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Research Article

Generalized Logarithmic Divergence Measure for Intuitionistic Fuzzy Matrix

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Abstract. For solving multi-criterion decision making problems, we in this paper propose a parametric generalized logarithmic divergence measure for intuitionistic fuzzy matrices. The validity of a symmetric divergence measure has been established for the proposed measure. Also, the properties (compliment, transitivity, concavity and symmetricity) of this measure for intuitionistic fuzzy matrices are studied. Application of the proposed measure has been illustrated through a decision-making problem in trade market.

Keywords. Intuitionistic fuzzy matrix, Mathematical operation, Generalized divergence measure, Decision-making problem

Mathematics Subject Classification (2020). 03B52, 03E72, 15B15

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1. Introduction

Atanassov [1] *introduced intuitionistic fuzzy sets* (IFSs) for linguistic evaluation to address the ambiguity in *fuzzy sets* (FSs). The purpose for this was also to obtain more reliable results than fuzzy sets because IFSs cover degrees of belongingness and non-belongingness. In terms of maintaining ambiguity and exaggeration, IFSs are more prominent than FSs in managing the imprecision and uncertainty of real-life problems. Fuzzy matrix theory was first introduced by Thomason [17] as a branch of fuzzy set theory. Fuzzy matrices are extremely useful in dealing with large amount of information such as Agriculture, Medicine and any other field dealing with uncertainty in information. Pal *et al*. [14] have introduced the concept of intuitionistic fuzzy matrices as an extension to the theory of ordinary fuzzy matrices, which is the hybridization of intuitionistic fuzzy set and matrix theory.

In information theory, there exist several information measures such as entropy, distance, divergence and similarity measure, which are playing an important role to manage the uncertain information. Among the existing information measures, divergence measure is an important tool to measure the degree of discrimination between two sets that can be effectively applied in different fields. Vlachos and Sergiadis [20] introduced an entropy and divergence measures for IFSs. Kullback and Leibler [9] obtained the measure of directed divergence between two probability distributions. Bhandari and Pal [3] presented some axioms to describe the measure of directed divergence between two fuzzy sets. Fan and Xie [6] introduced the divergence measure based on exponential operation. Ghosh *et al*. [7] gave an application in automated leukocyte recognition of fuzzy divergence. Montes *et al*. [13] studied the special classes of divergence measure. Mishra *et al*. [12] introduced a divergence measure for intuitionistic fuzzy sets based on ELECTRE method. Bajaj and Hooda [2] introduced the measure of fuzzy directed divergence measure. Bhatia and Singh [4] developed four fuzzy divergence measures. Tomar and Ohlan [18] introduced a generalised divergence measure based on exponential distribution. Verma and Sharma [19] introduced a generalised relative information measure for intuitionistic fuzzy sets.

Emam and Ragab [5] defined the determinant and adjoints of a square fuzzy matrix. Meenakshi and Kaliraja [11] applied the theory of interval-valued intuitionistic fuzzy matrices in medical area. Emam and Fndh [5] proved some results of max-min and min-max compositions of bifuzzy matrices. Khan and Pal [8] introduced the generalised inverses of intuitionistic fuzzy matrices. For fuzzy soft matrices, Sharma *et al*. [16] suggested a logarithmic entropy measure for fuzzy soft matrix and provided its application in decision making and data reduction problems. Inspired by the above work in the field of fuzzy matrices, in this paper, here we propose a parametric generalized logarithmic divergence measure for intuitionistic fuzzy matrices. The validity of a symmetric divergence measure has been established for the proposed measure. Also, the properties (compliment, transitivity, concavity and symmetricity) of this measure for intuitionistic fuzzy matrices are studied. Application of the proposed measure has been illustrated through a decision-making problem in trade market.

In Section 2, some basic definitions of intuitionistic fuzzy matrix and some mathematical operations on intuitionistic fuzzy matrices are defined. In Section 3, a generalized divergence measure for intuitionistic fuzzy matrix is proposed. Further, some properties of generalized divergence measure for intuitionistic fuzzy matrix are also discussed. In Section 4, an application and comparison of the proposed measure are discussed in decision making problems in trade market.

