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Research Article

A New Approach of Centroid based Ranking Fuzzy Numbers and its Comparative Reviews

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Abstract. The numeric values represented by fuzzy numbers are vague and ranking them according to their location on the real axis is not adequate and logical. Ranking of fuzzy numbers plays a vital role in measuring the degree of importance of different alternatives in decision-making under the fuzzy environment and the comparison of fuzzy numbers reflects that of alternatives. Although a lot of methods for ranking fuzzy numbers exist in the literature, even then none of them is superior to all others. This paper proposes a new method for ranking fuzzy numbers using the coordinates of the centroid point of the fuzzy numbers. It suggests a ranking score for the fuzzy number that multiplies the ordinate and the exponential value of the ratio of the ordinate to the abscissa of the centroid point. The proposed ranking score method can rank two or more fuzzy numbers simultaneously irrespective of their linear or non-linear membership functions. Furthermore, it consistently ranks symmetric fuzzy numbers of the same or different altitudes, images of fuzzy numbers, and the fuzzy numbers that describe the compensation of areas. Comparative reviews show an edge of the proposed method over several representative approaches.

Keywords. Centroid point, Ranking fuzzy numbers, Ranking score, Exponential value of ordinate

Mathematics Subject Classification (2020). 03E72, 94D05

1. Introduction

The ranking of fuzzy numbers is a significant feature of its applications in real-world scenarios. The fuzzy numbers do not always display a completely ordered set as can be done with the real numbers due to the presence of left-right fuzziness as a part of them. Several authors have put forth various techniques that produce a linearly ordered collection or ranking to resolve the problems of comparing fuzzy numbers. The notion of ranking imprecise quantities that are represented as fuzzy subsets has been first proposed by Jain [9]. Yager [20, 21] introduced the applicable ideas and concepts of ranking fuzzy subsets over the unit interval in which he proposed the centroid value on the horizontal axis. Since then, the literature has acknowledged a large number of suggested methods for ranking fuzzy numbers using various notions. These methods are classified based on the different parameters of the fuzzy numbers such as centroid methods, distance methods, area methods, lexicographical methods, methods based on the decision maker's viewpoint, and left-right spreads. Liou and Wang [10] introduced an indexing technique for ranking fuzzy numbers in which they suggested total integral value with an indicator of optimism. Fortemps and Roubens [8] suggested a method based on the compensation of areas of fuzzy numbers. Cheng [4] presented the formulae for the abscissa and ordinate of the centroid point of an L-R type fuzzy number and suggested the distance of the centroid point from the original point for comparing fuzzy numbers. However, the distance method [4] fails to differentiate the image of the fuzzy number. Before Cheng [4] the centroid value on the horizontal axis is the most important index for ranking fuzzy numbers. However, the centroid value on the vertical axis is taken into account only in special cases where the values on the horizontal axis are the same as in the case of symmetrical fuzzy numbers. Incidentally, the centroid formulae of Cheng [4] were found incorrect by Wang *et al*. [19] to prevent misapplications, and they presented the correct centroid formulae for ranking fuzzy numbers. In the year 2002, Chu and Tsao [5] suggested a rectangular area between the centroid point and the original point for ranking fuzzy numbers. Wang and Lee [19] in another approach pointed out a shortcoming of Chu and Tsao [5] and suggested that the x-coordinate of the centroid point indicates the representative location of the fuzzy number on the real line whereas the y-coordinate represents an average height of the fuzzy number and the importance of the degree of representative location is higher than the average height. Abbasbandy and Asady [1] suggested the sign distance from the fuzzy origin for distinguishing the fuzzy numbers. Asady and Zendehnam [3] advised a point nearest to the origin as a de-fuzzified value for the fuzzy number. A review of centroid index ranking methods was reported by Ramli and Mohamad [16]. Various centroid ranking methods are considered and compared showing the fact that no single method in the centroid concept is superior to all other methods since each method appears to have some advantages as well as disadvantages. Nasseri *et al*. [11] presented a technique for ranking fuzzy quantities based on the angle between the reference functions of the fuzzy numbers. Yu and Dat [22] presented a modification in the area integrals of Liou and Wang [10] to overcome the shortcomings. Chutia and Chutia [7] suggested a method based on the value and ambiguity of the fuzzy numbers to differentiate them. Nguyen [13] defined a unified index that multiplies the weighted mean and the weighted area of the fuzzy number as two different discriminatory components and obtained very good comparative results. Patra [14] presented a fuzzy risk analysis technique based on ranking generalized fuzzy numbers using the mean, area, and perimeter of the fuzzy number. Prasad and Sinha [18] suggested a unified integral that multiplies the core value and the left-right area integrals. Prasad and Sinha [15] also suggested points on the left and the right reference functions that divide them in the same ratio $m:n$. The horizontal mean of the two points used a ranking tool for the fuzzy numbers.

