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Research Article

Incident Vertex π-Coloring of Graphs

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Abstract. We defined the concept of π-coloring of graphs and incident vertex π coloring of graphs. The incident vertex π coloring number ($IV\pi CN$) of graphs is different from all existing coloring techniques. The $IV\pi CN$ of complete graph (K_n) is n. $IV\pi CN$ of wheel, star, double star graph are $(n+1)$. Also, $IV\pi CN$ of friendship, diamond and fan graphs are $\Delta+1$. The $IV\pi CN$ of double fan graph is $\Delta+2$. The $IV\pi CN$ of complete bipartite graphs $K_{m,n}$ is $(m+n)$. The $IV\pi CN$ of bipartite graph is bounded. Moreover, some results associated to enumeration of the number of graphs having equal incident vertex π chromatic number of few families are proved.

Keywords. π coloring of graphs, Incident vertex π coloring, Fan graphs, Friendship graphs, Wheel graphs, Diamond graphs, Star and Double Star graphs

Mathematics Subject Classification (2020). 05C15, 05C69

1. Introduction

Graph theory was started with the famous Seven Bridges of Königsberg problem. The problem was to start at any point, walks through all seven bridges on the Pregel river only one time and come back to beginning point. In 1936, Leonhard Euler [12] gave the solution to this problem with the help of the graph. There is no such closed walk that exists for this problem. Hence the concept of the Eulerian circuit was introduced in the Graph theory. "*The graph is Eulerian if and only if the degree of all vertices in a connected graph is an even*", this was the first paper considered and the development of Graph theory started.

Later, Euler gave the planer graph formula based on the invariant of polyhedron in algebraic topology (Richeson [21]). A polyhedron having a, b, c vertices, edges and closed regions, then

$a+c-b$ is invariant and for planar graphs, $a+c-b=2$. Euler indeed gave great contributions to the field of mathematics and physics, especially for the start of developing Graph theory.

Graph theory came into focus with the very famous problem, four color conjecture in 1850. This was an open challenging problem unsolved for almost 127 years. Many mathematicians and researchers worked on this problem but they not did succeed. Finally, in 1977 the proof of this problem was given by Appel and Haken [1, 2], and Appel *et al*. [3] with the help of 1200 hours of computer time. In the 20th century Graph theory started to develop with major concepts because Graph theory has many applications in the field of sciences, engineering, medical sciences, economics, and psychology. Hence in 21st century Graph theory is identified as one independent subject of mathematics.

Now-a-days many mathematicians and researchers are making an effort for searching new concepts with their real-time applications in various fields. We observe that many researchers worked on graph coloring such that adjacent vertices should have different color, but no one has worked to avoid repeating patterns while proper coloring of vertices. This paper defines new vertex coloring method to avoid the repeating patterns while proper coloring of graph's vertices.

1.1 Basics of Graph

Definition 1.1. A graph H is an order pair of sets (V,E), such that V is vertex-set of non-empty components and E is an edge set containing unordered couples of vertices from vertex set, recognized as edges of H (Deo [10], Harary [15], and West [25]).

If uv is an edge then vertices u,v are adjacent. Total number of edges at a vertex u is known as degree, denoted by $d(u)$. If degree is zero and one then it is a pendent vertex and isolated vertex, respectively. The degree of a graph means highest degree vertex number and it is denoted by Δ. The order of graph means cardinality of set V. All vertices nearby vertex u, is known as neighbor-set, it is represented as $N(u)$ and also called an open neighborhood. Closed neighborhood means a vertex u and it's all nearby vertices, represented as $N[u]$. Null graph means a graph that has n vertices but no edges (N_n). Complete graph (K_n) is one in which for every vertex u, $N[u]$ is vertex set of graph. Friendship graph (F_n) is formed by connecting n duplicates of cycle C_3 to a common vertex. A star graph ($K_{1,n}$) formed by connecting n pendent vertices to a common vertex. The double star graph ($K_{1,n,n}$) is formed by connecting n duplicates of path P_2 to a common vertex. Reader may refer books of Bondy and Murthy [9], Deo [10], Harary [15], Hond *et al*. [17], Kuble [19], Tremblay and Manohar [24], West [25], and Wilson [27] for more details about basic concepts of graph and its applications.

2. Literature Review

2.1 Coloring of Graphs

Graph coloring problem started with famous four color conjecture in 1850, which states that two countries with common borders should be colored differently. Francis Guthrie (1831-1899 AD) predicted this problem when he was tried to color a map of England's countries; observed that four colors are sufficient. He discussed this problem with brother Fredrick Guthrie (1833-1886 AD). Fredrick asked question to his teacher De Morgan, *Is this true for all maps*? Morgan was unable to respond to his question. De Morgan wrote letter to Cayley

Arthur and W. R. Hamilton in 1852. In 1879, Cayley Arthur published first time 4CC in London Royal Mathematical Society (Kubale [19]). Kempe [18] provided the first proof of 4CC in 1879. However, in 1890, Heawood [16] discovered a flaw in his proof and proved the theorem for five colors. In the meantime, work on coloring graph elements in progress. Appel and Haken [1, 2], and Appel *et al*. [3] used a computer with 1200 hours of computer-time to finally prove the Four-Color problem in 1977. The famous mathematics problem was solved comprehensively for the first time in history using a computer. Following this, many mathematicians proved 4CC in various ways. Ore and Stemple [20] demonstrated 4CC for maps with fewer than 40 countries using a numerical method. 4CC was demonstrated by Robertson *et al*. [22] using 633 unavoidable reducible configurations. Bhapkar and Salunke [7] demonstrated 4CC by using PNR (*Pivot Region Number*) of a graph.

