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Research Article

Cyclic Prime Graphicable Algebras

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Abstract. This study constitutes the continuation of innovative research in *Discrete Mathematics* introduced in earlier research on algebras, in general regarding the utilization of graphs to contemplate the specific instance of graphicable algebras, which form a subset of evolution algebras. Group theory, stochastic processes, and dynamical systems are all naturally linked to evolution algebras, which makes them extremely interesting. In light of a new line of research initiated previously by some of the authors, a depiction on primeness of a certain type of graphicable and subgraphicable algebra is given.

Keywords. Evolution algebra, Wheel graphicable algebra, Friendship graphicable algebra, Flower graphicable algebra, Jahangir graphicable algebra

Mathematics Subject Classification (2020). 05C78, 05C76, 17D99

1. Introduction

In this work, we discuss the relation between Tian graphicable algebras and graph labelings [7, 8]. These algebras are called evolution algebras. In particular, it is known as graphicable algebra. The main goal is to investigate the primeness of cyclic related graphicable algebras, in this we assigned a set of positive integers (known as weights) to $(c_{ik},\, k \in N(i))$ of $A(G)$.

Evolution algebra is a notion that bridges the gap between algebra and dynamical systems. Evolution algebras are non-associative Banach algebras on the algebraic system and they reflect discrete dynamical systems on the dynamical system. Graph theory, group theory, stochastic processes, mathematical physics and other fields of mathematics have many interconnections with evolution algebras. Evolution algebras do not belong to any of the well-known non-associative algebra classes, such as Lie, alternative, or Jordan algebras, because they are not defined by identities. These algebras are flexible and commutative, but they are not power associative. Evolution algebra's direct sum is also an evolution algebra. For more details on graphicable algebras (see [4,7,8],[1]).

Rosa [6], who created a labeling called β-valuation in 1967, is mostly significant for graph labelings. It is a crucial aspect of graph theory that assigns numerical values to vertices, edges or both subject to certain conditions. Social psychology, conflict resolution, energy cries, coding theory, the design of good radar location codes, missile guidance codes, synch-set codes and convolutional codes with optimal auto-correlation properties, x-ray crystallography, circuit design, radio astronomy and communication networks are linked with graph labelings. In recent years, numerous scholars have become interested in evolution algebras and graphs.

2. Preliminaries

This section is dedicated to recalling some basic concepts related to the topics discussed in this article. The reader might turn to [2,5],[1] for a more broad understanding of evolution algebras and graph theory, respectively.

Definition 2.1 ([1]). Let $A=(A,\cdot)$ be an algebra over a field K equipped with multiplication and let $S:=\{e_1,e_2,\dots,e_n,\dots\}$ be a basis of A. We say that A is an evolution algebra if

$$\left\{e_i\cdot e_i=\sum c_{ik}e_k \text{ for any } i \text{ and } e_i\cdot e_j=0 \text{ if } i\neq j\right\}. \tag{2.1}$$

The scalars $c_{ik}\in K$ are called the structure constants of A relative to S.

A basis S satisfying identity (2.1) *is called natural basis of A. If $K=R$ then A is real and is non negative if the structure constants c_{ik} are non negative.*

Tian present the concept of graphicable algebra in his book [7] as follows:

Definition 2.2. A commutative non associative algebra $G=(V,E)$, where V is the set of vertices and E the set of edges of G is called *graphicable algebra*, if it has the set of generators $V=\{e_1,e_2,e_3,\dots,e_r\}$ with the two defining relations:

$$R=\left\{\begin{array}{l} e_i^2=\sum_{e_k\in N(e_i)} e_k; \\ e_i\cdot e_j=0 \text{ if } i\neq j \text{ for } i,\ j=1,2,\dots,r \end{array}\right\},$$

where $N(e_i)$ is the neighborhood of e_i.

[1] J. Núñez, M. Silverio and M. T. Villar, Graph theory: a tool to study evolution algebras, *preprint*, (2012).

It is immediate to see that any graphicable algebra $A(G)$ is an evolution algebra, although the converse is not true in general.