Contrary to a large number of ranking indices based on the centroid point and other parameters of the fuzzy numbers, there is a wide scope for further studies and comparative reviews because none of them is superior to all others. For instance, Patra [14] incorporated the mean, area, and perimeter of the fuzzy number leading to counterintuitive ranking results for the fuzzy numbers having different degrees of representative locations on the horizontal axis. Numerical illustrations are shown in Example 4.2. Using the abscissa and ordinate of the centroid point of the fuzzy numbers, this paper suggested a ranking score that multiplies the abscissa and the exponential value of the ratio of ordinate to the absolute value of the abscissa. The proposed technique consistently ranks symmetric fuzzy numbers, fuzzy numbers that describe the compensation of areas, and crisp numbers.

The remaining sections of the paper are fragmented into the following four groups, excluding the introduction. Section 2 provides the preliminary definitions that are involved with the proposed work. The proposed ranking score for the fuzzy number and the ordering procedure are described in Section 3. Section 4 comprises comparative reviews with some representative approaches that exist in the literature. Section 5 finishes with conclusions.

2. Preliminaries

This section recalls basic definitions of different types of fuzzy numbers which are related to the forward study, [13] followed.

2.1 Generalized Fuzzy Number

A fuzzy subset A of the real line R is said to be a generalized fuzzy number if its membership function $f_A(x)$ holds the following conditions for $a,b,c,d \in R$, $(a \le b \le c \le d)$:

(i) $f_A(x)$ is a piece-wise continuous function from the real line R to the closed interval $[0,\omega]$ where ω is constant and $0 \le \omega \le 1$.

(ii) $f_A(x) = 0$, for all $x \in\,]-\infty, a]$,

(iii) $f_A(x)$ is strictly increasing on $[a,b]$,

(iv) $f_A(x) = \omega$, for all $x \in [b,c]$,

(v) $f_A(x)$ is strictly decreasing on $[c,d]$,

(vi) $f_A(x) = 0$, for all $x \in [d, \infty[$.

Conveniently, the generalized fuzzy number is represented by $A = (a,b,c,d;\omega)$, and its membership function $f_A(x)$ is expressed as:

$$f_A(x) = \begin{cases} f_A^L(x), & x \in [a,b], \\ \omega, & x \in [b,c], \\ f_A^R(x), & x \in [c,d], \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $f_A^L(x) : [a,b] \to [0,\omega]$ and $f_A^R(x) : [c,d] \to [0,\omega]$ are respectively, known as the left and the right membership functions of the fuzzy number A. $f_A^L(x)$ is continuous and strictly increasing on $[a,b]$, whereas $f_A^R(x)$ is also continuous but strictly decreasing on $[c,d]$.

2.2 Image of the Generalized Fuzzy Number

The image of a generalized fuzzy number $A = (a,b,c,d;\omega)$ concerned with the membership axis is denoted by A' and defined as $A' = (-d,-c,-b,-a;\omega)$, where $0 \le \omega \le 1$. Its membership function $f_{A'}(x)$ can be expressed by:

$$f_{A'}(x) = \begin{cases} f^L_{A'}(x), & x \in [a,b], \\ \omega, & x \in [b,c], \\ f^R_{A'}(x), & x \in [c,d], \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

where $f_{A'}(x) : [-d,-c] \to [0,\omega]$ and $f_{A'}(x) : [-b,-a] \to [0,\omega]$ are respectively known as the left and the right membership functions of A'. $f^L_{A'}(x)$ is continuous and strictly increasing on $[-d,-c]$, whereas $f^R_{A'}(x)$ is also continuous but strictly decreasing on $[-b,-a]$.

