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Research Article

Rough Statistical Convergence of Double Sequences in Probabilistic Normed Spaces

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Abstract. In this paper, we have defined rough convergence and rough statistical convergence of double sequences in probabilistic normed spaces which is more generalized version than the rough statistical convergence of double sequences in normed linear spaces. Also, we have defined rough statistical cluster points of double sequences and then, investigated some important results associated with the set of rough statistical limits of double sequences in these spaces. Moreover, in the same spaces, we have proved an important relation between the set of all rough statistical cluster points and rough statistical limits under certain condition.

Keywords. Probabilistic normed space, Rough statistical convergence of double sequences, Rough statistical cluster points of double sequences

Mathematics Subject Classification (2020). 40A35, 40A05, 40A99, 26E50, 40G99

1. Introduction

In 1951, the concept of usual convergence of real sequences was extended to statistical convergence of real sequences based on the natural density of a set by Fast [9], and Steinhaus [30] independently. Later on, this idea has been studied in different directions and in different spaces by many authors e.g, Connor [6], Çakallı and Savaş [7], Nuray *et al*. [8], Fridy [11], Fridy and Orhan [12], Mursaleen [20], Mursaleen and Edely [21], Nuray and Savaş [25], Šalát [32], Sarabadan and Talebi [34] and references cited therein.

In 2001, Phu [27] has initially introduced the concept of rough convergence of sequences in finite dimensional normed linear spaces which is basically a generalization of usual convergence and, in the same paper he has investigated that r-limit set is bounded, closed, convex and many more interesting results and later on, this concept has been extended to infinite dimensional normed linear spaces (Phu [29]). Also, He [28] has defined the notion of rough continuity of linear operators. Later, Ayter [3] extended this notion to rough statistical convergence based on natural density of a set. Malik and Maity [23, 24] has defined rough convergence and rough statistical convergence of double sequences in normed linear spaces. After that, the research work on this concept is still being carried out in different directions by Antal *et al*. [4], Ghosal and Banerjee [13], Hossain and Banerjee [14], Kişi and Ünal [18], Özcan and Or [26] and references cited therein.

In 1942, Menger [19] first proposed the concept of statistical metric space, now called probabilistic metric space, which is an interesting and important generalization of the notion of metric space. This concept, later on, was studied by Schweizer and Sklar [33]. Combining the idea of statistical metric space and normed linear space, Šerstnev [31] introduced the idea of probabilistic normed space. In 1993, Alsina *et al*. [1] gave a new definition of probabilistic normed space whic is basically a special case of the definition of Šerstnev [31]. Recently, Antal *et al*. [5] defined the notion of rough convergence and rough statistical convergence in probabilistic normed spaces. In this space, we have presented the notion of rough statistical convergence of double sequences and investigated some interesting results associated with the sets of rough statistical cluster points and rough statistical limits of double sequences.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and set of reals respectively. First, we recall some basic definitions and notations.

Definition 2.1 ([33])**.** A triangular norm, briefly t-norm, is a binary operation on $[0,1]$ which is continuous, commutative, associative, non decreasing and has 1 as unit element, i.e., it is the continuous mapping $\diamond : [0,1]\times[0,1] \to [0,1]$ such that for all $a,b,c,d \in [0,1]$:

(i) $a \diamond 1 = a$;

(ii) $a \diamond b = b \diamond a$;

(iii) $a \diamond b \geq c \diamond d$ whenever $a \geq c$ and $b \geq d$;

(iv) $a \diamond (b \diamond c) = (a \diamond b) \diamond c$.

Example 2.1 ([15])**.** The following are the examples of t-norms:

(i) $x \diamond y = \min\{x, y\}$;

(ii) $x \diamond y = x \cdot y$;

(iii) $x \diamond y = \max\{x + y - 1, 0\}$. This t-norm is known as Lukasiewicz t-norm.

Definition 2.2 ([10])**.** A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is said to be a distribution function if it is non decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. We denote D as the set of all distribution functions.

Definition 2.3 ([10])**.** A triplet $(X,\vartheta,\diamond)$ is called a *Probabilistic Normed Space* (shortly *PNS*) if X is a real vector space, v is a mapping from X into D (for $x \in X$, $t \in (R)$, $\vartheta(x;t)$ is the value of the distribution function $\vartheta(x)$ at t) and $\diamond$ is a t-norm satisfying the following conditions:

(i) $\vartheta(x;0) = 0$;

(ii) $\vartheta(x;t) = 1$, for all $t > 0$ iff $x = \theta$, θ being the zero element of X;

(iii) $\vartheta(\alpha x;t) = \vartheta\big(x;\frac{t}{|\alpha|}\big)$, for all $\alpha \in \mathbb{R}\setminus\{0\}$ and for all $t > 0$;

(iv) $\vartheta(x+y;s+t) \geq \vartheta(x;t)\diamond\vartheta(y;s)$, for all $x,y \in X$, and for all $s,t \in \mathbb{R}_0^+$.