In particular, if $e_{ij} = e_{ji}$, $i \neq j$ *and* $e_{ii} = 0$ *for every* $i \in A(G)$, *we call this algebra as an S-graphicable algebra*.

Definition 2.3. If the greatest common divisor of $(c_{ik}, k \in N(i)) = 1$ for $1 \le i \le n$, then the graphicable algebra $A(G)$ is a prime graphicable algebra.

In this paper, the structural constants c_{ik} represent the weights of edges.

3. Wheel Graphicable Algebra

The connection of a single universal vertex to each vertex of a cycle C_n $(n \ge 3)$ is wheel graph with $n+1$ vertices and $2n$ edges.

Our aim is to connect this family of W_n with a family of $A(W_n)$.

Definition 3.1. For $n \ge 3$, $A(W_n) \in W_n$ if $A(W_n)$ satisfies the following inequalities:

$$e_1^2 = e_2 + e_n + e_{n+1};$$
$$e_i^2 = e_{i-1} + e_{i+1} + e_{n+1}, \text{ for } 2 \le i \le n-1;$$
$$e_n^2 = e_{n-1} + e_{n+1} + e_1 \text{ and}$$
$$e_{n+1}^2 = \sum_{i=1}^{n} e_i.$$

Theorem 3.1. *The generalized wheel graphicable algebra* $A(W_n)$, *for* $n \ge 3$ *is prime.*

Proof. To show that all types of wheel graphicable algebras are prime, we start with minimum $n = 3$ and then generalize the case by induction. The graphicable algebra $A(W_3)$ associated to the graph W_3 has four generators viz., e_1, e_2, e_3 and e_4 which satisfies the conditions $e_1^2 = e_2 + 3e_3 + 4e_4$; $e_2^2 = e_1 + 2e_3 + 5e_4$; $e_3^2 = 2e_2 + 3e_1 + 6e_4$ and $e_4^2 = 4e_1 + 5e_2 + 6e_3$, and same is true for W_4 and the higher values of n.

We now define the function $f : \{c_{ik}, k \in N(i)\} \to \{1, 2, 3, \dots, 2n\}$ as

$$f(c_{12}) = 1;\ f(c_{1n}) = n;$$
$$f(c_{1(n+1)}) = n+1;$$
for $2 \le i \le n-1$
$$f(c_{i(i-1)}) = i-1;$$
$$f(c_{i(i+1)}) = i;$$
$$f(c_{i(n+1)}) = n+i;$$
$$f(c_{n(n-1)}) = n-1 \text{ and}$$
$$f(c_{n(n+1)}) = 2n.$$

Clearly, f is bijective and the following inequalities are obtained:

$$e_1^2 = e_2 + ne_n + (n+1)e_{n+1};$$
$$e_i^2 = (i-1)e_{i-1} + ie_{i+1} + (n+i)e_{n+1}, \text{ for } 2 \le i \le n-1;$$
$$e_n^2 = (n-1)e_{n-1} + 2ne_{n+1} + ne_1 \text{ and}$$
$$e_{n+1}^2 = \sum_{i=1}^{n}(n+i)e_i.$$

Thus, it is easy to check that the greatest common divisor of the coefficients of each term in the above result, i.e., the gcd of $(1,n+1,n),(i-1,i,n+i),(n-1,n,2n),(n+1,n+2,\dots,2n)$ are all equal to 1. Hence, the wheel graphicable algebra $A(W_n)$ is prime.

Figure 1 is wheel graphicable algebra $A(W_n)$ with n spokes of one cycle C_n.

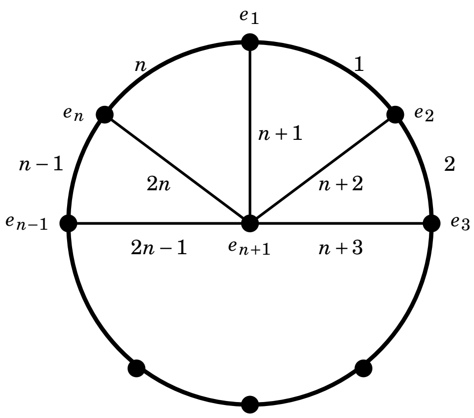

Figure 1. Wheel graphicable algebra $A(W_n)$

The subgraphicable algebra $A(H)$ is a graphicable algebra that corresponds to a subgraph H of G. This subgraphicable algebra is prime if the prime relations on $A(G)$ is also prime on the set of weights of structural constants of $A(H)$. One can see that the vertex deletion case of subgraphicable algebra by defining the vertex deleted graphicable algebra as follows:

