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Research Article

On the Bounded Region for the Stratified Shear Flows in β-plane

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Abstract. We consider incompressible, inviscid stratified shear flows in β-plane. We obtained a bounded and unbounded instability regions which depends on various parameters, and obtained a criterion for stability. Also, we obtained an upper bound for the growth rate, amplification factor of an unstable mode.

Keywords. Shear flows, Incompressible fluids, Stratified fluids, Inviscid fluids, β-plane

Mathematics Subject Classification (2020). 76E05

1. Introduction

The study of stability analysis of stratified shear flows under normal mode approach has been studied extensively (see Yih [15], Drazin and Reid [1], Schmid and Henningson [13]). Parallel shear flow problem is a standard classical problem of hydrodynamic stability. Kuo [6] considered incompressible, inviscid, homogeneous parallel shear flows in β-plane known as Kuo problem. For this problem, Kuo [6] derived Rayleigh inflexion point theorem. Pedlosky [12] proved the phase velocity lies inside the upper half of the semicircle which is the extension of Howard [4] semicircle for parallel shear flows. Hickernell [3] derived upper bound for the growth rate. Padmini and Subbiah [11] obtained two parabolic instability regions intersecting Howards semicircle. Thenmozhy and Vijayan [14] followed their work and obtained two parabolic instability regions. But, both their works depends on conditions. Lavanya *et al*. [8] derived parabolic instability region by removing conditions. Lavanya *et al*. [7] obtained upper bound for amplification factor, growth rate.

For incompressible, inviscid, stratified shear flows, Miles [10] derived a sufficient condition for stability. Howard [4] derived a semicircle theorem. Kochar and Jain [5] derived a semi ellipse theorem which depends on Richardson number. Gupta *et al*. [2] derived unbounded instability region depending on condition.

In this paper, we consider Taylor-Goldstein problem in β-plane. For this problem, we obtained an instability region which intersect with semi ellipse under certain condition. This has been illustrated with examples. Furthermore, we obtained a bounded instability regions depending on Coriolis parameter, stratification parameter, basic velocity profile, curvature and vorticity function. Also, we obtained a condition for stability and upper bound for amplification factor and growth rate.

2. Taylor-Goldstein Problem in β-plane

The Taylor-Goldstein problem in β-plane is given by

$$D^2(\phi)+\left[\frac{N^2}{(U-c)^2}-\frac{D^2(U)-\beta}{U-c}-k^2\right]\phi=0, \tag{2.1}$$

with boundary conditions

$$\phi(z_1)=0=\phi(z_2), \tag{2.2}$$

where U is the basic velocity profile, ϕ is the eigen function, $c=c_r+ic_i$ phase velocity, $k>0$ is the wave number, $\beta=\frac{2\Omega}{a}\cos\theta$ is the Coriolis parameter, a is the radius of earth, Ω is the earth's rotation rate, θ is the latitude (cf. Pedlosky [12], and Lindzen [9]).

Applying the transformation $\phi=(U-c)^{\frac{1}{2}}\psi$, we get

$$D[(U-c)D(\psi)]-\frac{1}{4}\frac{(D(U))^2}{(U-c)}\psi+\frac{N^2}{(U-c)}\psi-k^2(U-c)\psi-\left(\frac{D^2(U)}{2}-\beta\right)\psi=0, \tag{2.3}$$

with boundary conditions

$$\psi(z_1)=0=\psi(z_2). \tag{2.4}$$

3. Parabolic Instability Region

Theorem 3.1. *If* (c,ψ) *is a solution of* (2.3), (2.4) *and* $U_m=\frac{U_{\min}+U_{\max}}{2}$, *then*

$$c_i^2\le\lambda\left[c_r-\frac{U_{\min}}{4}+\frac{U_{\max}}{4}\right],$$

where

$$\lambda=\frac{2\left|\frac{(D(U))^2}{4}-N^2\right|_{\max}}{\left|\frac{3U_{\min}+U_{\max}}{2}\right|\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]+\left|\frac{D^2(U)}{2}-\beta\right|_{\min}}.$$

Proof. Multiplying (2.3) by ψ^*, integrating, applying (2.4) and comparing real and imaginary parts, we get

$$\int_{z_1}^{z_2}(U-c_r)[|D(\psi)|^2+k^2|\psi|^2]dz+\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2dz$$

$$+\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(U-c_r)|\psi|^2dz=0, \tag{3.1}$$

and

$$-c_i\int_{z_1}^{z_2}[|D(\psi)|^2+k^2|\psi|^2]dz+c_i\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}|\psi|^2dz=0. \tag{3.2}$$