Example 2.2 ([2])**.** For a real normed space $(X,\|\cdot\|)$, we define the probabilistic norm ϑ for $x \in X$, $t \in \mathbb{R}$ as $\vartheta(x;t) = \frac{t}{t+\|x\|}$. Then $(X,\vartheta,\diamond)$ is a *PNS* under the t-norm $\diamond$ defined by $x\diamond y = \min\{x,y\}$. Also, $x_n \xrightarrow{\|\cdot\|} \xi$ if and only if $x_n \xrightarrow{\vartheta} \xi$.

Definition 2.4 ([2])**.** Let $(X,\vartheta,\diamond)$ be a *PNS*. For $r > 0$, the open ball $B(x,\lambda;r)$ with center $x \in X$ and radius $\lambda \in (0,1)$ is the set

$$B(x,\lambda;r) = \{y \in X : \vartheta(y-x;r) > 1-\lambda\}.$$

Similarly, the closed ball is the set $\overline{B(x,\lambda;r)} = \{y \in X : \vartheta(y-x;r) \geq 1-\lambda\}$.

Definition 2.5 ([17])**.** Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$. Then $\{x_{mn}\}$ is said to be convergent to $\xi \in X$ with respect to the probabilistic norm ϑ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer n_0 such that $\vartheta(x_{mn}-\xi;\varepsilon) > 1-\lambda$ whenever $m,n \geq n_0$. In this case, we write ϑ_2-$\lim x_{mn} = \xi$ or $x_{mn} \xrightarrow{\vartheta_2} \xi$.

Definition 2.6. Let $K \subset \mathbb{N}$. Then, the natural density $\delta(K)$ of K is defined by

$$\delta(K) = \lim_{n\to\infty}\frac{1}{n}|\{k \leq n : k \in K\}|,$$

provided the limit exists.

It is clear that if K is finite then $\delta(K) = 0$.

Definition 2.7 ([5])**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an $PNS(X,\vartheta,\diamond)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough convergent to $\xi \in X$ with respect to the probabilistic norm ϑ if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and some non negative number r there exists $n_0 \in \mathbb{N}$ such that $\vartheta(x_n-\xi;r+\varepsilon) > 1-\lambda$ for all $n > n_0$. In this case, we write r_ϑ-$\lim_{n\to\infty} x_n = \xi$ or $x_n \xrightarrow{r_\vartheta} \xi$ and ξ is called r_ϑ-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.8 ([5])**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an $PNS(X,\vartheta,\star)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough statistically convergent to $\xi \in X$ with respect to the probabilistic norm ϑ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ and some non negative number r, $\delta(\{n \in \mathbb{N} : \vartheta(x_n-\xi;r+\varepsilon) \leq 1-\lambda\}) = 0$. In this case, we write r-St_ϑ-$\lim_{n\to\infty} x_n = \xi$ or $x_n \xrightarrow{r\text{-}St_\vartheta} \xi$.

Definition 2.9 ([21])**.** The double natural density of the set $K \subseteq \mathbb{N}\times\mathbb{N}$ is defined by

$$\delta_2(K) = \lim_{m,n\to\infty}\frac{|\{(i,j) \in K : i \leq m \text{ and } j \leq n\}|}{mn},$$

where $|\{(i,j)\in K : i\le m \text{ and } j\le n\}|$ denotes the number of elements of K not exceeding m and n, respectively. It can be observed that if K is finite, then $\delta_2(K)=0$. Also, if $A\subseteq B$, then $\delta_2(A)\le\delta_2(B)$.

Definition 2.10 ([17]). Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$. Then $\{x_{mn}\}$ is said to be statistically convergent to $\xi\in X$ with respect to the probabilistic norm ϑ if for every $\varepsilon>0$ and $\lambda\in(0,1)$, $K=\{(m,n), m\le i, n\le j : \vartheta(x_{mn}-\xi;\varepsilon)\le 1-\lambda\}$ has double natural density zero, that is, if $K(i,j)$ become the numbers of (m,n) in K:

$$\lim_{i,j}\frac{K(i,j)}{ij}=0.$$

In this case, we write $st_2^{\vartheta}\text{-}\lim x_{mn}=\xi$ or $x_{mn}\xrightarrow{st_2^{\vartheta}}\xi$.

Definition 2.11 ([24]). A subsequence $x'=\{x_{j_pk_q}\}$ of a double sequence $\{x_{jk}\}$ is called a dense subsequence, if $\delta_2(\{(j_pk_q)\in\mathbb{N}\times\mathbb{N} : p,q\in\mathbb{N}\})=1$.

3. Main Results

First we define rough convergence and rough statistical convergence of double sequences in probabilistic normed spaces.

Definition 3.1. Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough convergent to $\beta\in X$ with respect to the probabilistic norm ϑ if for every $\varepsilon>0$, $\lambda\in(0,1)$ there exists $n_0\in\mathbb{N}$ such that $\vartheta(x_{mn}-\beta;r+\varepsilon)>1-\lambda$ for all $m,n\ge n_0$. In this case, β is called r_2^{ϑ}-limit of $\{x_{mn}\}$ and we write $x_{mn}\xrightarrow{r_2^{\vartheta}}\beta$.