2. Preliminaries

Some basic definition and mathematical operation for intuitionistic fuzzy matrix are discussed in this section as follows:

Definition 2.1 (Intuitionistic Fuzzy Matrix [17])**.** Let P^* be an intuitionistic fuzzy matrix of order $m \times n$ which defined as

$$P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}.$$

Here, $\widetilde{\eta}_{P^*}(\alpha_{ij})$ and $\widetilde{\theta}_{P^*}(\alpha_{ij})$ are membership and non-membership values of intuitionistic fuzzy matrix, respectively. These values satisfy the condition $0 \le \widetilde{\eta}_{P^*}(\alpha_{ij}) \le 1$, $0 \le \widetilde{\theta}_{P^*}(\alpha_{ij}) \le 1$ and $0 \le \widetilde{\eta}_{P^*}(\alpha_{ij}) + \widetilde{\theta}_{P^*}(\alpha_{ij}) \le 1$.

The matrix representation of intuitionistic fuzzy matrix P^* as follows:

$$\begin{aligned} P^* &= [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n} \\ &= \begin{bmatrix} \widetilde{\eta}(\alpha_{11}), \widetilde{\theta}(\alpha_{11}) \widetilde{\eta}(\alpha_{12}), \widetilde{\theta}(\alpha_{12}) & \cdots & \widetilde{\eta}(\alpha_{1n}), \widetilde{\theta}(\alpha_{1n}) \\ \vdots & \ddots & \vdots \\ \widetilde{\eta}(\alpha_{m1}), \widetilde{\theta}(\alpha_{m1}) \widetilde{\eta}(\alpha_{m2}), \widetilde{\theta}(\alpha_{m2}) & \cdots & \widetilde{\eta}(\alpha_{mn}), \widetilde{\theta}(\alpha_{mn}) \end{bmatrix}_{m\times n}. \end{aligned}$$

Definition 2.2 (Boolean Intuitionistic Fuzzy Matrix [14])**.** An intuitionistic fuzzy matrix $P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}$ is said to be Boolean if its all elements are either 0 or 1.

Definition 2.3 (Most Intuitionistic Fuzzy Matrix [14])**.** An intuitionistic fuzzy matrix $P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}$ is said to be Boolean if it's all elements is 0.5.

Definition 2.4 (Rectangle Intuitionistic Fuzzy Matrix [14])**.** An intuitionistic fuzzy matrix $P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}$ is said to be rectangular intuitionistic fuzzy matrix if $m \neq n$.

Definition 2.5 (Square Intuitionistic Fuzzy Matrix [14])**.** An intuitionistic fuzzy matrix $P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}$ is said to be square intuitionistic fuzzy matrix if $m = n$.

2.1 Operations of Intuitionistic Fuzzy Matrices

Let $P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}$ and $Q^* = [\widetilde{\eta}_{Q^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij})]_{m\times n}$ be two intuitionistic fuzzy matrices. Here, we define some mathematical operation as follows:

Addition of two IFM

$$\begin{aligned} P^* + Q^* &= [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n} + [\widetilde{\eta}_{Q^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij})]_{m\times n} \\ &= \{\max(\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\eta}_{Q^*}(\alpha_{ij})), \min(\widetilde{\theta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij}))\} \quad \forall\ i,j. \end{aligned}$$

Subtraction of two IFM

$$\begin{aligned} P^* - Q^* &= [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n} - [\widetilde{\eta}_{Q^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij})]_{m\times n} \\ &= \{\min(\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\eta}_{Q^*}(\alpha_{ij})), \max(\widetilde{\theta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij}))\} \quad \forall\ i,j. \end{aligned}$$

Product of two IFM

$$\begin{aligned} P^* \times Q^* &= [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n} \times [\widetilde{\eta}_{Q^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij})]_{m\times n} \\ &= \{\max(\min(\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\eta}_{Q^*}(\alpha_{ij}))), \min(\max(\widetilde{\theta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{Q^*}(\alpha_{ij})))\} \quad \forall\ i,j. \end{aligned}$$