Multiplying (3.2) by $\frac{(c_r+U_m)}{c_i}$ and subtracting from (3.1), we get

$$\int_{z_1}^{z_2}(U+U_m)[|D(\psi)|^2+k^2|\psi|^2]dz+\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2dz$$
$$+\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(U-2c_r-U_m)|\psi|^2dz=0. \tag{3.3}$$

Applying Rayleigh-Ritz inequality, we get

$$\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]\int_{z_1}^{z_2}(U+U_m)|\psi|^2dz+\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2dz$$
$$+\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(U-2c_r-U_m)|\psi|^2dz\le 0.$$

Since $|U-c|^2\ge c_i^2$, we have

$$\int_{z_1}^{z_2}\frac{\left[\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right](U+U_m)+\left(\frac{D^2(U)}{2}-\beta\right)\right]c_i^2+\left(\frac{(D(U))^2}{4}-N^2\right)(U-2c_r-U_m)}{|U-c|^2}|\psi|^2dz\le 0.$$

This implies

$$\left[(U_{\min}+U_m)\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]+\left|\frac{D^2(U)}{2}-\beta\right|_{\min}\right]c_i^2\le\left|\frac{(D(U))^2}{4}-N^2\right|_{\max}[2c_r-U_{\min}+U_m],$$

i.e.,

$$c_i^2\le\lambda\left[c_r-\frac{U_{\min}}{4}+\frac{U_{\max}}{4}\right], \tag{3.4}$$

where

$$\lambda=\frac{2\left|\frac{(D(U))^2}{4}-N^2\right|_{\max}}{\left|\frac{3U_{\min}+U_{\max}}{2}\right|\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]+\left|\frac{D^2(U)}{2}-\beta\right|_{\min}}.$$

The region is unbounded, hence we proved that it intersects with semi ellipse region under certain condition. This has been done in the following theorem. □

Theorem 3.2. *If $\lambda<\lambda_c$, then the parabola given by* (3.4) *intersect with semi ellipse region*

$$\left[c_r-\frac{U_{\min}+U_{\max}}{2}\right]^2+\frac{2c_i^2}{1+\sqrt{1-4J_m}}\le\left[\frac{U_{\max}-U_{\min}}{2}\right]^2.$$

Proof. The semi ellipse region (cf. Kochar and Jain [5]) is given by

$$\left[c_r-\frac{U_{\min}+U_{\max}}{2}\right]^2+\frac{2c_i^2}{1+\sqrt{1-4J_m}}\le\left[\frac{U_{\max}-U_{\min}}{2}\right]^2. \tag{3.5}$$

Substituting (3.4) in (3.5), we get

$$c_r^2+\left[\frac{2\lambda}{1+\sqrt{1-4J_m}}-U_{\min}-U_{\max}\right]c_r+(U_{\min}U_{\max})+\frac{2\lambda}{1+\sqrt{1-4J_m}}\left[\frac{U_{\max}}{4}-\frac{U_{\min}}{4}\right]\le 0.$$

For real roots, the discriminant part of the above equation becomes

$$\left(\frac{2\lambda}{1+\sqrt{1-4J_m}}\right)^2-\frac{2\lambda}{1+\sqrt{1-4J_m}}(U_{\min}+3U_{\max})+(U_{\max}-U_{\min})^2\ge 0.$$

Solving, we get

$$\lambda=\frac{\left(1+\sqrt{1-4J_m}\right)(U_{\min}+3U_{\max})\pm\sqrt{5U_{\max}^2-3U_{\min}^2+14U_{\min}U_{\max}}}{4}.$$

λ with positive sign leads to $c_r<U_{\min}$ and hence if $\lambda<\lambda_c$, where

$$\lambda_c=\frac{\left(1+\sqrt{1-4J_m}\right)(U_{\min}+3U_{\max})-\sqrt{5U_{\max}^2-3U_{\min}^2+14U_{\min}U_{\max}}}{4}$$

then the parabola (3.4) intersects semi-ellipse (3.5). □

Example 1. Let us consider the flow $U=\left[z-\frac{1}{2}\right]^2$, $z\in[0,1]$, $\beta=z$, $N^2=z$.

In this case, $U_{\min}=0$, $U_{\max}=0.25$, $\lambda=\frac{0.5}{0.125[\pi^2+k^2]+1}$, $\lambda_c=0.0954$.

For $k\ge 5$, $\lambda<\lambda_c$ and hence the parabola $c_i^2\le 0.0933[c_r+0.0625]$, intersects with semi-ellipse.