Definition 3.2. Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough statistical convergent to $\beta\in X$ with respect to the probabilistic norm ϑ if for every $\varepsilon>0$, $\lambda\in(0,1)$, $\delta_2(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\beta;r+\varepsilon)\le 1-\lambda\})=0$. In this case, β is called $r\text{-}st_2^{\vartheta}$-limit of $\{x_{mn}\}$ and we write $r\text{-}st_2^{\vartheta}\text{-}\lim_{m,n\to\infty}x_{mn}=\beta$ or $x_{mn}\xrightarrow{r\text{-}st_2^{\vartheta}}\beta$.

Remark 3.1. (a) If we put $r=0$ in Definition 3.1, then the notion of rough convergence of a double sequence with respect to the probabilistic norm ϑ coincides with notion of ordinary convergence of the double sequence with respect to the probabilistic norm ϑ.

(b) From Definition 3.1, it is clear that r_2^{ϑ}-limit of a double sequence may not be unique. So, we denote $LIM_{x_{mn}}^{r\vartheta}$ to mean the set of all r_2^{ϑ}-limit of $\{x_{mn}\}$ with respect to the probabilistic norm ϑ.

(c) If we put $r=0$ in Definition 3.2, then the notion of rough statistical convergence of a double sequence with respect to the probabilistic norm ϑ coincides with statistical convergence of the double sequence with respect to the probabilistic norm ϑ. So, our whole discussion is on the fact $r>0$.

(d) From Definition 3.2, it is clear that r-st_2^{ϑ}-limit of a double sequence may not be unique. So, we denote st_2^{ϑ}-$LIM_{x_{mn}}^{r}$ to mean the set of all r-st_2^{ϑ}-limit of $\{x_{mn}\}$ with respect to the probabilistic norm ϑ.

The sequence $\{x_{mn}\}$ is said to be r_2^{ϑ}-convergent if $LIM_{x_{mn}}^{r_{\vartheta}} \neq \emptyset$. But, if the sequence is unbounded with respect to the probabilistic norm ϑ then $LIM_{x_{mn}}^{r_{\vartheta}} = \emptyset$ although in this case st_2^{ϑ}-$LIM_{x_{mn}}^{r} \neq \emptyset$ may be happened which has been shown in the following example.

Example 3.1. Let $(X,\|\cdot\|)$ be a real normed linear space and let $\vartheta(x;t) = \frac{t}{t+\|x\|}$ for $x \in X$ and $t > 0$. Then $(X,\vartheta,\diamond)$ is a *PNS* under the t-norm $\diamond$ defined by $x \diamond y = \min\{x,y\}$. For all $m,n \in \mathbb{N}$, we define a sequence $\{x_{mn}\}$ by

$$x_{mn} = \begin{cases} (-1)^{m+n}, & m,n \neq i^2 \ (i \in \mathbb{N}), \\ mn, & \text{otherwise.} \end{cases}$$

Then, we have

$$st_2^{\vartheta}\text{-}LIM_{x_{mn}}^{r} = \begin{cases} \emptyset, & r < 1, \\ [1-r, r-1], & \text{otherwise,} \end{cases}$$

and st_2^{ϑ}-$LIM_{x_{mn}}^{r} = \emptyset$ when $r = 0$. Also, $LIM_{x_{mn}}^{r_{\vartheta}} = \emptyset$ for any $r \geq 0$.

Remark 3.2. From Example 3.1, we have st_2^{ϑ}-$LIM_{x_{mn}}^{r} \neq \emptyset$ does not imply $LIM_{x_{mn}}^{r_{\vartheta}} \neq \emptyset$. But, $LIM_{x_{mn}}^{r_{\vartheta}} \neq \emptyset$ always implies $LIM_{x_{mn}}^{r_{\vartheta}} \neq \emptyset$ as $\delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} :$ either m or n runs over finite subsets of $\mathbb{N}\}) = 0$. So, $LIM_{x_{mn}}^{r_{\vartheta}} \subset st_2^{\vartheta}$-$LIM_{x_{mn}}^{r}$.

Example 3.2. We take the *PNS* in Example 3.1 and define the double sequence $\{x_{mn}\}$ by

$$x_{mn} = \begin{cases} mn, & m,n = i^2 \ (i \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Then, st_2^{ϑ}-$LIM_{x_{mn}}^{r} = [-r,r]$. Now, if we consider a subsequence $\{x_{m_j n_k}\}$ of $\{x_{mn}\}$ such that $m_j = j^2, n_k = k^2$, $j,k \in \mathbb{N}$, then st_2^{ϑ}-$LIM_{x_{m_j n_k}}^{r} = \emptyset$.

Remark 3.3. From Example 3.2, for any subsequence of a double sequence we do not conclude that st_2^{ϑ}-$LIM_{x_{mn}}^{r} \subseteq st_2^{\vartheta}$-$LIM_{x_{m_j n_k}}^{r}$.

But, this inclusion may be hold under certain condition which has been given in the following theorem.

Theorem 3.1. *Let $\{x_{m_j n_k}\}$ be a dense subsequence of $\{x_{mn}\}$ in a PNS$(X,\vartheta,\diamond)$. Then st_2^{ϑ}-$LIM_{x_{mn}}^{r} \subseteq st_2^{\vartheta}$-$LIM_{x_{m_j n_k}}^{r}$.*

Proof. The proof is obvious. Thus, we omit details. □

Definition 3.3. Let $\{x_{mn}\}$ be a double sequence in a *PNS*$(X,\vartheta,\diamond)$. Then $\{x_{mn}\}$ is said to be statistically bounded with respect to the probabilistic norm ϑ if for every $\lambda \in (0,1)$ there exists a positive real number G such that $\delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \vartheta(x_{mn};G) \leq 1-\lambda\}) = 0$.