Definition 3.2. Let $A(G)$ be the graphicable algebra that corresponds to a graph $G = (V, E)$, then for any vertex v of G, $A(G-v)$ is the vertex deleted subgraphicable algebra of $A(G)$ with vertex set $V - v$, whose edges are all those of G which are not incident with v.

Translating the above fact of vertex deletion into the language of graphicable algebras and from the weights assign in the Theorem 3.1 we obtain the following theorems:

Theorem 3.2. *$A(C_n)$, the cycle graphicable algebra is prime and Centre vertex deleted subgraphicable algebra of $A(W_n)$.*

Theorem 3.3. *A Fan graphicable $A(f_{(n-1)})$ is the nth vertex in the cycle of W_n deleted prime subgraphicable of $A(W_n)$.*

4. Friendship Graphicable Algebra

For ($n \geq 2$), a planar undirected graph with $2n+1$ vertices and $3n$ edges which is form by connecting n copies of the cycle graph C_3 with a common vertex (Centre vertex) is know as the friendship graph F_n (or dutch windmill graph) and is denoted by F_n. The graphicable algebra associated with this graph is defined as follows, using the same approach as before:

Definition 4.1. For $n \geq 2$, $A(F_n) \in F_n$ if $A(F_n)$ satisfies the following inequalities:

The relations of ith copy of C_3 in $A(F_n)$, for $1 \leq i \leq n$

$$e_{2i-1}^2 = e_{2i} + e_{2n+1} \text{ and } e_{2i}^2 = e_{2i-1} + e_{2n+1}.$$

and the relations on centre vertex is

$$e_{2n+1}^2 = \sum_{i=1}^{n} (e_{2i-1} + e_{2i}).$$

Theorem 4.1. *The friendship graphicable algebra $A(F_n)$ for $n \geq 2$ is prime.*

Proof. To prove that all kinds of friendship graphicable algebras are prime, we start with the simplest case of $n = 2$ and then generalize by induction. The graphicable algebra $A(F_2)$ has five generators viz. e_1, e_2, e_3, e_4 and e_5 that meets the required conditions $e_1^2 = 2e_2 + e_5$; $e_2^2 = 2e_1 + 3e_5$; $e_3^2 = 4e_5 + 5e_4$; $e_4^2 = 6e_5 + 5e_3$ and $e_5^2 = e_1 + 3e_2 + 4e_3 + 6e_4$. Same holds true for F_3 and so on.

As in the case of wheel graphicable algebras, we define the function $f : \{c_{ik}, k \in N(i)\} \to \{1, 2, 3, \dots, 3n\}$ as follows:

For $1 \leq i \leq n$

$$f(c_{(2i-1)(2i)}) = 3i - 1;$$

$$f(c_{(2i-1)(2n+1)}) = 3i - 2 \text{ and}$$

$$f(c_{(2i)(2n+1)}) = 3i.$$

Clearly, f is a bijective function and we obtain the following relations:

In ith copy C_3 of $A(F_n)$, for $1 \leq i \leq n$

$$e_{2i-1}^2 = (3i-1)e_{2i} + (3i-2)e_{2n+1} \text{ and}$$

$$e_{2i}^2 = (3i-1)e_{2i-1} + 3ie_{2n+1}.$$

At centre vertex

$$e_{2n+1}^2 = \sum_{i=1}^{n} ((3i-2)e_{2i-1} + 3ie_{2i}).$$

Similarly, one can easily check that the greatest common divisor of the coefficients of each term in the above result are all equal to 1. Thus, the friendship graphicable algebra is prime. Figure 2 is a friendship graphicable algebra $A(F_n)$ with n copies of C_3.