2.3 Generalized Trapezoidal Fuzzy Number

A generalized fuzzy number $A = (a,b,c,d;\omega)$ is said to be a generalized trapezoidal fuzzy number if its membership function $f_A(x)$ is given by

$$f_A(x) = \begin{cases} \dfrac{\omega(x-a)}{(b-a)}, & x \in [a,b], \\ \omega, & x \in [b,c], \\ \dfrac{\omega(x-d)}{(c-d)}, & x \in [c,d], \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

2.4 Generalized Triangular Fuzzy Number

A generalized fuzzy number $A = (a,b,c,d;\omega)$ is said to be a generalized triangular fuzzy number if $b = c$ and its membership function $f_A(x)$ is represented by

$$f_A(x) = \begin{cases} \dfrac{\omega(x-a)}{(b-a)}, & x \in [a,b], \\ \omega, & x = b, \\ \dfrac{\omega(x-d)}{(c-d)}, & x \in [b,c], \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

The triangular fuzzy number $A = (a,b,b,d;\omega)$ is simply expressed as $A = (a,b,d;\omega)$.

Remark 2.1. The notation of fuzzy numbers with linear or non-linear membership functions and generalized (normalized and non-normalized) fuzzy numbers are the same. However, they are characterized and identified differently by their respective membership functions.

3. Ranking Score Based on the Centroid Point of the Fuzzy Number

Let $(\overline{x}_A, \overline{y}_A)$ denotes the centroid point of the generalized fuzzy number $A = (a,b,c,d;\omega)$ as visualized in Figure 1.

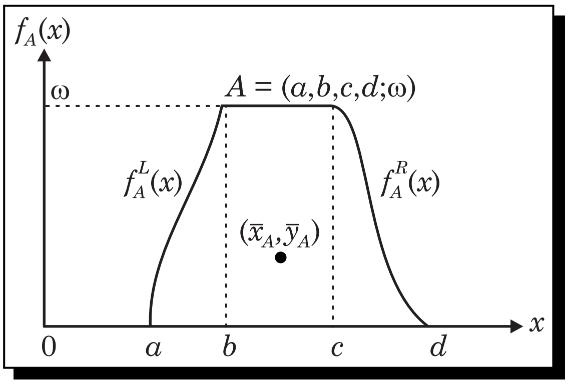

Figure 1. Visual representation of the general fuzzy number and its centroid point

The membership function of $A = (a,b,c,d;\omega)$ is displayed in eq. (2.1). Therefore

$$\bar{x}_A = \frac{\int_a^b (x\, f_A^L(x))\, dx + \int_b^c x\, dx + \int_c^d (x\, f_A^R(x))\, dx}{\int_a^b f_A^L(x)\, dx + \int_b^c dx + \int_c^d f_A^R(x)\, dx}, \tag{3.1}$$

$$\bar{y}_A = \frac{\int_0^\omega (y\, g_A^R(y))\, dy - \int_0^\omega (y\, g_A^L(y))\, dy}{\int_0^\omega g_A^R(y)\, dy - \int_0^\omega g_A^l(y)\, dy}, \tag{3.2}$$

where $g_A^L(y) : [0,\omega] \to [a,b]$ and $g_A^R(y) : [0,\omega] \to [c,d]$ are the inverse functions of $f_A^L(x) : [a,b] \to [0,\omega]$ and $f_A^R(x) : [c,d] \to [0,\omega]$, respectively. Clearly, $g_A^L(y) = a + (b-a)y$ and $g_A^R(y) = d - (d-c)y$.

Let us denote the ranking score of the generalized fuzzy number $A = (a,b,c,d;\omega)$ by $S(A)$, and defined as:

$$S(A) = (\bar{x}_A + \varepsilon) \cdot \exp\left(\frac{\bar{y}_A}{|\bar{x}_A + \varepsilon|}\right), \tag{3.3}$$

where $\varepsilon = 0$ for $\bar{x} \neq 0$, otherwise it will be a quantifiable and suitably small positive rational number, taken for comparing fuzzy numbers symmetrical about the membership axis.

Using the ranking score of the fuzzy numbers as defined in eq. (3.3), the ranking of any two fuzzy numbers $A_1 = (a_1, b_1, c_1, d_1; \omega_1)$ and $A_2 = (a_2, b_2, c_2, d_2; \omega_2)$ will be carried out as follows:

(i) If $S(A_1) > S(A_2)$, then $A_1 > A_2$.