In 2012, Birkhoff [8] introduced chromatic polynomial based on Gauss [13] fundamental theorem of algebra, each polynomial with complex coefficient of m degree has precisely m roots. For colors $\{1,2,3,\dots,k\}$ with $k \geq \chi(H)$, this polynomial provides numbers of ways of proper k-coloring. It has $(\chi(H)-1)$ zeros; $1,2,\dots,(\chi(H)-1)$. Birkhoff [8], Whitney [26], Rota [23], and other researcher contributed many result on chromatic polynomial. Some of the results are, chromatic polynomial is alternate in signs, coefficient of x^n is 1, coefficient of x^{n-1} is $-m$, that is number of edges, $P(H,\chi) \neq 0$, constant term in polynomial $P(H,x)$ is always zero, for planar graph H, $P(H,x) \neq 0$ is equitant to 4CC. Thus, Birkhoff attempted to solve the graph theory coloring problem using an algebraic method. Based on this, more than 550 papers have been published by many researchers since the advent to the chromatic polynomial till today. Benzer [5] identified the linear structure of the DNA molecule in 1955; as a result, Hajnal and Surányi [14] introduced and studied interval graphs, a subclass of chordal graphs, in 1958. In 1975, it was noticed that chordal graphs contain all of the roots from the set $\{1,2,\dots,(\chi(H)-1)\}$. Dmitriev [11] discovered that not only chordal graphs have this property, so the ultimate form of the characterization is: a graph H is chordal if and only if all the roots of the chromatic polynomial for every induced subgraph H' are integers from the set $\{1,2,\dots,(\chi(H)-1)\}$, the zeros of chromatic polynomial.

In graph theory, graph coloring is a subset of graph cataloging; the assignment of tags commonly referred as colors to the graph's elements subject to certain restrictions (Bondy and Murthy [9], Deo [10], Harary [15], Kuble [19], West [25], and Wilson [27]).

2.2 Basic Types of Colorings

Definition 2.1 (Vertex Coloring)**.** A technique for coloring of vertices so that two nearby vertices are colored differently is known as vertex coloring. The smallest number is chromatic number χ, of a graph (Harary [15], and Kubale [19]).

Definition 2.2 (Edge Coloring)**.** A technique for coloring of edges so that two nearby edges are colored differently is known as edge coloring. The smallest number is chromatic index χ', of a graph (Harary [15], and Kubale [19]).

Commonly vertex coloring was used to introduce graph coloring, because many coloring problems are straightaway converted into vertex coloring. A graph's edge coloring is basically its line graph's vertex coloring. A plane graph's face coloring is basically its dual graph's vertex coloring.

Definition 2.3 (Face Coloring or Region Coloring)**.** A technique for coloring of plane graph's faces so that two contiguous faces are colored differently is known as region coloring. The smallest number is region chromatic number of a graph (Appel and Haken [1], and Kubale [19]).

Definition 2.4 (Total Coloring)**.** The coloring to edges and vertices together of a graph with any two nearby elements are colored differently known as total coloring adjacent. Vertices, edges, or edges and their end-vertices are assigned the same color. The smallest number is total chromatic number χ''. This coloring technique was pioneered by Behzad [4] in 1965.

Definition 2.5 (Perfect Coloring)**.** The perfect coloring is a proper coloring to all elements of a planar graph in the sense that two nearby components colored differently. Such number of least possible colors necessary is known as perfect chromatic number $\chi^P(H)$. This coloring technique was developed by Bhange and Bhapkar [6], who collaborated to discover the kinds of perfect coloring.
There are many other coloring types that are defined by enforcing different condition while proper coloring of graph elements. But we observed that the focus is only on proper coloring with color patterns maybe repeats any number of times while coloring of graph elements.

2.3 Literature Gap

In the literature, the work of graph coloring is only the discussion of graph coloring with nearby vertices receiving different colors or labels and related aspects. No one has worked on estimating the proper vertex coloring of graph with different patterns of color sets for incident vertices at every edge.

3. Main Results

The quotient of perimeter to diameter of circle is called 'pi', denoted by Greek latter π. It is remains constant for any size of the circle. The decimal place numbers never ends and do not repeat themselves, hence it is irrational number. The concept of "*decimal places numbers differ in the pattern*" is used to define the new coloring of the graphs as *incident-vertex π-coloring of graphs* (*IVπCG*).

3.1 π-Coloring

Definition. Let H be a simple graph and set $X = \{X_1, \dots, X_r\}$ is a set of distinct subsets of some common characteristics or properties of H. Let set K having s distinct colors and $P(K)$ is its power set.
If there exists function $f : X \to P(K)$ such that assign different set of colors to each X_i that satisfies the condition, $f(X_i) \neq f(X_j)$, for all i, j, $i \neq j$, with some conditions. Such types of colorings are called *π-coloring of graphs*. The smallest value of s is named as *π-chromatic number* of graph H corresponding to function f. It is denoted by $\pi_f(H)$ or $\pi(H)$.
There are various functions f depending upon the different properties of the graph and so, there are different types of the π coloring.