Region of J_m for intersection:

$$0.16571\le J_m\le 0.25,\quad \text{for } k=0,$$

$$0.15988\le J_m\le 0.25,\quad \text{for } k=1,$$

$$0.14181\le J_m\le 0.25,\quad \text{for } k=2.$$

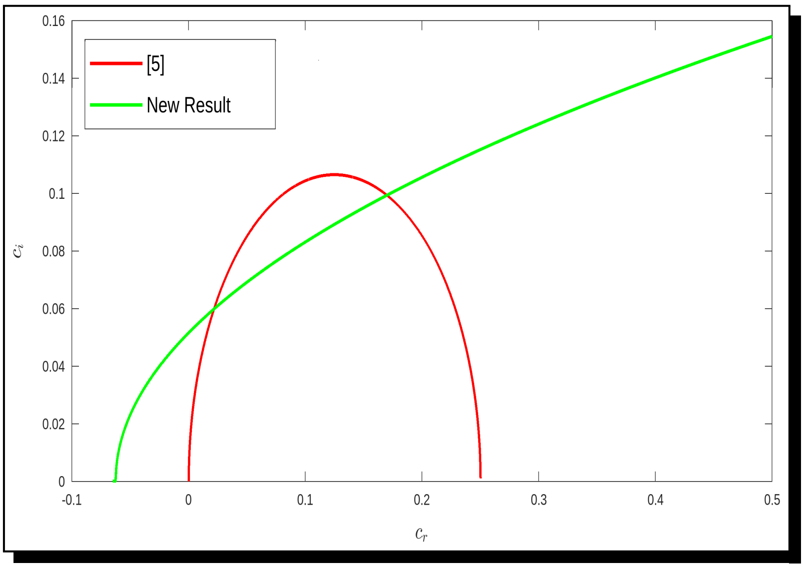

Figure 1. c_r versus c_i (intersection of parabola with semi ellipse)

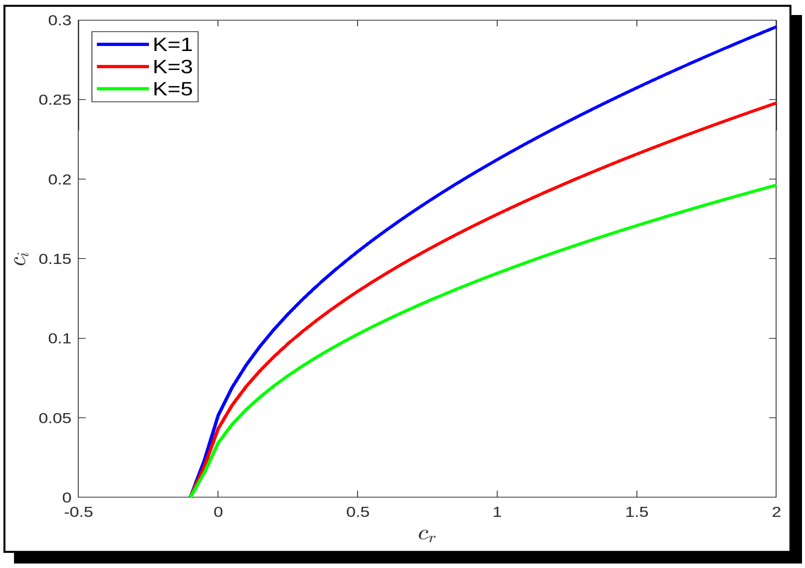

Figure 2. c_r versus c_i (parabolic instability region for different values of k)

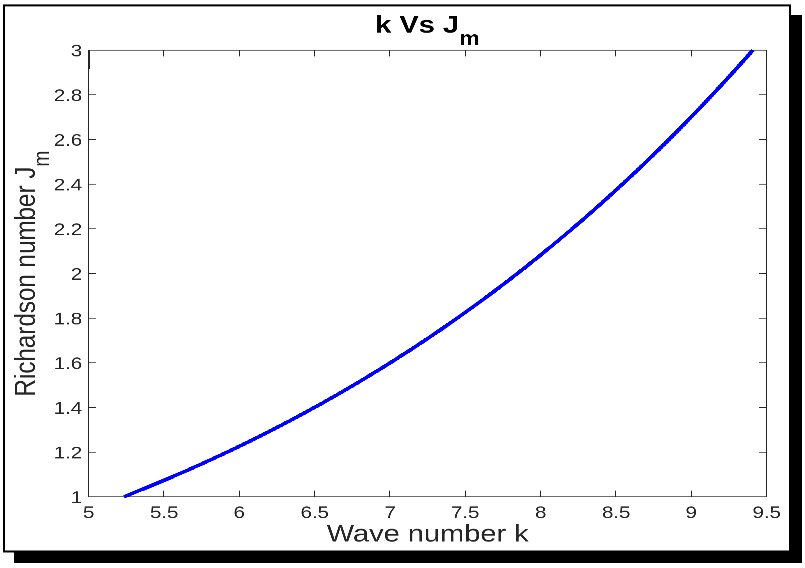

Figure 3. k versus J_m (wave number k versus Richardson number J_m)

Example 2. Let us consider the flow $U = \sin(z)$, $z \in [0,1]$, $\beta = \text{constant}$, $N^2 = z$.