(ii) If $S(A_1) < S(A_2)$, then $A_1 < A_2$.

(iii) If $S(A_1) = S(A_2)$, then $A_1 \sim A_2$.

4. Comparative Reviews

This section comprises the comparative ranking results of the suggested approach with some representative ranking methods using fuzzy numbers from the literature which are relevant for a wide range of numerical studies.

Example 4.1. Considering a pair of two triangular fuzzy numbers $A_1 = (1,4,5)$ and $A_2 = (2,3,6)$, taken from Nguyen [13]. The two fuzzy numbers are congruent and depict the compensation of areas as visualized in Figure 1, with their respective images $A_1' = (-5,-4,-1)$ and $A_2' = (-6,-3,-2)$ are displayed on the left of the membership axis. It is not an easy task for intuition to distinguish these two fuzzy numbers and their respective images due to flipping and sliding. Using formulae in eq. (3.3), the ranking score of both fuzzy numbers is evaluated and displayed in Table 1. The ordering results of the proposed method is $A_2' \prec A_1' \prec A_1 \prec A_2$, consistent with the representative approaches of Yu and Dat [22] and Nguyen [13]. Abbasbandy and Hajjari [2] are inconsistent with the proposed approach and yield the ranking results $A_1' \prec A_2' \prec A_2 \prec A_1$. However, the approaches of Abbasbandy and Asady [1], Asady and Zendehnam [3], Nasseri *et al*. [11], and Patra [14] failed to demonstrate any order. Hence the proposed approach is capable of ranking the fuzzy numbers and their respective images in a difficult situation for intuition.

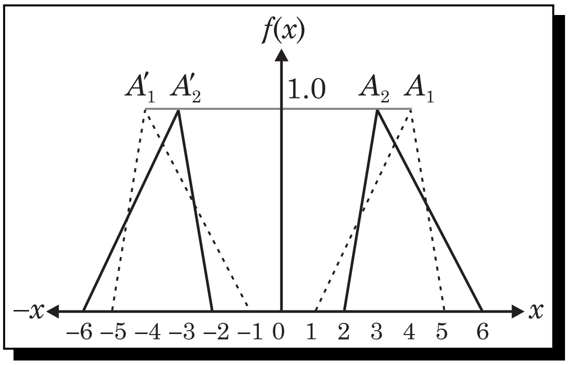

Figure 2. Visual representation of fuzzy numbers of Example 4.1 and their associated images

Table 1. Comparative ranking results of the fuzzy numbers in Example 4.1

Author	Ranking value of the fuzzy number				Ranking order
	A_1	A_2	A_1'	A_2'	
Abbasbandy and Asady [1] $P=1$	7.00	7.00	−7.00	−7.00	$A_1' \sim A_2' \prec A_2 \sim A_1$
$P=2$	5.2281	5.2281	−5.2281	−5.2281	$A_1' \sim A_2' \prec A_2 \sim A_1$
Asady and Zendehnam [3]	3.50	3.50	−3.50	−3.50	$A_1' \sim A_2' \prec A_2 \sim A_1$
Abbasbandy and Hajjari [2]	3.8334	3.1667	−3.8334	−3.1667	$A_1' \prec A_2' \prec A_2 \prec A_1$
Nasseri *et al*. [11]	6.7764	6.7764	−7.2237	−7.2237	$A_2' \sim A_1' \prec A_1 \sim A_2$
Yu and Dat [22] (*Me*)	3.4495	3.5506	−3.4495	−3.5506	$A_2' \prec A_1' \prec A_1 \prec A_2$
Nguyen [13] ($\lambda = 0.5$)	11.667	12.833	−11.667	−12.833	$A_2' \prec A_1' \prec A_1 \prec A_2$
Patra [14]	3.50	3.50	−3.50	−3.50	$A_1' \sim A_2' \prec A_2 \sim A_1$
Proposed Method	3.6839	4.0156	−3.0161	−3.3480	$A_2' \prec A_1' \prec A_1 \prec A_2$