Theorem 3.2. *Let $\{x_{mn}\}$ be a double sequence in a PNS$(X,\vartheta,\diamond)$. Then $\{x_{mn}\}$ is statistically bounded if and only if st_2^{ϑ}-$LIM^r_{x_{mn}} \neq \emptyset$ for some $r > 0$.*

Proof. Suppose that $\{x_{mn}\}$ is statistically bounded. Then, for every $\lambda \in (0,1)$ there exists a positive real number G such that $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn};G) \leq 1-\lambda\}) = 0$. Now, let $M = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn};G) \leq 1-\lambda\}$ and θ be the zero element in X. Now for $m,n \in M^c$ we have $\vartheta(x_{mn}-\theta;r+G) \geq \vartheta(x_{mn};G)\diamond\vartheta(\theta;r) > (1-\lambda)\diamond 1 = 1-\lambda$. This gives $\theta \in st_2^{\vartheta}$-$LIM^r_{x_{mn}}$ and consequently st_2^{ϑ}-$LIM^r_{x_{mn}} \neq \emptyset$.

Conversely, suppose that st_2^{ϑ}-$LIM^r_{x_{mn}} \neq \emptyset$. Let $\xi \in st_2^{\vartheta}$-$LIM^r_{x_{mn}} \neq \emptyset$. Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\delta_2(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\xi;r+\varepsilon) \leq 1-\lambda\}) = 0$. Therefore, almost all x_{mn} are contained in some ball with center ξ. This shows that $\{x_{mn}\}$ is statistically bounded. This completes the proof. □

Now, we give the algebraic characterization of rough statistically convergent double sequences in probabilistic normed spaces.

Theorem 3.3. *Let $\{x_{mn}\}$ and $\{y_{mn}\}$ be double sequences in a PNS$(X,\vartheta,\diamond)$. Then, for some $r > 0$ the following statements hold:*

(i) *If $x_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \beta$ and $y_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \eta$ then $x_{mn}+y_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \beta+\eta$.*

(ii) *If $x_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \beta$ and $\alpha(\neq 0) \in \mathbb{R}$ then $\alpha x_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \alpha\beta$.*

Proof. Let $\{x_{mn}\}$ and $\{y_{mn}\}$ be double sequences in PNS $(X,\vartheta,\diamond)$ and $r > 0$.

(i) Let $x_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \beta$ and $y_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \eta$. Let $\varepsilon > 0$. Now, for a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. So, $\delta_2(A) = 0$ and $\delta_2(B) = 0$ where $A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\beta;\frac{r+\varepsilon}{2}) \leq 1-s\}$ and $B = \{(m,n)\in\mathbb{N}\times\mathbb{N} : \vartheta(y_{mn}-\eta;\frac{r+\varepsilon}{2}) \leq 1-s\}$. Now for $(i,j) \in A^c \cap B^c$, we have $\vartheta(x_{ij}+y_{ij}-(\beta+\eta);r+\varepsilon) \geq \vartheta(x_{ij}-\beta;\frac{r+\varepsilon}{2})\diamond(y_{ij}-\eta;\frac{r+\varepsilon}{2}) > (1-s)\diamond(1-s) > 1-\lambda$, i.e. $A^c \cap B^c \subset \{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}+y_{ij}-(\beta+\eta);r+\varepsilon) > 1-\lambda\}$. Therefore, $\delta_2(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}+y_{ij}-(\beta+\eta);r+\varepsilon) \leq 1-\lambda\}) = 0$, which gives $x_{mn}+y_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \beta+\eta$.

(ii) Since $x_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \beta$ and $\alpha \neq 0$, then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\beta;\frac{r+\varepsilon}{|\alpha|}) \leq 1-\lambda\}) = 0$ i.e., $\delta_2(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \vartheta(\alpha x_{mn}-\alpha\beta;r+\varepsilon) \leq 1-\lambda\}) = 0$, which gives $\alpha x_{mn} \xrightarrow{r\text{-}st_2^{\vartheta}} \alpha\beta$. This completes the proof. □

We will discuss on some topological and geometrical properties of the set st_2^{ϑ}-$LIM^r_{x_{mn}}$.