In this case $U_{\min} = 0$, $U_{\max} = 0.84147$, $\lambda = \frac{0.5}{0.420735[\pi^2+k^2]+|1.84147|}$, $\lambda_c = 0.0321415$.

For $k \geq 0$, $\lambda < \lambda_c$ and hence the parabola $c_i^2 \leq 0.030280[c_r + 0.2103675]$, intersects with semi-ellipse.

Region of J_m for intersection:

$$0 \leq J_m \leq 0.25, \quad \text{for } k = 0,$$

$$0 \leq J_m \leq 0.25, \quad \text{for } k = 1,$$

$$0 \leq J_m \leq 0.25, \quad \text{for } k = 2.$$

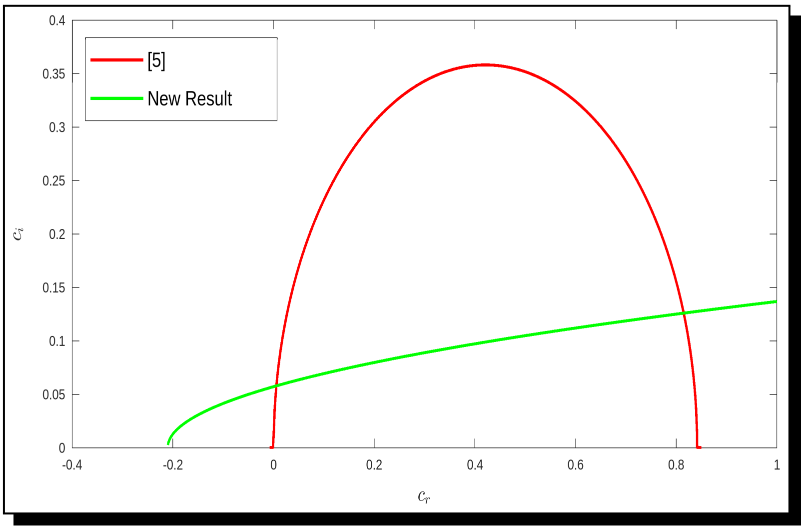

Figure 4. c_r versus c_i (intersection of parabola with semi ellipse)

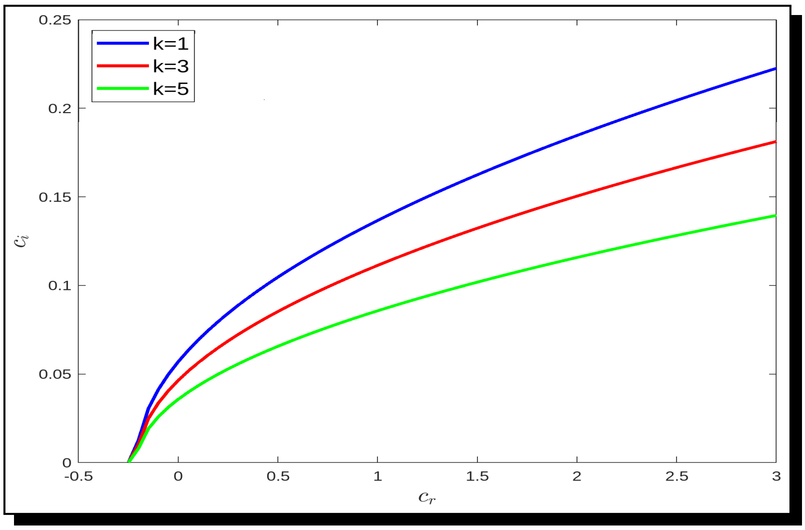

Figure 5. c_r versus c_i (parabolic instability region for different values of k)