Complement of Intuitionistic Fuzzy Matrix

Let $P^* = [\widetilde{\eta}_{P^*}(\alpha_{ij}), \widetilde{\theta}_{P^*}(\alpha_{ij})]_{m\times n}$ be an intuitionistic fuzzy matrix. The complement of the matrix is defined as

$$P^{*^c} = [\widetilde{\theta}_{P^*}(\alpha_{ij}), \widetilde{\eta}_{P^*}(\alpha_{ij})]_{m\times n}.$$

3. Divergence Measure for Intuitionistic Fuzzy Matrix

In fuzzy, information measures are applicable to deal with the problem related to uncertainty. In this section, firstly we review some divergence measure for intuitionistic fuzzy sets are as following:

Firstly, Vlachos and Sergiadis [20] introduced a divergence measure for IFSs as

$$\mathcal{D}_{VS}(P^*,Q^*)=\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\ln\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\eta}_{P^*}(\alpha_{ij})+\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)+\widetilde{\theta}_{P^*}(\alpha_{ij})\ln\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\theta}_{P^*}(\alpha_{ij})+\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\right].$$

Zhang and Jiang [22] introduced a divergence measure for IFSs as

$$\mathcal{D}_{ZJ}(P^*,Q^*)=\sum_{i=1}^{n}\left[\frac{(\widetilde{\eta}_{P^*}(\alpha_{ij})+1-\widetilde{\theta}_{p^*}(\alpha_{ij})}{2}\ln\left(\frac{(\widetilde{\eta}_{P^*}(\alpha_{ij})+1-\widetilde{\theta}_{p^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{p^*}(\alpha_{ij})+2+\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\right.$$
$$\left.+\frac{(\widetilde{\theta}_{P^*}(\alpha_{ij})+1-\widetilde{\eta}_{p^*}(\alpha_{ij})}{2}\ln\left(\frac{\widetilde{\theta}_{p^*}(\alpha_{ij})+1-(\widetilde{\eta}_{P^*}(\alpha_{ij}))}{\frac{1}{2}(\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{p^*}(\alpha_{ij})+2+\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right].$$

Mao *et al.* [10] defined a divergence measure as follows:

$$\mathcal{D}_{JM}(P^*,Q^*)=\sum_{i=1}^{n}\left[\widetilde{\vartheta}_{P^*}(\alpha_{ij})\ln\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\vartheta}_{P^*}(\alpha_{ij})+\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\right.$$
$$\left.+|\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})|\ln\left(\frac{|\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})|}{\frac{1}{2}(|\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})|-|\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij})|)}\right)\right].$$

Wei and Ye [21] proposed a intuitionistic fuzzy divergence measure as

$$\mathcal{D}_{WY}(P^*,Q^*)=\frac{1}{n}\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\eta}_{P^*}(\alpha_{ij})+\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\theta}_{P^*}(\alpha_{ij})+\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\right.$$
$$\left.+\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{\frac{1}{2}(\widetilde{\vartheta}_{P^*}(\alpha_{ij})+\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\right].$$

Generalized Logarithmic Divergence Measure for Intuitionistic Fuzzy Matrix

$$\mathcal{D}_{\lambda}(P^*,Q^*)=\left[\frac{2^{\lambda}}{mn\log 2}\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right.\right.\right.$$
$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)$$
$$\left.\left.+\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\right]\right]$$
$$+\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\right.\right.$$
$$+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)$$
$$\left.\left.\left.+\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\right]\right]\right].$$

Properties 3.1. *Let $\mathcal{D}_{\lambda}:[IFM]_{m\times n}\times[IFM]_{m\times n}\to K$ be the divergence measure function defined on intuitionistic fuzzy matrices. P^*, Q^* and R^* be intuitionistic fuzzy matrices which satisfies the axioms.*

(i) $\mathcal{D}_{\lambda}(P^*,Q^*)\geq 0$.

(ii) *For $\lambda\neq 1$, $\mathcal{D}_{\lambda}(P^*,Q^*)=0$ if only if $P^*=Q^*$.*

(iii) *For $\lambda=1$, $\mathcal{D}_{\lambda}(P^*,Q^*)=0$.*

(iv) $\mathcal{D}_{\lambda}(P^*,Q^*)=\mathcal{D}_{\lambda}(Q^*,P^*)$.