Theorem 3.4. *Let $\{x_{mn}\}$ be a double sequence in a PNS$(X,\vartheta,\diamond)$. Then the set st_2^{ϑ}-$LIM^r_{x_{mn}}$ is closed.*

Proof. If st_2^{ϑ}-$LIM^r_{x_{mn}} = \emptyset$ then we have nothing to prove. So, let st_2^{ϑ}-$LIM^r_{x_{mn}} \neq \emptyset$. Suppose that $\{y_{mn}\}$ is a double sequence in st_2^{ϑ}-$LIM^r_{x_{mn}}$ such that $y_{mn} \xrightarrow{\vartheta_2} \beta$. For a given $\lambda \in (0,1)$ choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. Then, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that

$\vartheta(y_{mn}-\beta;\frac{\varepsilon}{2})>1-s$ for all $m,n>n_0$. Suppose $i,j>n_0$. Then $\vartheta(y_{ij}-\beta;\frac{\varepsilon}{2})>1-s$. Again, since $\{y_{ij}\}\in st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$, then $\delta_2(P)=0$ where $P=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{mn}-y_{ij};r+\frac{\varepsilon}{2})\le 1-s\}$. Now for $(s,t)\in P^c$, we have $\vartheta(x_{st}-\beta;r+\varepsilon)\ge\vartheta(x_{st}-y_{ij};r+\frac{\varepsilon}{2})\diamond\vartheta(y_{ij}-\beta;\frac{\varepsilon}{2})>(1-s)\diamond(1-s)>1-\lambda$. Therefore, $\{(s,t)\in(N)\times\mathbb{N}:\vartheta(x_{st}-\beta;r+\varepsilon)\le 1-\lambda\}\subset P$. Since $\delta_2(P)=0$, therefore $\delta_2(\{(s,t)\in(N)\times\mathbb{N}:\vartheta(x_{st}-\beta;r+\varepsilon)\le 1-\lambda\})=0$. Consequently $\beta\in st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$. So, $st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$ is closed. This completes the proof. □

Theorem 3.5. *Let $\{x_{mn}\}$ be a double sequence in a PNS$(X,\vartheta,\diamond)$. Then the set $st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$ is convex for some $r>0$.*

Proof. Let $\xi_1,\xi_2\in st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$ and $t\in(0,1)$. Suppose $\lambda\in(0,1)$. Choose $s\in(0,1)$ such that $(1-s)\diamond(1-s)>1-\lambda$. Then $\delta_2(A)=0$ and $\delta_2(B)=0$ where $A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{mn}-\xi_1;\frac{r+\varepsilon}{2(1-t)})\le 1-s\}$ and $B=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{mn}-\xi_2;\frac{r+\varepsilon}{2t})\le 1-s\}$. Now for $i,j\in A^c\cap B^c$ we have $\vartheta(x_{ij}-[(1-t)\xi_1+t\xi_2];r+\varepsilon)\ge\vartheta((1-t)(x_{ij}-\xi_1);\frac{r+\varepsilon}{2})\diamond\vartheta(t(x_{ij}-\xi_2);\frac{r+\varepsilon}{2})=\vartheta(x_{ij}-\xi_1;\frac{r+\varepsilon}{2(1-t)})\diamond\vartheta(x_{ij}-\xi_2;\frac{r+\varepsilon}{2t})>(1-s)\diamond(1-s)>1-\lambda$. Therefore, $\{(i,j)\in\mathbb{N}\times N:\vartheta(x_{ij}-[(1-t)\xi_1+t\xi_2];r+\varepsilon)\le 1-\lambda\}\subset A\cup B$. Hence $\delta_2(\{(i,j)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{ij}-[(1-t)\xi_1+t\xi_2];r+\varepsilon)\le 1-\lambda\})=0$, i.e. $(1-t)\xi_1+t\xi_2\in st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$. Therefore, $st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$ is convex. This completes the proof. □

Theorem 3.6. *A double sequence $\{x_{mn}\}$ in a PNS$(X,\vartheta,\diamond)$ is rough statistically convergent to $\xi\in X$ with respect to the probabilistic norm ϑ for some $r>0$ if there exists a double sequence $\{y_{mn}\}$ in X such that $st_2^{\vartheta}\text{-}\lim y_{mn}=\xi$ and for every $\lambda\in(0,1)$, $\vartheta(x_{mn}-y_{mn};r)>1-\lambda$ for all $m,n\in\mathbb{N}$.*

Proof. Let $\varepsilon>0$ be given. For a given $\lambda\in(0,1)$, choose $s\in(0,1)$ such that $(1-s)\diamond(1-s)>1-\lambda$. Suppose that $st_2^{\vartheta}\text{-}\lim y_{mn}=\xi$ and $\vartheta(x_{mn}-y_{mn};r)>1-\lambda$ for all $m,n\in\mathbb{N}$. Then $\delta_2(A)=0$ where $A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(y_{mn}-\xi;\varepsilon)\le 1-s\}$. Now for $(i,j)\in A^c$, we have $\vartheta(x_{ij}-\xi;r+\varepsilon)\ge\vartheta(x_{ij}-y_{ij};r)\diamond\vartheta(y_{ij}-\xi;r)>(1-s)\diamond(1-s)>1-\lambda$. Therefore, $\{(i,j)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{ij}-\xi;r+\varepsilon)\le 1-\lambda\}\subset A$. Hence $\delta_2(\{(i,j)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{ij}-\xi;r+\varepsilon)\le 1-\lambda\})=0$. Consequently, $x_{mn}\xrightarrow{r\text{-}st_2^{\vartheta}}\xi$. This completes the proof. □

Theorem 3.7. *Let $\{x_{mn}\}$ be a double sequence in a PNS$(X,\vartheta,\diamond)$. Then there do not exist $x_1,x_2\in st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$ for some $r>0$ and every $\lambda\in(0,1)$ such that $\vartheta(x_1-x_2;mr)\le 1-\lambda$ for $m(\in\mathbb{R})>2$.*