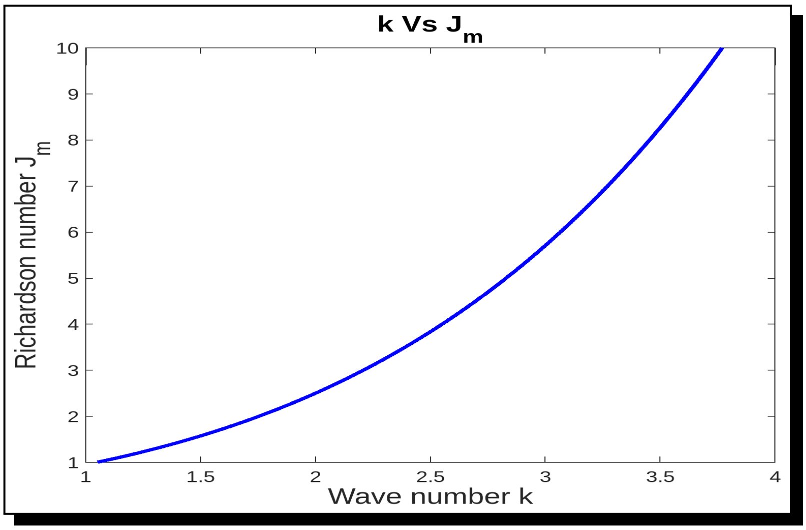

Figure 6. k versus J_m (wave number k versus Richardson number J_m)

4. Bounded Instability Regions

Theorem 4.1. *If* $|D(U)|^2_{\min} > 0$ *then the range of* (c_r, c_i) *is given by*

$$\left[c_r - \frac{U_{\min}+U_{\max}}{2}\right]^2 + c_i^2 + \frac{J_m c_i^2}{(1+A_1)^2} \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

where

$$A_1^2 = \frac{|D^2(U)-\beta|_{\max}[U_{\max}-U_{\min}] + |N^2|_{\max}}{|D(U)|^2_{\min}}.$$

Proof. Multiplying (2.1) by ϕ^*, integrating, applying (2.2) and comparing real parts, we get

$$\int_{z_1}^{z_2} [|D(\phi)|^2 + k^2|\phi|^2]dz + \int_{z_1}^{z_2} \frac{(D^2(U)-\beta)(U-c_r)}{|U-c|^2}|\phi|^2 dz - \int_{z_1}^{z_2} \frac{N^2[(U-c_r)^2 - c_i^2]}{|U-c|^4}|\phi|^2 dz = 0. \tag{4.1}$$

Using triangular inequality, we get

$$\int_{z_1}^{z_2} [|D(\phi)|^2 + k^2|\phi|^2]dz \le \int_{z_1}^{z_2} \frac{(D^2(U)-\beta)(U-c_r)}{|U-c|^2}|\phi|^2 dz + \int_{z_1}^{z_2} \frac{N^2}{|U-c|^4}[(U-c_r)^2 - c_i^2]|\phi|^2 dz. \tag{4.2}$$

Applying, $|U-c_r| \le |U_{\max}-U_{\min}|$, $(U-c_r)^2 - c_i^2 \le |U-c|^2$, we have

$$\int_{z_1}^{z_2} [|D(\phi)|^2 + k^2|\phi|^2]dz \le [|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}| + |N^2|_{\max}] \int_{z_1}^{z_2} \frac{|\phi|^2}{|U-c|^2} dz. \tag{4.3}$$

Using the Transformation

$$\phi = (U-c)\varphi, \tag{4.4}$$

we get

$$|D(\phi)|^2 \ge |U-c|^2|D(\varphi)|^2 - 2|U-c||D(U)||\varphi||D(\varphi)| + |D(U)|^2|\varphi|^2. \tag{4.5}$$

Using Cauchy-Schwartz inequality, we get

$$\int_{z_1}^{z_2} |U-c||D(U)||\varphi||D(\varphi)|dz \le BC, \tag{4.6}$$

where

$$B^2 = \int_{z_1}^{z_2} |D(U)|^2|\varphi|^2 dz, \tag{4.7}$$

$$C^2 = \int_{z_1}^{z_2} |U-c|^2[|D(\varphi)|^2 + k^2|\varphi|^2]dz. \tag{4.8}$$

Using (4.5), (4.6), (4.7) and (4.8), we get

$$\int_{z_1}^{z_2} [|D(\varphi)|^2 + k^2|\varphi|^2]dz \ge [C-B]^2. \tag{4.9}$$

Substituting (4.9), (4.4) in (4.3), we get

$$[C-B]^2 \le [|D^2(U)-\beta|_{\max}(U_{\max}-U_{\min}) + |N^2|_{\max}] \int_{z_1}^{z_2} |\varphi|^2 dz. \tag{4.10}$$

From (4.7), we have

$$B^2 \ge |D(U)|^2_{\min} \int_{z_1}^{z_2} |\varphi|^2 dz,$$

i.e.,

$$\frac{B^2}{|D(U)|^2_{\min}} \ge \int_{z_1}^{z_2} |\varphi|^2 dz. \tag{4.11}$$

Substituting (4.11) in (4.10), we get

$$[C-B]^2 \le A_1^2 B^2,$$

where

$$A_1^2 = \frac{|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}| + |N^2|_{\max}}{|D(U)|^2_{\min}}.$$