3.2 Incident Vertex π Coloring of the Graph

In this coloring type, we assign different colors to an incident vertex of the edges, resulting in a properly colored graph with different pattern.

Definition. Let H be a simple graph, $X = \{H_1,\dots,H_r\}$ is collection of distinct two element subsets of vertex set of H where $H_i = \{e_i = (u,v) \mid u,v \text{ in } V,\ e_i \text{ in } E\}$, $P(K)$ is the power set for a set K having s distinct colors.
If there exists function $f : X \to P(K)$, such that assign proper colors to an incident vertices that satisfies the condition, $f(H_i) \neq f(H_j)$, for each i,j, $i \neq j$. This is called *incident vertex π coloring of graph H*. The least value of s is called *incident vertex π chromatic number of H* corresponds to f, and represented by $IV\pi CN(H)$.

Example. Let H be a graph and $X = \{H_1,\dots,H_8\}$. Define, $H_1 = \{h_1,h_2\}$, $H_2 = \{h_1,h_3\}$, $H_3 = \{h_2,h_3\}$, $H_4 = \{h_2,h_4\}$, $H_5 = \{h_3,h_5\}$, $H_6 = \{h_4,h_5\}$, $H_7 = \{h_4,h_6\}$, $H_8 = \{h_5,h_6\}$.
Now, allot colors to each vertices of graph H (see Figure 1). $h_1 \to 1$, $h_2 \to 2$, $h_3 \to 3$, $h_4 \to 4$, $h_5 \to 5$ and $h_6 \to 1$.

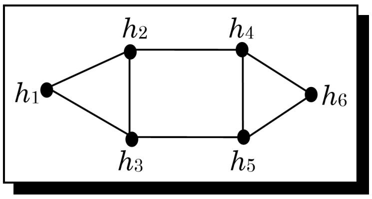

Figure 1. Graph H

As the result,

$$f(H_1) = \{1,2\},\quad f(H_2) = \{1,3\},\quad f(H_3) = \{2,3\},\quad f(H_4) = \{2,4\},\quad f(H_5) = \{3,5\},$$
$$f(H_6) = \{4,5\},\quad f(H_7) = \{1,4\},\quad f(H_8) = \{1,5\}.$$

Here, all $f(H_i)$ are distinct with minimum number of colors required 5. Therefore, the *incident vertex π chromatic number of H* is 5. That means $IV\pi CN(H) = 5$.

Note. For graph H, chromatic number is $\chi(H) = 3$ and edge chromatic number is $\chi'(H) = 3$. Both are different from the *incident vertex π chromatic number of H*, $IV\pi CN(H) = 5$. So, the concept of *incident vertex π chromatic number* is different.

Theorem 3.1. *For path P_n if $IV\pi CN(P_n) = m$, then*

$$\max(n) = \begin{cases} \frac{m^2-2m+4}{2}, & \text{if } m \text{ is even}, \\ \frac{m(m-1)}{2}, & \text{if } m \text{ is odd}. \end{cases}$$

Proof. *Case* I: m is even.
The number of ways for selecting two elements from m elements is ${}^mC_2$ ways. In ${}^mC_2$, each color appears $(m-1)$-times, which is odd. In any path graph P_n, we can use odd number of colors only twice in H_i for the two pendent vertices and for intermediate (non-pendent) vertices, use even number of times. Use any two colors for pendent vertices, and $(m-2)$ colors to remaining vertices $(m-2)$-times each.

Therefore, the number of edges is $= \frac{(m-2)(m-2)+2(m-1)}{2} = \frac{2-2m+m^2}{2}$.

Hence, maximum value of n is $= \frac{2-2m+m^2}{2} + 1 = \frac{m^2-2m+4}{2}$.

Case II: m is odd.

Each color appears $(m-1)$-times, which is even. Therefore, use any two colors with pendent vertices $(m-2)$-times and remaining $(m-2)$ colors to remaining vertices $(m-1)$-times each.

Thus, the number of edges is $= \frac{2(m-2)+(m-2)(m-1)}{2} = \frac{-2-m+m^2}{2}$.

Hence, maximum value of n is $= \frac{-2-m+m^2}{2} + 1 = \frac{m(m-1)}{2}$. □

Theorem 3.2. *There are N graphs having $IV\pi CN(P_n) = k$, where* $N = \begin{cases} \frac{k}{2}+1, & \textit{if } k \textit{ is even}, \\ \frac{3k-7}{2}, & \textit{if } k \textit{ is odd}. \end{cases}$

Proof. Let $IV\pi CN(P_n) = k$. There are two cases for k is either even or odd as given below.

Case I: k is an even number.

If $k = 2r$, by Theorem 3.1, maximum value of n is $N_1 = \frac{4-2k+k^2}{2}$. Now $k-1 = 2r-1$ which is an odd number. Therefore, maximum number of vertices having $IV\pi CN(P_n) = k-1$, is $N_2 = \frac{(k-1)^2-(k-1)}{2} = \frac{2-3k+k^2}{2}$.