Proof. On contrary, we assume that there exist $x_1,x_2\in st_2^{\vartheta}\text{-}LIM_{x_{mn}}^r$ for which $\vartheta(x_1-x_2;mr)\le 1-\lambda$ for $m(\in\mathbb{R})>2$. Let $\varepsilon>0$ be given. For a given $\lambda\in(0,1)$, choose $s\in(0,1)$ such that $(1-s)\diamond(1-s)>1-\lambda$. Then $P=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{mn}-x_1;r+\frac{\varepsilon}{2})\le 1-s\}$ and $Q=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(x_{mn}-x_2;r+\frac{\varepsilon}{2})\le 1-s\}$ have double natural density zero. Now for $(i,j)\in P^c\cap Q^c$, we have $\vartheta(x_1-x_2;2r+\varepsilon)\ge\vartheta(x_{ij}-x_1;r+\frac{\varepsilon}{2})\diamond\vartheta(x_{ij}-x_2;r+\frac{\varepsilon}{2})>(1-s)\diamond(1-s)>1-\lambda$. Therefore,

$$\vartheta(x_1-x_2;2r+\varepsilon)>1-\lambda. \tag{3.1}$$

Now, if we choose $\varepsilon=mr-2r$, $m>2$, in eq. (3.1) then we have $\vartheta(x_1-x_2;mr)>1-\lambda$, $m>2$, which is a contradiction. This completes the proof. □

Definition 3.4 (c.f. [16])**.** Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$. Then a point $\xi \in X$ is said to be statistical cluster point of $\{x_{mn}\}$ with respect to the probabilistic norm ϑ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\xi;\varepsilon) > 1-\lambda\}) > 0$.

We denote $\Lambda_{(x_{mn})}(st_2^{\vartheta})$ to mean ordinary statistical cluster points of $\{x_{mn}\}$ with respect to the probabilistic norm ϑ.

Definition 3.5. Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$. Then a point $\xi \in X$ is said to be rough statistical cluster point of $\{x_{mn}\}$ with respect to the probabilistic norm ϑ if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and some $r > 0$, $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\xi;r+\varepsilon) > 1-\lambda\}) > 0$. The set of all rough statistical cluster points of $\{x_{mn}\}$ is denoted as $\Lambda^r_{(x_{mn})}(st_2^{\vartheta})$.

Remark 3.4. If $r=0$, then $\Lambda^r_{(x_{mn})}(st_2^{\vartheta}) = \Lambda_{(x_{mn})}(st_2^{\vartheta})$.

Theorem 3.8. *Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$. Then, the set $\Lambda^r_{(x_{mn})}(st_2^{\vartheta})$ is closed for some $r > 0$.*

Proof. If $\Lambda^r_{(x_{mn})}(st_2^{\vartheta}) = \emptyset$ then we have nothing to prove. So, let $\Lambda^r_{(x_{mn})}(st_2^{\vartheta}) \neq \emptyset$. Suppose that $\{\omega_{mn}\}$ is a double sequence in $\Lambda^r_{(x_{mn})}(st_2^{\vartheta})$ such that $\omega_{mn} \xrightarrow{\vartheta_2} \zeta$. Now for given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. Then, for every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\vartheta(\omega_{mn}-\zeta;\frac{\varepsilon}{2}) > 1-s$ for all $m,n \geq k_\varepsilon$. Now choose $m_0,n_0 \in \mathbb{N}$ such that $m_0,n_0 > k_\varepsilon$. Then $\vartheta(\omega_{m_0n_0}-\zeta;\frac{\varepsilon}{2}) > 1-s$. Again, since $\{\omega_{m_0n_0}\} \in \Lambda^r_{(x_{mn})}(st_2^{\vartheta})$, then $\delta_2(K) > 0$ where $K = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\omega_{m_0n_0};r+\frac{\varepsilon}{2}) > 1-s\}$. Now, for $(i,j) \in K$, we have $\vartheta(x_{ij}-\zeta;r+\varepsilon) \geq \vartheta(x_{ij}-\omega_{m_0n_0};r+\frac{\varepsilon}{2}) \diamond \vartheta(\omega_{m_0n_0}-\zeta;\frac{\varepsilon}{2}) > (1-s)\diamond(1-s) > 1-\lambda$. Therefore, $\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}-\zeta;r+\varepsilon) > 1-\lambda\} \subset K$. Since $\delta_2(K) > 0$, $\delta_2(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}-\zeta;r+\varepsilon) > 1-\lambda\}) > 0$. So, $\zeta \in \Lambda^r_{(x_{mn})}(st_2^{\vartheta})$. Hence $\Lambda^r_{(x_{mn})}(st_2^{\vartheta})$ is closed. □

Theorem 3.9. *Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$ and $r > 0$. Then, for an arbitrary $\zeta \in \Lambda_{(x_{mn})}(st_2^{\vartheta})$ and $\lambda \in (0,1)$, $\vartheta(\gamma-\zeta;r) > 1-\lambda$ for all $\gamma \in \Lambda^r_{(x_{mn})}(st_2^{\vartheta})$.*