Example 4.2. Consider the following two symmetrical triangular fuzzy numbers and three trapezoidal fuzzy numbers with different degrees of representative location on the horizontal axis, given (Wang and Lee [19]) $A_1 = (3,5,5,7;1)$, $A_2 = (3,5,5,7;0.8)$, $B_1 = (5,7,9,10;1)$, $B_2 = (6,7,9,10;0.6)$, $B_3 = (7,8,9,10;0.4)$. Figure 3. presents the visual interpretation of the membership functions of these fuzzy numbers. From Figure 3, it is observed that the two fuzzy numbers A_1 and A_2 are symmetrical about the line $x = 5$ and have the same support

but different weights. Therefore, based on the weight of the fuzzy numbers, the intuitive preference will be $A_1 \succ A_2$. Figure 3 demonstrates that the fuzzy numbers B_1, B_2 and B_3 are different degrees of representative location on the horizontal axis with different altitudes. Hence, based on the degrees of representative location on the horizontal axis, their logical order will be $B_1 \prec B_2 \prec B_3$. Using formulae in eq. (3), the ranking scores of these fuzzy numbers are evaluated and placed in Table 2. From Table 2, the ranking results for the two symmetrical fuzzy numbers A_1 and A_2 by the proposed method are the same as the intuitive outcome. The other three methods Wang and Lee [19], Nasseri *et al*. [11], and Patra [14] demonstrated the same ranking result, whereas, Abbasbandy and Asady [1], Asady and Zendehnam [3] and Abbasbandy *et al*. [2] have failed to discriminate any order and yields $A_1 \sim A_2$. Also, from Table 2, the ranking outcome for the trapezoidal fuzzy numbers by the proposed method is the same as the intuitive results. The other three methods, Wang and Lee [19], Abbasbandy and Asady [1], Asady and Zendehnam [3], Abbasbandy *et al*. [2], and Nasseri [11] are congruent with the proposed method. Patra's method [14] is inconsistent and yields $B_1 \succ B_2 \succ B_3$.

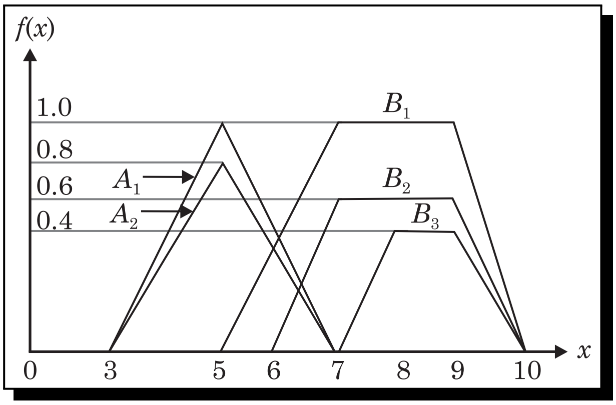

Figure 3. Visual representation of the fuzzy numbers of Example 4.2

Table 2. The comparative ranking order of the fuzzy numbers of Example 4.2

Author	Ranking value of the fuzzy number					Ranking order
	A_1	A_2	B_1	B_2	B_3	
Abbasbandy and Asady [1] $P = 1$	10.00	10.00	15.50	16.00	17.00	$A_1 \sim A_2; B_1 \prec B_2 \prec B_3$
$P = 2$	7.26	7.26	11.26	11.52	12.11	$A_1 \sim A_2; B_1 \prec B_2 \prec B_3$
Asady and Zendehnam [3]	5.00	5.00	7.75	8.00	8.50	$A_1 \sim A_2; B_1 \prec B_2 \prec B_3$
Abbasbandy *et al*. [2]	5.00	5.00	7.917	8.00	8.50	$A_1 \sim A_2; B_1 \prec B_2 \prec B_3$
Wang and Lee [19]	0.50	0.40	7.714	8.00	8.5	$A_1 \succ A_2; B_1 \prec B_2 \prec B_3$
Nasseri *et al*. [11]	9.70	9.66	15.34	15.84	16.80	$A_1 \succ A_2; B_1 \prec B_2 \prec B_3$
Patra [14]	5.00	4.90	7.75	5.36	2.81	$A_1 \succ A_2; B_1 \succ B_2 \succ B_3$
Proposed Method	5.00	4.00	7.70	8.00	8.50	$A_1 \succ A_2; B_1 \prec B_2 \prec B_3$