Hence, $N_1 - N_2 = \frac{4-2k+k^2}{2} - \frac{2-3k+k^2}{2} = \frac{1}{2}k + 1$.

Case II: k is an odd number.

If $k = 2r+1$, by Theorem 3.1, maximum value of n is $N_3 = \frac{k^2-k}{2}$. Here $k-1 = 2r$, hence maximum number of vertices with $IV\pi CN(P_n) = k-1$ is $N_4 = \frac{(k-1)^2-2(k-1)+4}{2} = \frac{7-4k+k^2}{2}$.

Thus, $N_3 - N_4 = \frac{k^2-k}{2} - \frac{7-4k+k^2}{2} = \frac{3k-7}{2}$. Hence theorem is proved. □

Theorem 3.3. *The incident vertex π chromatic number of $K_{1,n}$ is $n+1$.*

Proof. The star graph $K_{1,n}$, as shown in Figure 2.

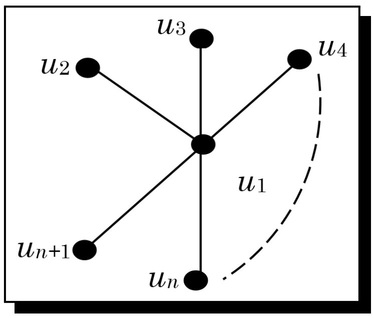

Figure 2. Star graph $K_{1,n}$

In star graph there are $n+1$ vertices, $V(K_{1,n}) = \{u_1, u_2, u_3, \dots, u_{n+1}\}$ and there are n number of edges. In view of that, the set $X = \{H_1, \dots, H_n\}$, where $H_i = \{u_1, u_{i+1}\}$, for $i = 1$ to n.

Define, $f : X \to P(K)$, such that assign different colors to incident vertices as: $u_1 \to 1, \dots, u_{n+1} \to n+1$.

Here color set $K = \{1, 2, \dots\}$ with its power set is $P(K)$.

Observe that, $f(H_i) = \{1, i+1\}$; $i = 1$ to n. Hence we needed $(n+1)$ different colors to color the sets of X such that $f(H_i) \neq f(H_j)$; for i, j with $i \neq j$. As a result, $IV\pi CN(K_{1,n}) = n+1$. □

Theorem 3.4. *The incident vertex π chromatic number of $K_{1,n,n}$ is $n+1$.*

Proof. There are $(2n+1)$ vertices and $2n$ edges in double star graph. Let us denote central vertex by v and vertices adjacent to v are $u_1, u_2, \dots, u_n$ and the vertices $w_1, w_2, \dots, w_n$ are adjacent to $u_1, u_2, \dots, u_n$, respectively (Figure 3). Therefore, $X = \{H_1, H_2, \dots, H_n, H_{n+1}, \dots, H_{2n}\}$, where $H_i = \{v, u_i\}$, $i = 1, 2 \dots, n$ and $H_{n+j} = \{u_j, w_j\}$, $j = 1, 2 \dots, n$.
Define, $f : X \to P(K)$, such that assign different colors to incident vertices as follows.
$v \to 1$, $u_1 \to 2$, $u_2 \to 3$, $\dots$, $u_n \to n+1$, $w_1 \to 3$, $w_2 \to 4$, $w_3 \to 5$, $\dots$, $w_{n-1} \to n+1$, and $w_n \to 2$.

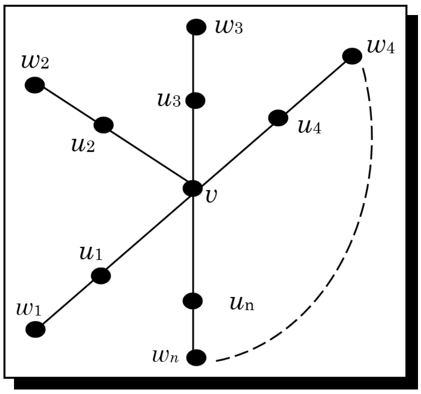

Figure 3. Double star graph $K_{1,n,n}$

Here, $f(H_i) = \{1, i+1\}$; $i = 1, 2 \dots, n$; $f(H_{n+j}) = \{j+1, j+2\}$, $j = 1, 2 \dots, n-1$ and $f(H_{2n}) = \{n+1, 2\}$. As a result, $f(H_p) \neq f(H_q)$, for all p and $q, p \neq q$. Thus, $IV\pi CN(K_{1,n,n}) = (n+1)$. □

Theorem 3.5. *The incident vertex π chromatic Number of complete graph is n.*

Proof. In complete graph, every vertex adjacent to remaining all, so there are ${}^nC_2$ edges. Let $V(K_n) = \{v_1, \dots, v_n\}$ is vertex set and $X = H(K_n) = \{H_1, H_2, \dots, H_r\}$, where $r = {}^nC_2$. $H_i = \{(u, v)/u, v \text{ in } V\}$. Also, set $K = \{1, 2, \dots, n\}$ of distinct colors with its power set is $P(K)$.
Thus, we have $f : X \to P(K)$, is a function that assign different colors to incident vertices as: $v_i \to i$, $i = 1, 2, \dots, n$. Here, $f(H_i) \neq f(H_j)$, for all i, j, $i \neq j$.
Therefore, an *incident vertex π chromatic number of K_n* is n. That means $IV\pi CN(K_n) = n$. □