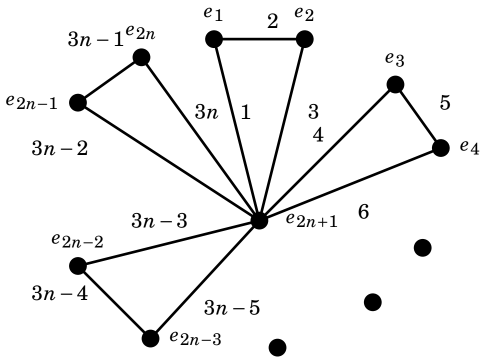

Figure 2. Friendship graphicable algebra $A(F_n)$

5. Flower Graphicable Algebra

A graph with $n(m-1)$ vertices and nm edges formed by an n-cycle and n sets of $m-2$ vertices forming m-cycles around the n-cycle, each m-cycle intersecting the n-cycle on a single edge is called an $n \times m$ flower graph and indicated by $f_{n\times m}$. The petals are the m-cycles, whereas the n-cycle is known as the center of $f_{n\times m}$. All of the n vertices that make up the center have a degree of 4, whereas the rest have a degree of 2.

Definition 5.1. For $n \geq 3$, $m \geq 3$, $A_{(f_{n\times m})} \in f(n\times m)$ if $A(f_{n\times m})$ with the basis $\{e_1, e_2, e_3, \dots, e_{n(m-1)}\}$ $(\bmod\, n(m-1))$ satisfies the following inequalities:

In the centre of $A(f_{n\times m})$;

$$e_1^2 = e_m + e_2 + e_{n(m-1)} + e_{(n-1)(m-1)+1} \text{ and}$$

$$e_{(t(m-1)+1)}^2 = e_{(m-1)(t-1)+1} + e_{t(m-1)+m} + e_{t(m-1)} + e_{t(m-1)+2}, \quad \text{for } 1 \leq t \leq n-1.$$

Now for $m-2$ generators in the tth petal of $A(f_{n\times m})$, for $1 \leq t \leq n$;

$$e_{(m(t-1)-t+i)}^2 = e_{(m-1)t-m+i-1} + e_{(m-1)t-m+i+1}, \quad \text{for } 1 \leq i \leq m-2.$$

Theorem 5.1. *The flower graphicable algebra $A(f_{n\times m})$ for $n \geq 3$, $m \geq 3$ is prime.*

Proof. As commented before, the graphicable algebra associated to the flower graph $f_{3\times 3}$ has 6 generators viz. e_1, e_2, e_3, e_4, e_5 and e_6 which satisfies the following conditions $e_1^2 = e_2 + 7e_3 + 9e_5 + 6e_6$; $e_2^2 = e_1 + 2e_3$; $e_3^2 = 7e_1 + 2e_2 + 3e_4 + 8e_5$; $e_4^2 = 4e_5 + 3e_3$; $e_5^2 = 8e_3 + 4e_4 + 5e_6 + 9e_1$ and $e_6^2 = 5e_5 + 6e_1$.

The same as applicable for $f_{3\times 4}$ and so on. As a consequence, we define the function $f : \{c_{ik}, k \in N(i)\} \to \{1, 2, 3, \dots, nm\}$ as follows:

$$f(c_{1m})) = n(m-1)+1;$$

$$f(c_{12})) = 1;$$

$$f(c_{1n(m-1)}) = n(m-1);$$

$$f(c_{1((n-1)(m-1)+1))}) = nm;$$

for $1 \le t \le n-1$

$$f(c_{(t(m-1)+1)((m-1)(t-1)+1)}) = n(m-1)+t;$$

$$f(c_{(t(m-1)+1)((m-1)t+m)}) = n(m-1)+t+1;$$

$$f(c_{(t(m-1)+1)(m-1)t}) = (m-1)t.$$

In tth petal of $A(f_{n\times m})$ where $1 \le t \le n$ and $3 \le i \le m$

$$f(c_{((m(t-1)-t+i)((m-1)t-m+i-1)}) = (m-1)t-m+i-1 \text{ and}$$

$$f(c_{((m(t-1)-t+i)(m(t-1)-t+i+1)}) = m(t-1)-t+i.$$

Clearly, f is a bijective function and we obtain the following relations:

In the centre of $A(f_{n\times m})$ are

$$e_1^2 = (n(m-1)+1)e_m + e_2 + n(m-1)e_{n(m-1)} + nme_{(n-1)(m-1)+1};$$

$$e^2_{(t(m-1)+1)} = (n(m-1)+t)e_{(m-1)(t-1)+1} + (n(m-1)+t+1)e_{t(m-1)+m} + t(m-1)e_{t(m-1)} + (t(m-1)+1)e_{t(m-1)+2}, \text{ for } 1 \le t \le n-1.$$

In tth petal of $A(f_{n\times m})$ where $1 \le t \le n$ is

$$e^2_{(m(t-1)-t+i)} = ((m-1)t-m+i-1)e_{(m-1)t-m+i-1} + (m(t-1)-t+i)e_{(m(t-1)-t+i+1)}, \text{ for } 3 \le i \le m.$$

Thus, one can note that the greatest common divisor in above relations of any two successive natural numbers is 1. Therefore, with the above weights, the flower graphicable algebra with n petals is prime.

Figure 3 is a flower graphicable algebra $A(f_{n\times m})$ with n petals.

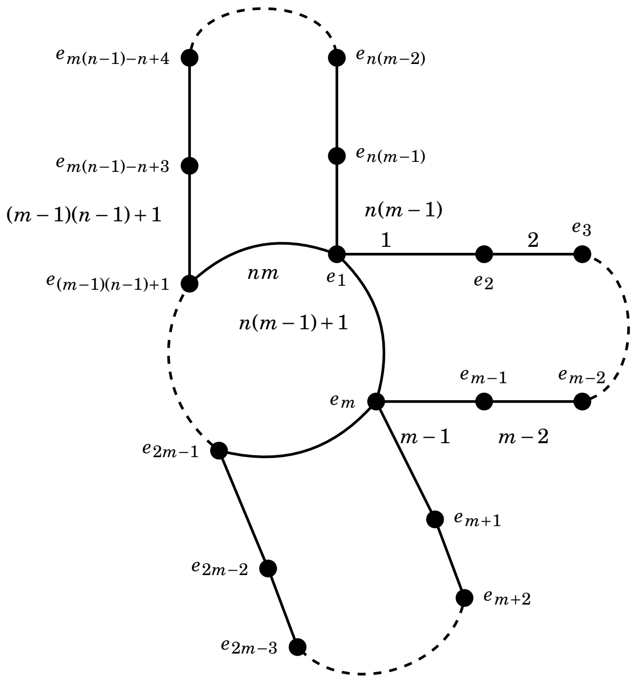

Figure 3. Flower graphicable algebra $A(f_{n\times m})$

Definition 5.2. Let $A(G)$ be the graphicable algebra that corresponds to a graph $G=(V,E)$, then for any edge e of G, $A(G-e)$ is the edge deleted subgraphicable algebra of $A(G)$ with vertex set V, and edge set $E-e$.

From Theorem 5.1 we obtained the following corollaries:

Corollary 5.1. *The deletion of any edge in the center of a flower graphicable algebra $A(f_{n\times m})$ results in an edge deleted flower subgraphicable algebra $A(H)$. Thus, one can see that there exists n-edge deleted prime subgraphicable algebras of $A(f_{n\times m})$.*

Corollary 5.2. *The deletion of n edges of centre cycle of a flower graphicable algebra $A(f_{n\times m})$ results in an edge deleted subgraphicable algebra $A(C_{(n-1)m})$, which is a cycle. Clearly, by the above weights the subgraphicable algebra $A(C_{(n-1)m})$ is also prime.*

6. Jahangir Graphicable Algebra

A Jahangir graph $J_{n,\,m}$ for $n\geq 2$, $m\geq 3$ is a graph consisting of a cycle C_{nm} with one additional vertex that is neighboring to m vertices of C_{nm} at a distance n on C_{nm}.