Example 4.3. Consider the following five trapezoidal fuzzy numbers with different degrees of representative locations on the horizontal axis and of different heights, followed from Nasseri [11]. $A_1 = (1,3,4,6;0.6)$, $A_2 = (1,3,4,6;0.9)$, $A_3 = (6,8,9,11;1)$, $A_4 = (5,9,12,15;0.2)$, $A_5 = (5,9,12,15;0.7)$. Figure 3. presents the visual understanding of the membership functions of these fuzzy numbers. Based on the degree of representative locations on the horizontal axis, the logical ranking of these fuzzy numbers is $A_1 \prec A_2 \prec A_3 \prec A_4 \prec A_5$. Using the formula in eq. (3.3), the ranking scores of these fuzzy numbers are evaluated as $S(A_1) = 3.5082$, $S(A_2) = 3.5123$, $S(A_3) = 8.5331$, $S(A_4) = 10.2135$, and $S(A_5) = 10.5123$ which yields the ranking results of the proposed method the same as the intuitive outcome. The following Table 3 shows the result of the proposed method and some convenient methods for ordering the above fuzzy numbers.

Table 3. The comparative ranking order of the fuzzy numbers of Example 4.3

Author	Ranking Results
Cheng [4]	$A_1 \prec A_2 \prec A_3 \prec A_4 \prec A_5$
Nasseri [11]	$A_1 \prec A_2 \prec A_3 \prec A_4 \prec A_5$
Nguyen [13] for moderate choice	$A_1 \prec A_2 \prec A_4 \prec A_3 \prec A_5$
Patra [14]	$A_2 \prec A_1 \prec A_3 \prec A_5 \prec A_4$
Proposed Method	$A_1 \prec A_2 \prec A_3 \prec A_4 \prec A_5$

To sum up, the proposed approach is convenient, reasonable, and effective for ranking fuzzy numbers with different altitudes.

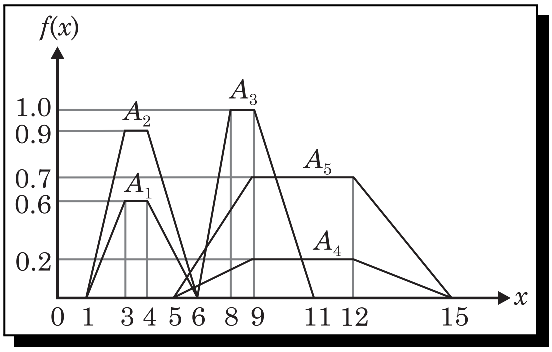

Figure 4. Visual interpretation of the membership functions of the fuzzy numbers of Example 4.3

Example 4.4. Considering intuitively, a triangular fuzzy number $A_1 = (1,5,5,7;1)$ overlapped with a trapezoidal fuzzy number $A_2 = (1,3,5,9;1)$. The visual representation of the two fuzzy numbers is shown in Figure 3. Their partnered image $A'_1 = (-7,-5,-5,-1;1)$ and $A'_2 = (-9,-5,-3,-1;1)$ is leftward of the membership axis. The intuition is not clear to distinguish these fuzzy numbers. Using formulae in eq. (3.3), the ranking score of these two fuzzy numbers and their images are obtained and shown in Table 4.

Table 4. Comparative ranking results of the fuzzy numbers of Example 4.4

Author	Ranking value of the fuzzy number				Ranking order
	A_1	A_2	A_1'	A_2'	
Abbasbandy and Asady [1] $P = 1$	9.00	9.00	−9.00	−9.00	$A_1' \sim A_2' \prec A_2 \sim A_1$
$P = 2$	6.83	7.39	−6.83	−7.39	$A_2' \prec A_1' \prec A_1 \prec A_2$
Asady and Zendehnam [3]	4.50	4.50	−4.50	−4.50	$A_1' \sim A_2' \prec A_2 \sim A_1$
Abbasbandy and Hajjari [2]	4.83	4.17	−4.83	−4.17	$A_1' \prec A_2' \prec A_2 \prec A_1$
Nasseri *et al.* [12]	8.62	8.62	−9.38	−9.38	$A_2' \sim A_1' \prec A_1 \sim A_2$
Chutia and Chutia [7], $\alpha = 0.5$	3.58	3.17	−3.58	−3.17	$A_1' \prec A_2' \prec A_2 \prec A_1$
K. Patra [14]	2.04	4.50	−2.04	−4.50	$A_2' \prec A_1' \prec A_1 \prec A_2$
Proposed Method	4.34	4.61	−4.34	−4.61	$A_2' \prec A_1' \prec A_1 \prec A_2$