Theorem 3.6. *The incident vertex π chromatic number of friendship graph F_n is $\Delta + 1$.*

Proof. Let v is central vertex and $u_1, \dots, u_{2n}$ are remaining vertices of F_n as shown in Figure 4. Therefore, $X = \{H_1, \dots, H_{3n}\}$, where $H_i = \{v, u_i\}$, for $i = 1, \dots, 2n$ and $H_{2n+j} = \{u_{2j-1}, u_{2j}\}$, for $j = 1, \dots, n$.
Define $f : X \to P(K)$, such that assign different colors to incident vertices as: $v \to 1$, $u_t \to (t+1)$, for $t = 1, 2, \dots, 2n$.
Here, $f(H_i) = \{1, i+1\}$ for $i = 1, 2, \dots, 2n$; $f(H_{2n+j}) = \{2j, 2j+1\}$, for $j = 1, 2, \dots, n$.
Therefore, $f(H_p) \neq f(H_q)$, for all p and $q, p \neq q$.

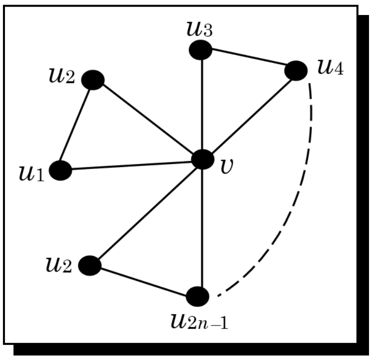

Figure 4. Friendship graph F_n

Hence, number of colors required are $1+2n = 1+\Delta$, where $\Delta = 2n$ = maximum degree of F_n. Thus, $IV\pi CN(F_n) = \Delta + 1$. □

Theorem 3.7. *The incident vertex π chromatic number of wheel graph W_{n+1}, is $n+1$.*

Proof. W_n is a wheel graph with v is a centered vertex and $u_1, u_2, \dots, u_n$ be remaining vertices as shown in Figure 5.

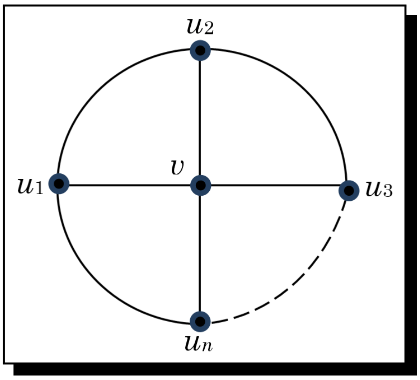

Figure 5. Wheel graph W_{n+1}

Therefore, $X = \{H_1, H_2, \dots, H_n, \dots, H_{2n}\}$, where, $H_p = \{u_p, u_{p+1}\}$, for $p = 1,2,\dots,n-1$, $H_n = \{u_n, u_1\}$, and $H_{n+q} = \{u_q, v\}$ for $q = 1,2,\dots,n$.
Define function $f : X \to P(K)$, such that assign different colors to incident vertices as: $v \to 1$, $u_1 \to 2$, $u_2 \to 3, \dots, u_n \to n+1$. Here, $f(H_p) = \{p+1, p+2\}$ for $p = 1,2,\dots,n-1$, $f(H_n) = \{n+1, 2\}$ and $f(H_{n+q}) = \{q+1, 1\}$ for $q = 1,2,\dots,n$.
Therefore, $f(H_p) \neq f(H_q)$, for p and $q, p \neq q$. Thus, $IV\pi CN(W_{n+1}) = (n+1)$. □

Theorem 3.8. *The incident vertex π chromatic number of fan graph $F_{1,n}$ is $\Delta + 1$.*

Proof. $F_{1,n}$ is fan graph where vertex v of degree n, and $u_1, \dots, u_n$ be remaining vertices as shown in Figure 6.

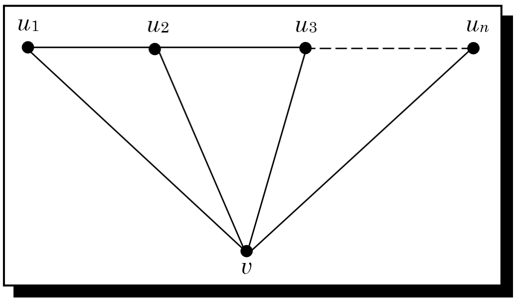

Figure 6. Fan graph $F_{1,n}$

For graph $F_{1,n}$, $\Delta(F_{1,n}) = n$. Therefore, $X = \{H_1, H_2, \dots, H_n, \dots, H_{2n-1}\}$, where $H_i = \{u_i, u_{i+1}\}$, $i = 1$, to $n-1$ and $H_{n+j} = \{u_{j+1}, v\}$, $j = 0, \dots, n-1$.
Define $f : X \to P(K)$, such that allot different colors to incident vertices as: $u_1 \to 1$, $u_2 \to 2, \dots, u_n \to n$, and $v \to n+1$.
Here, $f(H_i) = \{i, i+1\}$, $i = 1, 2 \dots, n-1$; $f(H_{n+j}) = \{j+1, n+1\}$, $j = 0, 1, \dots, n-1$.
Therefore, $f(H_p) \neq f(H_q)$, for all p and $q, p \neq q$. Thus, $n+1$ color needed for *incident vertex π* coloring. Hence, $IV\pi CN(F_{1,n}) = \Delta + 1$. □

Theorem 3.9. *The incident vertex π chromatic Number of double fan graph is $\Delta + 2$.*

Proof. Consider double fan graph $F_{2,n}$ having vertices u, v with $d(u) = d(v) = n$, and $u_1, u_2, \dots, u_n$ be remaining vertices. There are $(3n-1)$ edges in $F_{2,n}$ as shown in Figure 7.