Proof. For given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. Since $\zeta \in \Lambda_{(x_{mn})}(st_2^{\vartheta})$, then for every $\varepsilon > 0$, $\delta_2(M) > 0$ where $M = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\zeta;\varepsilon) > 1-s\}$. Now, we will show that if $\gamma \in X$ satisfying $\vartheta(\gamma-\zeta;r) > 1-s$ then $\gamma \in \Lambda^r_{(x_{mn})}(st_2^{\vartheta})$. Now, for $(i,j) \in M$ we have $\vartheta(x_{ij}-\gamma;r+\varepsilon) \geq \vartheta(x_{ij}-\zeta;\varepsilon) \diamond \vartheta(\gamma-\zeta;r) > (1-s)\diamond(1-s) > 1-\lambda$. Therefore, $M \subset \{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}-\gamma;r+\varepsilon) > 1-\lambda\}$. Hence $\delta_2(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}-\gamma;r+\varepsilon) > 1-\lambda\}) > 0$, Consequently, $\gamma \in \Lambda^r_{(x_{mn})}(st_2^{\vartheta})$. This completes the proof. □

Theorem 3.10. *Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$ and $r > 0$. Then, for $\lambda \in (0,1)$ and fixed $x_0 \in X$, $\Lambda^r_{(x_{mn})}(st_2^{\vartheta}) = \bigcup_{x_0 \in \Lambda_{(x_{mn})}(st_2^{\vartheta})} \overline{B(x_0,\lambda,r)}$.*

Proof. For $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. Let $\gamma \in \bigcup_{x_0 \in \Lambda_{(x_{mn})}(st_2^{\vartheta})} \overline{B(x_0,\lambda,r)}$. Then, there exists a $x_0 \in \Lambda_{(x_{mn})}$ such that $\vartheta(x_0-\gamma;r) > 1-s$. Since $x_0 \in \Lambda_{(x_{mn})}$, then for every

$\varepsilon > 0$, $\delta_2(Z) > 0$ where $Z = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn} - x_0;\varepsilon) > 1-s\}$. Now for $(i,j) \in Z$, we have $\vartheta(x_{ij} - \gamma; r+\varepsilon) \geq \vartheta(x_{ij} - x_0;\varepsilon) \diamond \vartheta(x_0 - \gamma; r) > (1-s)\diamond(1-s) > 1-\lambda$. Therefore, $Z \subset \{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij} - \gamma; r+\varepsilon) > 1-\lambda\}$. Since $\delta_2(Z) > 0$, $\delta_2(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij} - \gamma; r+\varepsilon) > 1-\lambda\}) > 0$. Hence $\gamma \in \Lambda^r_{(x_{mn})}(st_2^\vartheta)$ and so, $\bigcup_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)} \subset \Lambda^r_{(x_{mn})}(st_2^\vartheta)$.

Conversely, suppose that $\gamma \in \Lambda^r_{(x_{mn})}(st_2^\vartheta)$. Now, we show that $\gamma \in \bigcup_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)}$. If possible, let $\gamma \notin \bigcup_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)}$. Then for every $x_0 \in \Lambda_{(x_{mn})}(st_2^\vartheta)$, $\vartheta(\gamma - x_0; r) < 1-\lambda$, which contradicts the fact of Theorem 3.9. Hence $\gamma \in \bigcup_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)}$. Therefore, $\Lambda^r_{(x_{mn})}(st_2^\vartheta) \subset \bigcup_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)}$. This completes the proof. □

Theorem 3.11. *Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$. Then, for some $r > 0$ and any $\lambda \in (0,1)$, the following statements hold:*

(i) *If $x_0 \in \Lambda_{(x_{mn})}(st_2^\vartheta)$, then st_2^ϑ-$LIM^r_{x_{mn}} \subseteq \overline{B(x_0,\lambda,r)}$.*

(ii) st_2^ϑ-$LIM^r_{x_{mn}} = \bigcap_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)} = \{y_0 \in X : \Lambda_{(x_{mn})}(st_2^\vartheta) \subseteq \overline{B(y_0,\lambda,r)}\}$

Proof. Suppose $\{x_{mn}\}$ is a double sequence in a $PNS(X,\vartheta,\diamond)$ and $r > 0$.

(i) Now, for a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. Let $\beta \in st_2^\vartheta$-$LIM^r_{x_{mn}}$. Then, for every $\varepsilon > 0$, we have $\delta_2(K_1) = 0$ and $\delta_2(k_2) > 0$ where $K_1 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn} - \beta; r+\varepsilon) \leq 1-s\}$ and $K_2 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn} - x_0;\varepsilon) > 1-s\}$. Now for $(i,j) \in K_1^c \cap K_2$, we have $\vartheta(\beta - x_0; r) \geq \vartheta(x_{ij} - \beta; r+\varepsilon) \diamond \vartheta(x_{ij} - x_0;\varepsilon) > (1-s)\diamond(1-s) > 1-\lambda$. Therefore, $\beta \in \overline{B(x_0,\lambda,r)}$. Hence st_2^ϑ-$LIM^r_{x_{mn}} \subseteq \overline{B(x_0,\lambda,r)}$.