Proof. (i) $\mathcal{D}_\lambda(P^*,Q^*) \geq 0$:
Consider,

$$\begin{aligned}\mathcal{D}_\lambda(P^*,Q^*) = \Bigg[\frac{2^\lambda}{mn\log 2}\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right) \\ +\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right) \\ +\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\Bigg]\Bigg] \\ +\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right) \\ +\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right) \\ +\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\Bigg].\end{aligned}$$

By the definition, the prove is obvious because all membership, non-membership and hesitancy values are lies between 0 and 1 and logarithmic values is also positive. Here, the divergence measure is also

$$\mathcal{D}_\lambda(P^*,Q^*) \geq 0. \tag{3.1}$$

(ii) For $\lambda \neq 1$, $\mathcal{D}_\lambda(P^*,Q^*) = 0$ if only if $P^* = Q^*$
Assume, $P^* = Q^* \Rightarrow \widetilde{\eta}_{P^*} = \widetilde{\eta}_{Q^*}$, $\widetilde{\theta}_{P^*} = \widetilde{\theta}_{Q^*}$, $\widetilde{\vartheta}_{P^*} = \widetilde{\vartheta}_{Q^*}$.
Consider,

$$\begin{aligned}\mathcal{D}_\lambda(P^*,Q^*) = \Bigg[\frac{2^\lambda}{mn\log 2}\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right) \\ +\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right) \\ +\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg] \\ +\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right) \\ +\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right) \\ +\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\Bigg] \\ = \frac{2^\lambda}{mn\log 2}[0] = 0. \end{aligned} \tag{3.2}$$

(iii) For $\lambda=1$, $\mathcal{D}_\lambda(P^*,Q^*)=0$
Consider, the divergence measure for $\lambda=1$, we get

$$\begin{aligned}
\mathcal{D}_\lambda(P^*,Q^*) &= \Bigg[\frac{2^\lambda}{mn\log 2}\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-1)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-1)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-1)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\\
&\quad+\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-1)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-1)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-1)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\Bigg]\\
&= \frac{2^\lambda}{mn\log 2}[0]=0.
\end{aligned} \tag{3.3}$$

(iv) $\mathcal{D}_\lambda(P^*,Q^*)=\mathcal{D}_\lambda(Q^*,P^*)$
Consider,

$$\begin{aligned}
\mathcal{D}_\lambda(P^*,Q^*) &= \Bigg[\frac{2^\lambda}{mn\log 2}\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\\
&\quad+\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\Bigg]\\
&= \Bigg[\frac{2^\lambda}{mn\log 2}\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)\\
&\quad+\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]
\end{aligned}$$

$$+\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right.\right.$$
$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)$$
$$\left.\left.+\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\right]\right]\Bigg]\Bigg]$$
$$=\mathcal{D}_\lambda(Q^*,P^*).$$

Theorem 3.1. *Let P^* and Q^* be intuitionistic fuzzy matrices then*

$$\mathcal{D}_\lambda(P^*\cup Q^*,P^*\cap Q^*)=\mathcal{D}_\lambda(P^*,Q^*).$$

To prove this theorem, we consider the set in the form of two separate subsets which defined as follows.

Let P^* and Q^* be two intuitionistic fuzzy matrices defined over the universe of discourse $A=\{\alpha_{ij}\mid\alpha_{ij}\in A\}$. We consider two submatrices which defined as

$$A_1=\{P^*(\alpha_{ij})\in P^* \text{ or } Q^*(\alpha_{ij})\in Q^*;\ P^*\le Q^*\},$$
$$A_2=\{P^*(\alpha_{ij})\in P^* \text{ or } Q^*(\alpha_{ij})\in Q^*;\ P^*\ge Q^*\}.$$

When $P^*(\alpha_{ij})$, $Q^*(\alpha_{ij})\in A_1$, $P^*\le Q^*$

$$\widetilde{\eta}_{P^*}(\alpha_{ij})\le\widetilde{\eta}_{Q^*}(\alpha_{ij}) \text{ and } \widetilde{\theta}_{P^*}(\alpha_{ij})\ge\widetilde{\theta}_{Q^*}(\alpha_{ij})$$

When $P^*(\alpha_{ij})$, $Q^*(\alpha_{ij})\in A_1$, $P^*\ge Q^*$

$$\widetilde{\eta}_{P^*}(\alpha_{ij})\ge\widetilde{\eta}_{Q^*}(\alpha_{ij}) \text{ and } \widetilde{\theta}_{P^*}(\alpha_{ij})\le\widetilde{\theta}_{Q^*}(\alpha_{ij})$$

Proof.