The ranking scores demonstrate the ranking results as $A_2' \prec A_1' \prec A_1 \prec A_2$, consistent with the ranking results of Abbasbandy *et al.* [1] for $p = 2$, and Patra [14]. Abbasbandy *et al.* [2] and Chutia *et al.* [6] distinguish these fuzzy numbers differently and yield $A_1' \prec A_2' \prec A_2 \prec A_1$. However, Asady *et al.* [3], and Nasseri *et al.* [11] failed to demonstrate any preference and yield results $A_1' \sim A_2' \prec A_2 \sim A_1$. Hence, the proposed method can rank fuzzy numbers and their images in a blurred situation for intuition.

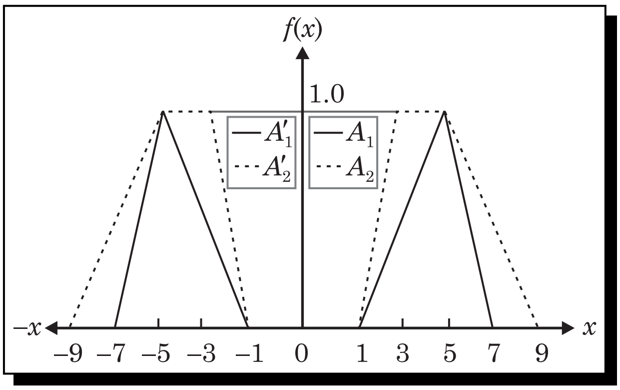

Figure 5. Visualization of fuzzy quantities and their partnered images of Example 4.4

Example 4.5. Taken from Nguyen [13], a triangular fuzzy number $A_1 = (1,2,2,5;1)$ and a general fuzzy number $A_2 = (1,2,2,4;1)$ with non-linear membership function $f_{A_2}(x)$, given by:

$$f_{A_2}(x) = \begin{cases} f_{A_2}^L(x) = \sqrt{1-(x-2)^2}, & 1 \le x \le 2, \\ f_{A_2}^R(x) = \sqrt{1-\frac{1}{4}(x-2)^2}, & 2 \le x \le 4, \\ 0, & \text{otherwise} \end{cases}$$

The visual interpretation of their membership functions is displayed in Figure 6. Human intuition realizes on $A_1 \succ A_2$ $(A_1' \prec A_2')$ based on the right spread of the fuzzy numbers.

For the non-linear fuzzy number $A_2=(1,2,2,4;1)$, using eq. (3.1) and eq. (3.2), signify that:

$$\overline{x}_{A_2}=\frac{\int_1^2(x\sqrt{1-(x-2)^2})\,dx+\int_2^4(x\sqrt{1-\frac{1}{4}(x-2)^2})\,dx}{\int_1^2(\sqrt{1-(x-2)^2})\,dx+\int_2^4(\sqrt{1-\frac{1}{4}(x-2)^2})\,dx}=2.4244,$$

$$\overline{y}_{A_2}=\frac{\int_0^1 y(2+2\sqrt{1-y^2})\,dy-\int_0^1 y(2-\sqrt{1-y^2})\,dy}{\int_0^1(2+2\sqrt{1-y^2})\,dy-\int_0^1(2-\sqrt{1-y^2})\,dy}=0.4244.$$

Using these values of $\overline{x}_{A_2}$ and $\overline{y}_{A_2}$ in eq. (3.3), the ranking score $S(A_2)$ for the fuzzy number A_2 is obtained and displayed in Table 5. Based on eq. (3), the ranking outcome of the proposed approach is found as $A_2 \prec A_1$ ($A'_1 \prec A'_2$) which is in support of intuitive perception. Considering index approaches from the literature for comparing and validating the results of the proposed method.

From Table 5, we find that the ranking results of the proposed approach coincide with the neutral decision of Liou and Wang [10] and Nguyen [13]. The ranking result of Chutia *et al*. [6] for all values of the indicator of optimism in the interval $[0,1]$ is inconsistent with the proposed approach. Patra [14] also coincides with the proposed approach. Hence, the proposed method is also consistent in discriminating the fuzzy numbers with non-linear membership functions.