(ii) By the previous part, we have st_2^ϑ-$LIM^r_{x_{mn}} \subseteq \bigcap_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)}$. Let $\gamma \in \bigcap_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)}$. Then, $\vartheta(\gamma - x_0; r) \geq 1-\lambda$ for all $x_0 \in \Lambda_{(x_{mn})}(st_2^\vartheta)$ and hence $\Lambda_{(x_{mn})}(st_2^\vartheta) \subseteq \overline{B(\gamma,\lambda,r)}$, i.e., $\bigcap_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)} \subseteq \{y_0 \in X : \Lambda_{(x_{mn})}(st_2^\vartheta) \subseteq \overline{B(y_0,\lambda,r)}\}$. Further, suppose that $\gamma \notin st_2^\vartheta$-$LIM^r_{x_{mn}}$. Then for $\varepsilon > 0$, $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn} - \gamma; r+\varepsilon) \leq 1-\lambda\}) \neq 0$, which gives that there exists a statistical cluster point x_0 of $\{x_{mn}\}$ for which $\vartheta(\gamma - x_0; r+\varepsilon) \leq 1-\lambda$. Therefore, $\Lambda_{(x_{mn})}(st_2^\vartheta) \nsubseteq \overline{B(\gamma,\lambda,r)}$ and $\gamma \notin \{y_0 \in X : \Lambda_{(x_{mn})}(st_2^\vartheta) \subseteq \overline{B(y_0,\lambda,r)}\}$. Therefore, $\{y_0 \in X : \Lambda_{(x_{mn})}(st_2^\vartheta) \subseteq \overline{B(y_0,\lambda,r)}\} \subseteq st_2^\vartheta$-$LIM^r_{x_{mn}}$. Therefore, st_2^ϑ-$LIM^r_{x_{mn}} = \bigcap_{x_0\in\Lambda_{(x_{mn})}(st_2^\vartheta)} \overline{B(x_0,\lambda,r)} = \{y_0 \in X : \Lambda_{(x_{mn})}(st_2^\vartheta) \subseteq \overline{B(y_0,\lambda,r)}\}$. This completes the proof. □

Theorem 3.12. *Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$ such that $x_{mn} \xrightarrow{st_2^\vartheta} \zeta$. Then, there exists $\lambda \in (0,1)$ such that st_2^ϑ-$LIM^r_{x_{mn}} = \overline{B(\zeta,\lambda,r)}$ for some $r > 0$.*

Proof. For given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\diamond(1-s) > 1-\lambda$. Since $x_{mn} \xrightarrow{st_2^\vartheta} \zeta$, then for every $\varepsilon > 0$, the set $Y = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{mn}-\zeta;\varepsilon) \le 1-s\}$ has the double natural density zero. Now, let $y_* \in \overline{B(\zeta,s,r)}$. Then $\vartheta(y_*-\zeta;r) \ge 1-s$. Now, for $(i,j) \in Y^c$, $\vartheta(x_{ij}-y_*;r+\varepsilon) \ge \vartheta(x_{ij}-\zeta;\varepsilon)\diamond\vartheta(\zeta-y_*;r) > (1-s)\diamond(1-s) > 1-\lambda$. Therefore $\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}-y_*;r+\varepsilon) \le 1-\lambda\} \subseteq Y$. Since $\delta_2(Y) = 0$, $\delta_2(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \vartheta(x_{ij}-y_*;r+\varepsilon) \le 1-\lambda\}) = 0$. Consequently, $y_* \in st_2^\vartheta\text{-}LIM^r_{x_{mn}}$. Hence $\overline{B(\zeta,\lambda,r)} \subseteq st_2^\vartheta\text{-}LIM^r_{x_{mn}}$. Again, since $x_{mn} \xrightarrow{st_2^\vartheta} \zeta$, $\Lambda_{(x_{mn})}(st_2^\vartheta) = \{\zeta\}$ and consequently, from Theorem 3.11 we have $st_2^\vartheta\text{-}LIM^r_{x_{mn}} \subseteq \overline{B(\zeta,\lambda,r)}$. Hence $st_2^\vartheta\text{-}LIM^r_{x_{mn}} = \overline{B(\zeta,\lambda,r)}$. This completes the proof. □

Theorem 3.13. *Let $\{x_{mn}\}$ be a double sequence in a $PNS(X,\vartheta,\diamond)$ such that $x_{mn} \xrightarrow{st_2^\vartheta} \eta$. Then $\Lambda^r_{(x_{mn})}(st_2^\vartheta) = st_2^\vartheta\text{-}LIM^r_{x_{mn}}$ for some $r > 0$.*

Proof. Since $x_{mn} \xrightarrow{st_2^\vartheta} \eta$, $\Lambda_{(x_{mn})}(st_2^\vartheta) = \{\eta\}$. By Theorem 3.10, for $\lambda \in (0,1)$, $\Lambda^r_{(x_{mn})}(st_2^\vartheta) = \overline{B(\eta,\lambda,r)}$. Again, from Theorem 3.12, $\overline{B(\eta,\lambda,r)} = st_2^\vartheta\text{-}LIM^r_{x_{mn}}$. Therefore, $\Lambda^r_{(x_{mn})}(st_2^\vartheta) = st_2^\vartheta\text{-}LIM^r_{x_{mn}}$. This completes the proof. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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