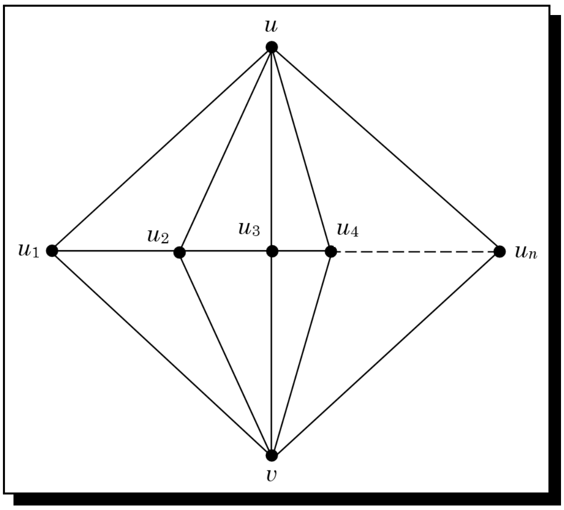

Figure 7. Double fan graph $F_{2,n}$

For double fan graph $F_{2,n}$, $\Delta = n$. Let $X = \{H_1, H_2, \dots, H_n, \dots, H_{3n-1}\}$, where $H_i = \{u_i, u_{i+1}\}$, $i = 1, \dots, n-1$, $H_{n+j} = \{u_{j+1}, u\}$, $j = 0, 1, \dots, n-1$ and $H_{(2n)+k} = \{v, u_{k+1}\}$, $k = 0, 1, \dots, n-1$.
Define $f : X \to P(K)$, such that allot different colors to incident vertices as: $u_1 \to 1$, $u_2 \to 2, \dots, u_n \to n$, $u \to n+1$, $v \to n+2$. Here, $f(H_i) = \{i, i+1\}$, $i = 1, 2, \dots, n-1$, $f(H_{n+j}) = \{j+1, n+1\}$, $j = 0, 1, \dots, n-1$ and $f(H_{2n+k}) = \{n+2, k+1\}$, $k = 0, 1, \dots, n-1$.
Therefore, $f(H_p) \neq f(H_q)$, for all p and $q, p \neq q$. Thus, $n+2$ colors needed for *incident vertex π* coloring. Hence, $IV\pi CN(F_{2,n}) = \Delta + 2$. □

Theorem 3.10. *For bipartite graph $K_{m,n}$, $IV\pi CN(K_{m,n}) = m+n$.*

Proof. Consider bipartite graph $K_{m,n}$, with the partition of vertex sets are $A = \{u_1, u_2, \dots, u_m\}$ and $B = \{v_1, v_2, \dots, v_n\}$ and $V = A \cup B$.
Let $X = \{H_{ij} = \{u_i, v_j\},\ i = 1,2,\dots,m$ and $j = 1,2\dots,n\}$, $K = \{1,2\dots,m+n\}$.
Define, $f : X \to P(K)$, such that assign different colors to incident vertices as: $u_i \to i$, $i = 1,2,\dots,m$ and $v_j \to m+j$, $j = 1,2,\dots,n$ thus we observe that $f(H_{ij}) = \{i, m+j\}$, and $f(H_p) \neq f(H_q)$, for all p and q, $p \neq q$. Thus, $IV\pi CN(K_{m,n}) = m+n$. □

Theorem 3.11. *$\Delta \leq IV\pi CN(B_{m,n}) \leq m+n$, where Δ maximum degree of bipartite graph.*

Proof. Consider bipartite graph $B_{m,n}$, its vertex set is $V = A \cup B$, where $A = \{u_1, \dots, u_m\}$ and $B = \{v_1, \dots, v_n\}$ and edge set E having edges with end vertices of each edge are from set A and B. Let v be any vertex with largest degree Δ. Since v has Δ different neighbors, so we needed minimum $\Delta+1$ different color for incident vertex π coloring. Also, by Theorem 3.10, we required atmost $(m+n)$ color for incident vertex π coloring.
Thus $\Delta \leq IV\pi CN(B_{m,n}) \leq m+n$. Hence result proved. □

Theorem 3.12. *The highest value of n for cycle graph C_n with $IV\pi CN(C_n) = k$, is $2r(r-1)$, if $k = 2r$ and $r(2r+1)$, if $k = 2r+1$.*

Proof. We will prove theorem with two cases for k.