Definition 6.1. For $n\geq 2$, $m\geq 3$, $A(J_{n,\,m})\in J_{n,\,m}$ if $A(J_{n,m})$ satisfies the following inequalities:

$$e_1^2=e_2+e_{nm}+e_{nm+1};$$
$$e_{jn+i}^2=e_{jn+i-1}+e_{jn+i+1}(\text{mod } nm),\ \text{for } 2\leq i\leq n,\ 0\leq j\leq m-1;$$
$$e_{jn+1}^2=e_{jn}+e_{jn+2}+e_{nm+1},\ \text{for } 1\leq j\leq m-1 \text{ and}$$
$$e_{nm+1}^2=\sum_{j=0}^{m-1}e_{nj+1}.$$

Theorem 6.1. *The Jahangir graphicable algebra $A(J_{n,m})$ for $n\geq 2$, $m\geq 3$ is prime.*

Proof. As discuss earlier, the graphicable algebra associated to the Jahangir graph $J_{2,3}$ has 7 generators viz. e_1, e_2, e_3, e_4, e_5, e_6 and e_7 which satisfies the following relations; $e_1^2=e_2+7e_7+6e_6$; $e_2^2=e_1+2e_3$; $e_3^2=2e_2+3e_4+8e_7$; $e_4^2=4e_5+3e_3$; $e_5^2=4e_4+5e_6+9e_7$; $e_6^2=5e_5+6e_1$ and $e_7^2=7e_1+8e_3+9e_5$. The same is true for $J_{2,4}$ and so on.

Define the function $f:((c_{ik},k\in N(i))\to\{1,2,3,\dots,(n+1)m\}$ as follows:

$$f(c_{12})=1;$$
$$f(c_{1(nm)})=nm;$$
$$f(c_{1(nm+1)})=nm+1;$$

for $2\leq i\leq n$, $0\leq j\leq m-1$;

$$f(c_{(jn+i)(jn+i-1)})=jn+i-1;$$
$$f(c_{(jn+i)(jn+i+1)}=jn+i\ (\text{mod } nm)$$

for $1 \le j \le m-1$;

$$f(c_{(nm+1)(nj+1)}) = nm + j + 1;$$

$$f(c_{(jn)(nj+1)}) = jn \text{ and}$$

$$f(c_{(jn+2)(nj+1)})) = jn + 1.$$

We have the following relations:

$$e_1^2 = e_2 + nme_{nm} + (nm+1)e_{nm+1};$$

$$e_{jn+i}^2 = (jn+i-1)e_{jn+i-1} + (jn+i)e_{jn+i+1} (\bmod nm) \text{ for } 2 \le i \le n,\ 0 \le j \le m-1;$$

$$e_{jn+1}^2 = jne_{jn} + (jn+1)e_{jn+2} + (nm+j+1)e_{nm+1}, \ \text{ for } 1 \le j \le m-1 \text{ and}$$

$$e_{nm+1}^2 = \sum_{j=0}^{m-1} (nm+j+1)e_{nj+1}.$$

Thus, one can note that the greatest common divisor in above relations of any two successive natural numbers is 1. Therefore, with the above weights, the Jahangir graphicable algebra is prime.

Figure 4 is Jahangir graphicable algebra $A(J_{n,\,m})$ for $m+1$ cycles with m spokes.

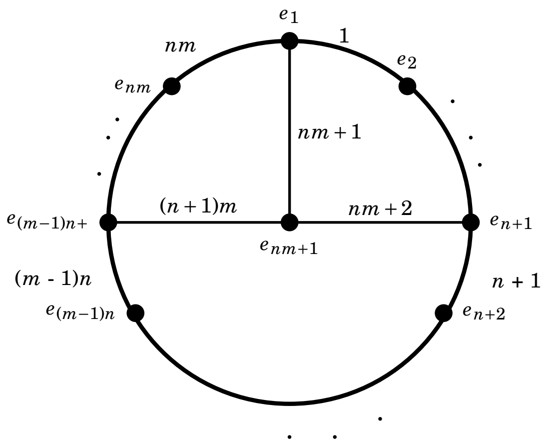

Figure 4. Jahangir graphicable algebra $A(J_{n,m})$

Remark 6.1. The cycle graphicable algebra $A(C_{nm})$ which is obtained by the deletion of centre vertex of a Jahangir graphicable algebra $A(J_{n,m})$ results, a vertex deleted prime subgraphicable algebra.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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