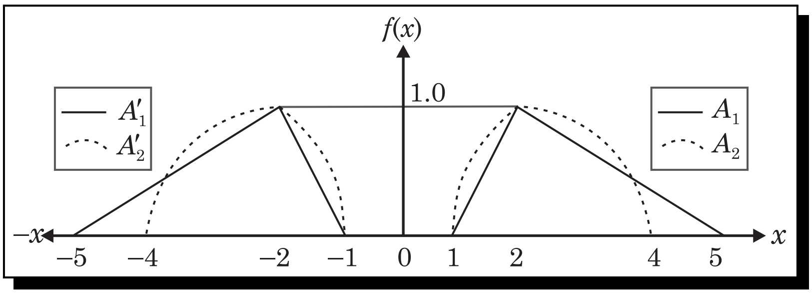

Figure 6. Visual interpretation of the fuzzy numbers and their images of Example 4.5

Table 5. Comparative ranking results of the fuzzy numbers in Example 4.5

Author	Ranking value of the fuzzy number				Ranking order
	A_1	A_2	A'_1	A'_2	
Liou and Wang [10] ($\alpha=0.5$)	2.50	2.40	−2.50	−2.40	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Nguyen [13] ($\alpha=0.5$)	6.67	5.80	−6.67	−5.80	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Chutia and Chutia [7] ($\alpha=0.5$)	1.667	1.7165	−1.667	−1.7165	$A'_2 \prec A'_1 \prec A_1 \prec A_2$
Patra [14]	2.122	1.835	−2.122	−1.835	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Proposed Method	3.0217	2.8882	−2.3533	−2.051	$A'_1 \prec A'_2 \prec A_2 \prec A_1$

Example 4.6. Consider the two sets of crisp numbers which are considered by Nguyen [13]. The first set consists $A_1=(1,1,1,1;0.5)$ and $A_2=(1,1,1,1;1.0)$ and the second set consists $B_1=(0.1,0.1,0.1,0.1;0.8)$ and $B_2=(-0.1,-0.1,-0.1,-0.1;1)$. These crisp numbers can be visualized in Figure 7. Using eq. (3.3), the ranking scores for these crisp numbers are obtained

as $S(A_1) = 1.003$, $S(A_2) = 1.005$, $S(B_1) = 0.1004$ and $S(B_2) = -0.1005$. Using eq. (3), A_1 and A_2 are ranked as $A_1 \prec A_2$ whereas, B_1 and B_2 are ranked as $B_1 \succ B_2$. Rezvani [17], Chutia and Chutia [6], and Nguyen [13] all come up with the same ranking results, indicating that the proposed method may be utilized with crisp numbers as well.

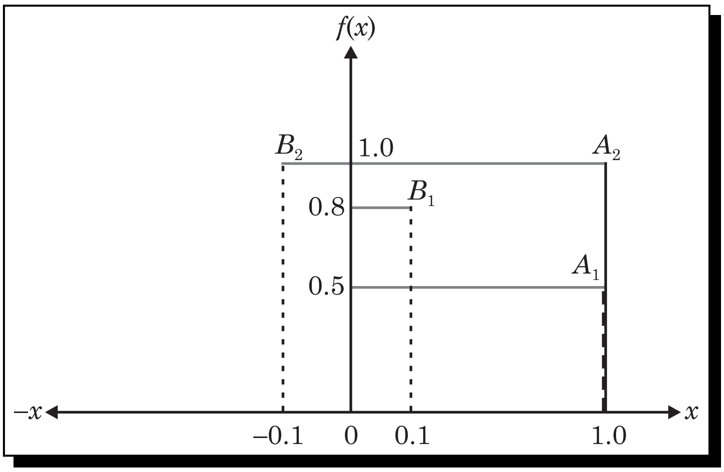

Figure 7. Visual representation of crisp numbers of Example 4.6

5. Conclusion

In this paper, a convenient and very effective ranking procedure for fuzzy numbers has been suggested. The proposed approach could not avoid the height of the fuzzy numbers but also has been taken into account together with the position on the horizontal axis. Numerical examples in this paper demonstrated that the suggested method yields logical ranking for symmetric fuzzy numbers and fuzzy numbers of different altitudes and with different degrees of representative location on the horizontal axis. Hence, the demonstrated ranking method is more balanced and robust in comparison to several other methods and may apply to the problems of risk analysis and decision-making in a fuzzy environment.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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