From the above equation, we get

$$B^2 \ge \frac{C^2}{[1+A_1]^2}. \tag{4.12}$$

Now,

$$\int_{z_1}^{z_2} N^2|\varphi|^2 dz \ge \frac{N^2}{|(D(U))^2|_{\min}} \int_{z_1}^{z_2} |D(U)|^2|\varphi|^2 dz,$$

i.e.,

$$\int_{z_1}^{z_2} N^2|\varphi|^2 dz \ge J_m B^2, \tag{4.13}$$

where

$$J_m = \left[\frac{N^2}{(D(U))^2}\right]_{\min}.$$

Substituting (4.12) in (4.13), we get

$$\int_{z_1}^{z_2} N^2|\varphi|^2 dz \ge \frac{J_m C^2}{[1+A_1]^2}. \tag{4.14}$$

Using the inequality $|U-c|^2 \ge c_i^2$ in (4.8), we have

$$C^2 \ge c_i^2 \int_{z_1}^{z_2} [|D(\varphi)|^2 + k^2|\varphi|^2] dz. \tag{4.15}$$

Substituting (4.15) in (4.14), we have

$$\int_{z_1}^{z_2} N^2|\varphi|^2 dz \ge \frac{J_m c_i^2}{(1+A_1)^2} \int_{z_1}^{z_2} [|D(\varphi)|^2 + k^2|\varphi|^2] dz. \tag{4.16}$$

From Howard [4], we have

$$\left[\left[c_r - \frac{U_{\min}+U_{\max}}{2}\right]^2 + c_i^2 - \left[\frac{U_{\max}-U_{\min}}{2}\right]^2\right] \int_{z_1}^{z_2} [|D(\varphi)|^2 + k^2|\varphi|^2] dz + \int_{z_1}^{z_2} N^2|\varphi|^2 dz \le 0. \tag{4.17}$$

Substituting (4.16) in (4.17), we get

$$\left[c_r - \frac{U_{\min}+U_{\max}}{2}\right]^2 + c_i^2 + \frac{J_m}{(1+A_1)^2} c_i^2 \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2.$$ □

Theorem 4.2. *The range of* (c_r, c_i) *is given by*

$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_0\left(1+\frac{A_2}{c_i}\right)^2 c_i^2 \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

where

$$A_2^2 = \frac{|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}| + |N^2|_{\max}}{\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]}.$$

Proof. Using Rayleigh Ritz inequality and $|U-c|^2 \geq c_i^2$, in (4.8), we get

$$\int_{z_1}^{z_2} |\varphi|^2 dz \leq \frac{C^2}{\frac{\pi^2}{(z_2-z_1)^2}+k^2}. \tag{4.18}$$

Substituting (4.18) in (4.10), we get

$$[C-B]^2 \geq \frac{A_2^2 C^2}{c_i^2},$$

where

$$A_2^2 = \frac{|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}| + |N^2|_{\max}}{\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]}.$$

From the above equation, we have

$$1-\frac{A_2}{c_i} \leq \frac{B}{C},$$

i.e.,

$$\text{if} \quad C^2 \leq \frac{B^2}{\left[1+\frac{A_2}{c_i}\right]^2} \leq \frac{B^2}{\left[1-\frac{A_2}{c_i}\right]^2}, \quad \text{then} \quad C^2\left[1+\frac{A_2}{c_i}\right]^2 \leq B^2. \tag{4.19}$$

Substituting (4.20) in (4.17), we get

$$\int_{z_1}^{z_2} N^2|\varphi|^2 dz \geq J_m C^2\left[1+\frac{A_2}{c_i}\right]^2. \tag{4.20}$$

Substitute (4.20) in (4.17), we get

$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_m\left(1+\frac{A_2}{c_i}\right)^2 c_i^2 \leq \left[\frac{U_{\max}-U_{\min}}{2}\right]^2. \qquad \square$$

Theorem 4.3. *The range of* (c_r, c_i) *is given by*

$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_m\left(1+\frac{A_3}{c_i}\right)^2 c_i^2 \leq \left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

where

$$A_3^2 = \frac{|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}| + |N^2|_{\max}}{k^2}.$$

Proof. From (4.8), dropping first term which is positive from RHS, we get

$$\int_{z_1}^{z_2} |\varphi|^2 dz \leq \frac{C^2}{k^2 c_i^2}. \tag{4.21}$$

Substituting (4.21) in (4.10), we get

$$[C-B]^2 \geq \frac{A_3^2 C^2}{c_i^2},$$

where

$$A_3^2 = \frac{|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}|+|N^2|_{\max}}{k^2}.$$