(I) k is even i.e. $k = 2r$.
In ${}^kC_2$, each color repeated $(k-1)$-times means odd number of times. But in cycle graph C_n, each color contributes to even number of edges. As $(k-1)$ is odd, every color used atmost $(k-2)$-times for $IV\pi C$, and so, maximum number of vertices in corresponding C_n is $N_1 = \frac{k(k-2)}{2} = 2r(r-1)$.

(II) k is odd, $k = 2r+1$.
In ${}^kC_2$, each color repeated $(k-1)$-times means even number of times. As $(k-1)$ is even, every color contributes atmost $(k-1)$-times in $IV\pi C$. Therefore, maximum number of vertices in corresponding C_n is $N_2 = \frac{k(k-1)}{2} = \frac{(2r+1)(2r+1-1)}{2} = r(2r+1)$. □

Theorem 3.13. *The number of C_n graphs having $IV\pi CN(C_n) = k$, are $r-1$ if $k = 2r$ and $3r$ if $k = 2r+1$.*

Proof. There are two cases for integer k as below.

(I) If k is even, $k = 2r$.
By Theorem 3.12, the highest C_n graph is at $N_1 = 2r(r-1)$. For $k-1 = 2r-1$, greatest value of n is at $N_2 = (r-1)\{2(r-1)+1\} = (r-1)(2r-1)$.
Hence, the required numbers of graphs are $N_1 - N_2 = (r-1)$.

(II) If k is odd, $k = 2r+1$.
Again by Theorem 3.12, the highest C_n graph is at $N_3 = r(2r+1)$. For $(k-1) = 2r$, the maximum value of n is at $N_4 = 2r(r-1)$.
Hence, the required numbers of graphs is $N_3 - N_4 = 3r$. □

Theorem 3.14. *If H_n is n copies of K_2 graph and $IV\pi CN(H_n) = k$, then ${}^{(k-1)}C_2 + 1 \leq n \leq {}^kC_2$, where $k > 2$.*

Proof. The theorem is proved by induction.
Let us denote $L = {}^{(k-1)}C_2 + 1$ is a least values and $G = {}^{k}C_2$ greatest value.
For $k = 3$, $L = G = 3$. So, $n = 3$. Thus theorem is true $k = 3$.
Consider now theorem is true for $k = r$. Therefore, $IV\pi CN(H_n) = r$, if ${}^{(r-1)}C_2 + 1 \le n \le {}^{r}C_2$.
Moreover, the greatest (G) and least values (L) of n are ${}^{r}C_2$ and ${}^{(r-1)}C_2 + 1$.
Let H_p be a graph with $IV\pi CN(H_p) = k = r + 1$.
Now, add one K_2 graph to H_p, we get one more color set color $\{1, r+1\}$. The smallest value of p is $L = {}^{r}C_2 + 1$. Continuously, adding K_2 graphs, we obtain color sets $\{t, r+1\}$, for $t = 2, 3, \dots, r$.
So, greatest value of p will be $G = {}^{r}C_2 + r = {}^{r+1}C_2$. Thus, ${}^{r}C_2 + 1 \le p \le {}^{r+1}C_2$.
Henceforth by induction method theorem is true for all values of k. □

Theorem 3.15. *If H_n is n copies of K_2 graph, then $IV\pi CN(H_n) = \left\lfloor \frac{3+\sqrt{8n-7}}{2} \right\rfloor$, where $n \ge 1$.*

Proof. If $IV\pi CN(H_n) = k$, then the smallest and the greatest values of n is $L = {}^{(k-1)}C_2 + 1$ and $G = {}^{k}C_2$, respectively. There are $(k-1)$ graphs having $IV\pi CN(H_n) = k$.
At $L = n = {}^{(k-1)}C_2 + 1$, the value of $k = \left\lfloor \frac{3+\sqrt{8n-7}}{2} \right\rfloor$ is integer. Now increasing n continuously up to $G = {}^{k}C_2$. We get, $\left\lfloor \frac{3+\sqrt{8n-7}}{2} \right\rfloor = k.xyz$.
This is a fraction value and its integer part is k. Hence $k = \left\lfloor \frac{3+\sqrt{8n-7}}{2} \right\rfloor$. □

Theorem 3.16. *If H_n is n copies of K_2 graph, then $IV\pi CN(H_n) = \left\lceil \frac{1+\sqrt{8n-1}}{2} \right\rceil$, where $n \ge 1$.*

Proof. Suppose $IV\pi CN(H_n) = k$, at $n = {}^{k}C_2$, the value of $\left\lceil \frac{1+\sqrt{8n-1}}{2} \right\rceil$ is an integer. Now, reduce n successively by 1 up to ${}^{(k-1)}C_2 + 1$, we get, $\left\lceil \frac{1+\sqrt{8n-1}}{2} \right\rceil = (k-1).xyz\dots$ and its greatest integer part is k. Hence, $k = \left\lceil \frac{1+\sqrt{8n-1}}{2} \right\rceil$. □

Theorem 3.17. *The $IV\pi CN$ of diamond graph D_n for $n = 4, 6, 8\dots$, is $\Delta(D_n) + 1$.*

Proof. The diamond graph D_n has n vertices namely $\{u, v, x_1, x_2, \dots, x_{n-2}\}$ and there are $(2n-3)$ edges as shown in Figure 8.

