]\in C^2$ with 250×250 grid points from a dynamic and graphical perspective. For stopping criteria, we take $|s_{n+1}-s_n|<10^{-16}$ and the maximum number of iterations are 100. Consider the polynomial functions in a complex plane are $f_1(z)=1-z^2$, $f_2(z)=1-z^3$ and $f_3(z)=1-z^4$. All the figures are illustrated by PYTHON programming.

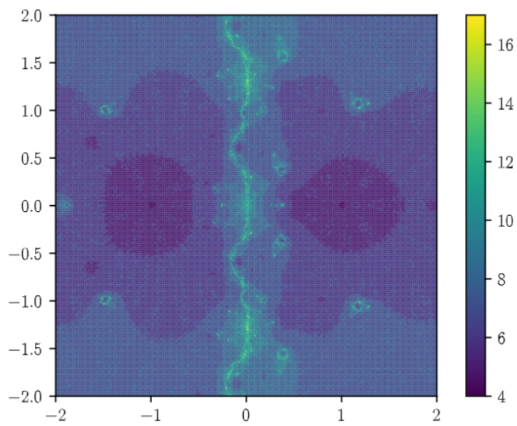
Example 1. $f_1(z)=1-z^2$.



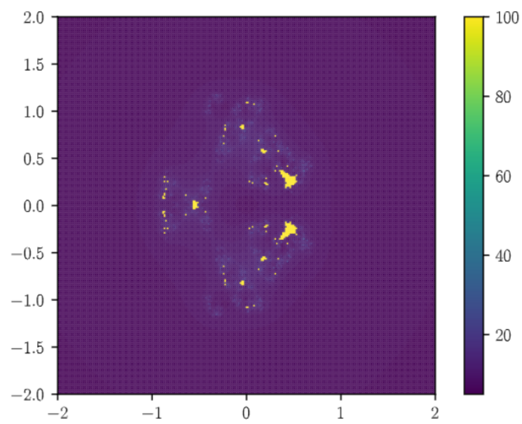
(a) (MSM)



(b) (SBM)

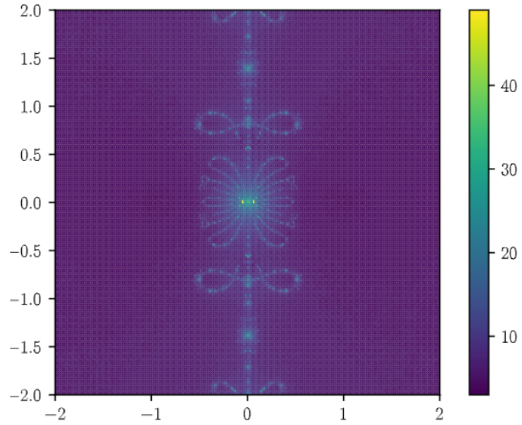


(c) (HBM)



(d) (NSM)

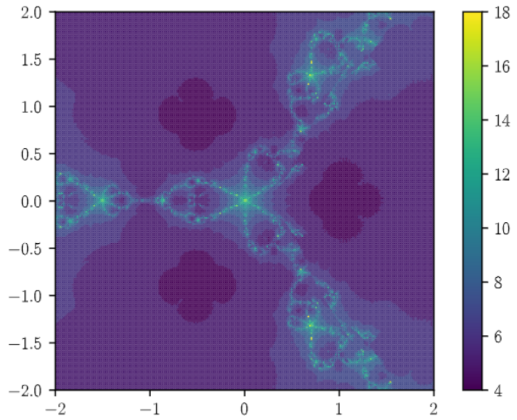
Figure *Contd.*



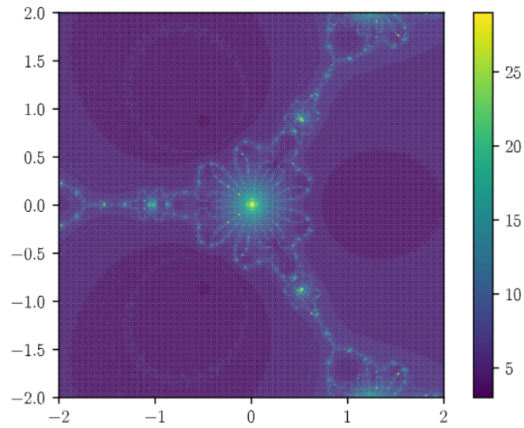
(e) (FCM)