If

$$C^2 \le \frac{B^2}{\left[1+\frac{A_3}{c_i}\right]^2} \le \frac{B^2}{\left[1-\frac{A_3}{c_i}\right]^2}, then\ C^2\left[1+\frac{A_3}{c_i}\right]^2 \le B^2. \tag{4.22}$$

Substituting (4.22) in (4.13), we get

$$\int_{z_1}^{z_2} N^2|\varphi|^2 dz \ge J_m C^2\left[1+\frac{A_3}{c_i}\right]^2. \tag{4.23}$$

Substitute (4.23) in (4.17), we get

$$\left[c_r-\left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_m\left(1+\frac{A_3}{c_i}\right)^2 c_i^2 \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2. \tag{4.24}$$

□

5. Criterion for Wave Number

Theorem 5.1. *If $k \le k_c$, where*

$$k_c^2 = \frac{J_m[|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}|+|N^2|_{\max}]}{\left(\frac{U_{\max}-U_{\min}}{2}\right)^2},$$

implies stability.

Proof. From (4.24), we have

$$\left[c_r-\left(\frac{U_{\max}+U_{\min}}{2}\right)\right]^2 + c_i^2(1+J_m)+2A_3J_mc_i \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2 - J_mA_3^2.$$

For stability,

$$J_mA_3^2 \ge \left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

i.e.,

$$J_m\left[\frac{|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}|+|N^2|_{\max}}{k^2}\right] \ge \left(\frac{U_{\max}-U_{\min}}{2}\right)^2.$$

From above equation, we get

$$k \le \sqrt{\frac{J_m|D^2(U)-\beta|_{\max}|U_{\max}-U_{\min}|+|N^2|_{\max}}{\left(\frac{U_{\max}-U_{\min}}{2}\right)}}.$$

From above, we have the statement of the theorem. □

Example 3. (1) $U = 1-z^2$, $N^2=\beta=z$, $z\in[0,1]$,
for the above flow, $k \le 1.732$ implies stability.

(2) $U=1-z^2$, $N^2=z$, $\beta=a$ constant, $z\in[0,1]$,
for the above flow, $k\le 2$ implies stability.

(3) $U = 1 - z^2$, $N^2 = \beta = a$ constant, $z \in [0,1]$,
for the above flow, $k \leq 2$ implies stability.

(4) $U = 1 - z^2$, $N^2 = a$ constant, $\beta = z$, $z \in [0,1]$,
for the above flow, $k \leq 1.732$ implies stability.

(5) $U = z$, $N^2 = \beta = a$ constant, $z \in [0,1]$,
for the above flow, $k \leq 2.828$ implies stability.

(6) $U = z$, $N^2 = a$ constant, $\beta = z$, $z \in [0,1]$,
for the above flow, $k \leq 2$ implies stability.

6. Growth Rate

Theorem 6.1. *The upper bound for the growth rate of an unstable mode is.*

$$k^2 c_i^2 \leq \frac{|D^2(U) - \beta|_{\max}\left[\frac{U_{\max}-U_{\min}}{4}\right] + \frac{|N^2|_{\max}}{4} - |N^2|_{\min}}{\left[\frac{\pi^2}{k^2(z_2-z_1)^2} + 1\right]}.$$

Proof. Using $\frac{(U-c_r)}{|U-c|^2} \leq \frac{1}{2c_i}$, $\frac{1}{|U-c|^2} \leq \frac{1}{c_i^2}$ and Rayleigh-Ritz inequality in (4.2), we have

$$\left[\frac{\pi^2}{(z_2-z_1)^2} + k^2\right]\int_{z_1}^{z_2} |\phi|^2 dz \leq \frac{|D^2(U) - \beta|_{\max}\frac{c_i}{2} + \frac{|N^2|_{\max}}{4} - |N^2|_{\min}}{c_i^2}\int_{z_1}^{z_2} |\phi|^2 dz.$$

Since

$$c_i \leq \left(\frac{U_{\max} - U_{\min}}{2}\right),$$

we have

$$k^2 c_i^2 \leq \frac{|D^2(U) - \beta|_{\max}\left[\frac{U_{\max}-U_{\min}}{4}\right] + \frac{|N^2|_{\max}}{4} - |N^2|_{\min}}{\left[\frac{\pi^2}{k^2(z_2-z_1)^2} + 1\right]}.$$ □