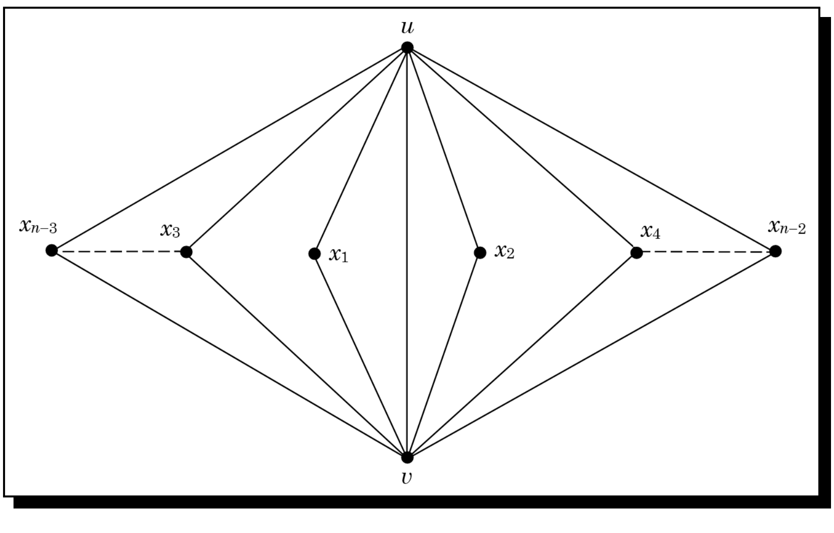

Figure 8. Diamond graph D_n

Let $H_i = \{u, x_i\}$, $i = 1,2\dots,(n-2)$, $H_{(n-2)+j} = \{x_j, v\}$, $j = 1,2\dots,(n-2)$, and $H_{2n-3} = \{u,v\}$. So, $X = \{H_1, H_2, \dots, H_n, \dots, H_{2n-3}\}$, and $K = \{1,2,\dots,n\}$.
Define $f : X \to P(K)$, such that assigns different colors to incident vertices.
$u \to 1$, $v \to 2$, $x_i \to 2+i$, where $i = 1,2\dots,(n-2)$.
Here, $f(H_i) = \{1, i+2\}$, $i = 1,2\dots,(n-2)$, $f(H_{(n-2)+j}) = \{j+2, 2\}$, $j = 1,2\dots,(n-2)$, and $f(H_{2n-3}) = \{1,2\}$.
Therefore, $f(H_p) \neq f(H_q)$, for all p and $q, p \neq q$. Thus, $IV\pi CN(D_n) = n$ and $\Delta(D_n) = n-1$.
Hence, $IV\pi CN(D_n) = \Delta(D_n) + 1$. □

Theorem 3.18. *The $IV\pi CN$ of the maximal planar graph H is $\Delta(H)+1$.*

Proof. It is the particular case of D_n. We can apply Theorem 3.17, to prove this result. □

Algorithm 3.19. An algorithm for finding incident vertex π chromatic number of the path P_n.
Consider path P_n having n edges as shown in Figure 9.
Then its vertex set is $V(P_n) = \{p_1, p_2, \dots, p_{n+1}\}$ and
Edge set is $E(P_n) = \{e_1, e_2, \dots, e_n\}$, where $e_i = (p_i, p_{i+1})$, $i = 1,2,\dots,n$.

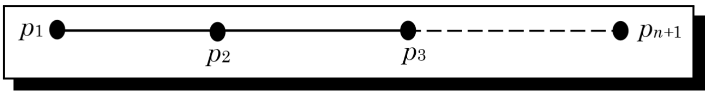

Figure 9. Path P_n

Let $X = \{X_t = e_t \mid t = 1 \text{ to } n\}$ and $K = \{1,2,\dots,m\}$ is a set having m distinct colors.
For incident vertex π chromatic number of path P_n use following steps for algorithm:

Step 1: Define $F : X \to P(K)$ such that assign different colors to incident vertices for each edge with $F(X_p) \neq F(X_q)$, for all p and $q, p \neq q$.

Step 2: Start to assign colors $1,2,\dots,$ from set K with $X_1 = \{p_1, p_2\} = \{1,2\}$, $X_1 = \{p_2, p_3\} = \{2,3\}$ etc., and denote color set $K_3 = \{1,2,3\}$.

Step 3: Now color next set X_q with available color from set K_3 and check $F(X_p) \neq F(X_q)$, for all $p = \{1,2,3\} < q$, if yes then use color from K_3 otherwise go to next step.

Step 4: If $F(X_p) = F(X_q)$, for existing available colors from K_3 then select new color from set K, add it to K_3 becomes now K_4 and start to colors with *Step* 3.

Step 4: If all set X_p are colored with $F(X_p) \neq F(X_q)$, for all p and $q, p \neq q$ with the color set is used $K_r = \{1,2,3,\dots,r\}$ and *Stop*.

Thus, we will get minimum number of colors and that is $IV\pi CN(P_n)$.

4. Conclusion

We introduced incident vertex π coloring ($IV\pi CN$). For some standard graph's families, $IV\pi CN$ are computed. We have obtained the $IV\pi CN$ bounds for some families. Also, worked on enumeration for some graph's families with their $IV\pi CN$ is fixed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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