Figure 1. The polynomiographs of MSM, SBM, HBM, NSM, FCM for $f_1(z)$

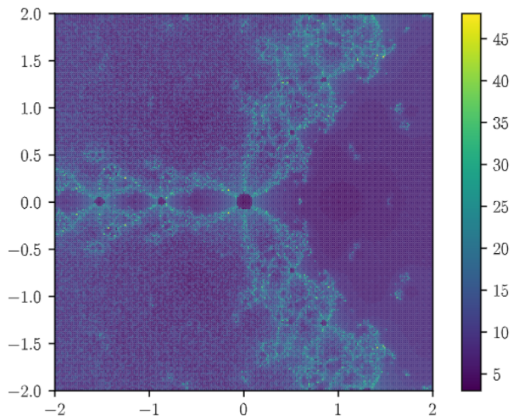
Example 2. $f_2(z) = 1 - z^3$.



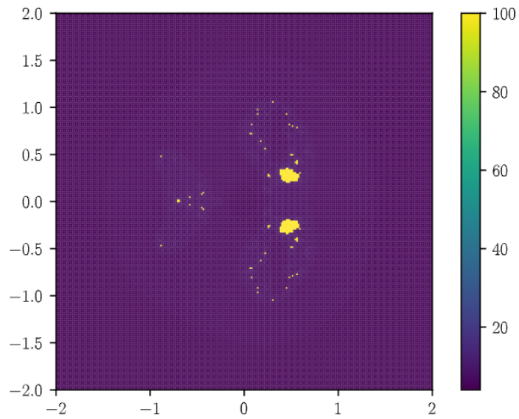
(a) (MSM)



(b) (SBM)

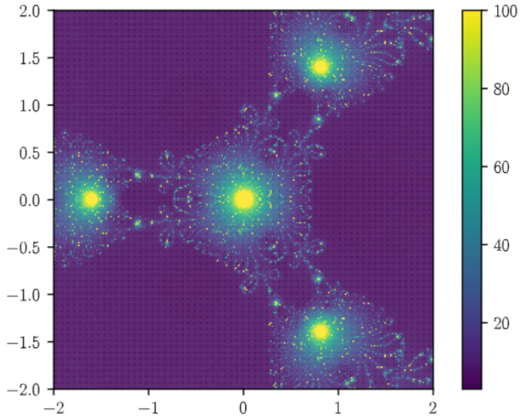


(c) (HBM)



(d) (NSM)