7. Amplification Factor

Theorem 7.1. *The upper bound for amplification factor is*

$$c_i^4 \leq \frac{\left[|N^2|_{\max}|D^2(U) - \beta|_{\max}\left[\frac{U_{\max}-U_{\min}}{2}\right] + |N^4|_{\max} + |D^2(U) - \beta|^2_{\max}\left[\frac{U_{\max}-U_{\min}}{2}\right]^2\right]}{\left[\frac{\pi^4}{(z_2-z_1)^4} + \frac{2k^2\pi^2}{(z_2-z_1)^2} + k^4\right]}.$$

Proof. Multiplying (2.1) by $(\phi^*)''$, integrating, applying (2.2), we get

$$\int_{z_1}^{z_2} |D^2(\phi)|^2 dz + k^2\int_{z_1}^{z_2} |D(\phi)|^2 dz + \int_{z_1}^{z_2}\left(\frac{N^2}{(U-c)^2} - \frac{(D^2(U)-\beta)}{U-c}\right)\phi\left(\phi^*\right)'' dz = 0. \tag{7.1}$$

From (2.1), taking complex conjugate, we have

$$(\phi^*)'' = \left(k^2 + \frac{(D^2(U)-\beta)}{U-c^*} - \frac{N^2}{(U-c^*)^2}\right)\phi^*. \tag{7.2}$$

Substituting (7.2) in (7.1) and equating real parts, we get

$$\int_{z_1}^{z_2}|D^2(\phi)|^2dz+k^2\int_{z_1}^{z_2}|D(\phi)|^2dz+k^2\int_{z_1}^{z_2}\left(\frac{N^2\left[(U-c_r)^2-c_i^2\right]}{|U-c|^4}\right)|\phi|^2dz$$
$$+2\int_{z_1}^{z_2}\frac{N^2[(D^2(U)-\beta)]}{|U-c|^4}(U-c_r)|\phi|^2dz-\int_{z_1}^{z_2}\frac{N^4}{|U-c|^4}|\phi|^2dz$$
$$-k^2\int_{z_1}^{z_2}\left(\frac{(D^2(U)-\beta)(U-c_r)}{|U-c|^2}\right)|\phi|^2dz-\int_{z_1}^{z_2}\left(\frac{(D^2(U)-\beta)^2}{|U-c|^4}\right)|\phi|^2dz=0. \tag{7.3}$$

Multiplying (4.1) by k^2 and adding with (7.3), we get

$$\int_{z_1}^{z_2}|D^2(\phi)|^2dz+2k^2\int_{z_1}^{z_2}|D(\phi)|^2dz+k^4\int_{z_1}^{z_2}|\phi|^2dz+2\int_{z_1}^{z_2}\frac{N^2[(D^2(U)-\beta)]}{|U-c|^4}(U-c_r)|\phi|^2dz$$
$$-\int_{z_1}^{z_2}\frac{N^4}{|U-c|^4}|\phi|^2dz-\int_{z_1}^{z_2}\left(\frac{(D^2(U)-\beta)^2}{|U-c|^4}\right)|\phi|^2dz=0.$$

Using Rayleigh Ritz inequality,

$$\frac{|U-c|}{|U-c|^2}\le\frac{1}{2c_i},\qquad \frac{1}{|U-c|^2}\le\frac{1}{c_i^2},$$

we get

$$\left[\frac{\pi^4}{(z_2-z_1)^4}+\frac{2k^2\pi^2}{(z_2-z_1)^2}+k^4\right]\int_{z_1}^{z_2}|\phi|^2dz$$
$$\le\left[\frac{|N^2|_{\max}}{c_i^3}|D^2(U)-\beta|_{\max}+\frac{|N^4|_{\max}}{c_i^4}+\frac{|D^2(U)-\beta|^2{}_{\max}}{c_i^2}\right]\int_{z_1}^{z_2}|\phi|^2dz.$$

Since $c_i\le\left(\frac{U_{\max}-U_{\min}}{2}\right)$, we have

$$c_i^4\le\frac{\left[|N^2|_{\max}|D^2(U)-\beta|_{\max}\left[\frac{U_{\max}-U_{\min}}{2}\right]+|N^4|_{\max}+|D^2(U)-\beta|^2{}_{\max}\left[\frac{U_{\max}-U_{\min}}{2}\right]^2\right]}{\left[\frac{\pi^4}{(z_2-z_1)^4}+\frac{2k^2\pi^2}{(z_2-z_1)^2}+k^4\right]}.$$ □

8. Conclusion

We consider Taylor-Goldstein problem in β-plane. First, we derived an unbounded instability region which intersects with semi ellipse region under certain condition. Second, we derived a bounded instability region depending on stratification parameter, Coriolis function, shear and vorticity function. Third, we obtained a criterion for stability. The stability results have been illustrated with standard examples. Fourth, we obtained an upper bound for amplification factor and growth rate of an unstable mode.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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