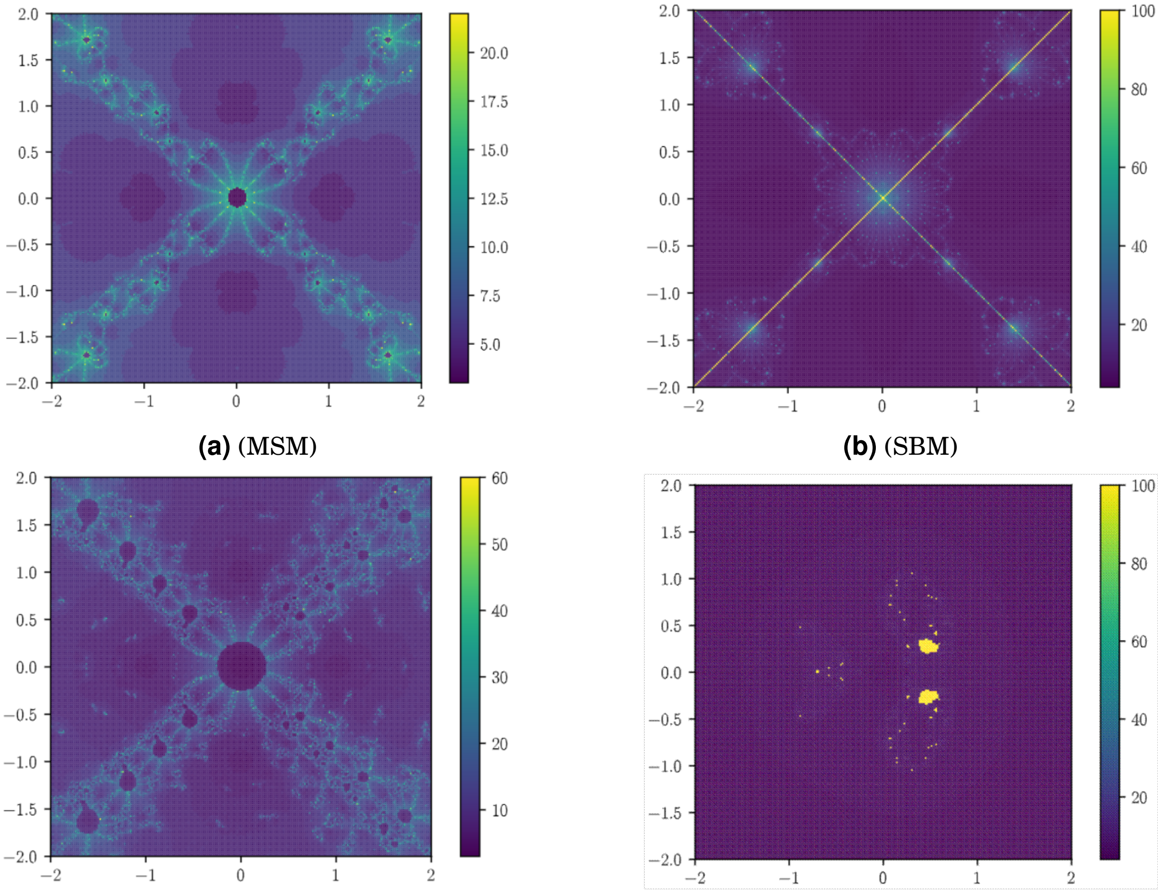
Figure *Contd*.



(e) (FCM)

Figure 2. The polynomiographs of MSM, SBM, HBM, NSM, FCM for $f_2(z)$

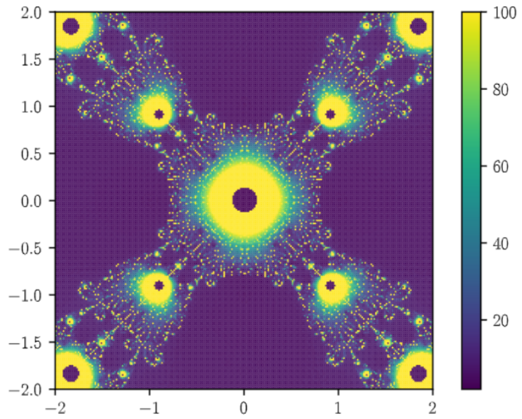
Example 3. $f_3(z) = 1 - z^4$.



(a) (MSM) **(b)** (SBM)

(c) (HBM) **(d)** (NSM)

Figure *Contd.*



(e) (FCM)

Figure 3. The polynomiographs of MSM, SBM, HBM, NSM, FCM for $f_3(z)$

Figures 1-3 show that the method (MSM) gives better results when compared to other methods. From the fractal graphs, the method MSM requires fewer iterations in the strong, moderate, and weakly convergent areas than the methods SBM, HBM, NSM, and FCM. Thus, the proposed technique MSM is the best for all three polynomials in terms of the number of iterations per convergent point.

6. Conclusion

In this paper, we suggested a seventh-order iterative method for locating the roots of nonlinear equations. To demonstrate the superiority of the proposed MSM approach, we examined existing and suggested methods used to solve various real-world issues. The results are shown in Tables 2-9. The introduced approach's dynamic behavior has been examined to analyze the stability. Figures 1-3 show our proposed iterative method's effectiveness and best convergence, which gives the best results compared to other methods.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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