$\mathcal{D}_\lambda(P^*\cup Q^*,P^*\cap Q^*)$

$$=\left[\frac{2^\lambda}{mn\log 2}\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij}))}\right)\right.\right.\right.$$
$$+\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij}))}\right)$$
$$\left.\left.+((1-\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij})\log\left(\frac{1-\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij})}{\begin{pmatrix}(\lambda(1-\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij}))\\+((1-\lambda)(1-\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij})))\end{pmatrix}}\right)\right]\right]$$
$$+\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij}))}\right)\right.\right.$$
$$+\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij}))}\right)$$
$$\left.\left.+(1-\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij}))}{\begin{pmatrix}(\lambda(1-\widetilde{\eta}_{P^*\cap Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cap Q^*}(\alpha_{ij}))\\+((1-\lambda)(1-\widetilde{\eta}_{P^*\cup Q^*}(\alpha_{ij})-\widetilde{\theta}_{P^*\cup Q^*}(\alpha_{ij})))\end{pmatrix}}\right)\right]\right]\Bigg]\Bigg]$$

$$= \left[\frac{2^{\lambda}}{mn\log 2}\left[\sum_{\alpha_{ij}\in A_1}\left[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\right.\right.\right.$$

$$+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)$$

$$\left.+(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))+((1-\lambda)(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})))}\right)\right]$$

$$+\sum_{\alpha_{ij}\in A_2}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right.$$

$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)$$

$$\left.\left.+(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))+((1-\lambda)(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij})))}\right)\right]\right]$$

$$+\left[\sum_{\alpha_{ij}\in A_1}\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right.$$

$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))+((1-\lambda)(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij})))}\right)$$

$$+\sum_{\alpha_{ij}\in A_2}\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)$$

$$+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)$$

$$\left.\left.\left.+1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))+((1-\lambda)(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})))}\right)\right]\right]\right]$$

$$=\mathcal{D}_{\lambda}(P^*,Q^*).$$ □

Theorem 3.2. *Let P^* and Q^* be intuitionistic fuzzy matrices and P^{c^*} and Q^{c^*} be compliment of matrices then*

$$\mathcal{D}_{\lambda}(P^*,Q^*)=\mathcal{D}_{\lambda}(P^{c^*},Q^{c^*}).$$

Proof. Consider,

$\mathcal{D}_{\lambda}(P^{c^*},Q^{c^*})$

$$= \left[\frac{2^{\lambda}}{mn\log 2}\left[\sum_{i=1}^{n}\left[\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\right.\right.\right.$$

$$+\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij})))+(1-\lambda)(1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]$$

$$+\left[\sum_{i=1}^{n}\left[\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)\right.\right.$$

$$+\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))}{(\lambda 1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))+(1-\lambda)(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})))}\right)\Bigg]\Bigg]\Bigg]$$

$$=\left[\frac{2^{\lambda}}{mn\log 2}\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right.\right.\right.$$

$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})))+(1-\lambda)(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))))}\right)\Bigg]\Bigg]$$

$$+\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\right.\right.$$

$$+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda 1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))+(1-\lambda)1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})}\right)\Bigg]\Bigg]\Bigg]$$

$$=\mathcal{D}_{\lambda}(P^*,Q^*).$$ □

Theorem 3.3. *Let P^* and Q^* be intuitionistic fuzzy matrices and P^{c^*} and Q^{c^*} be compliment of matrices then*

$$\mathcal{D}_{\lambda}(P^*,Q^{c^*})=\mathcal{D}_{\lambda}(P^{c^*},Q^*).$$

Proof. Consider,

$$\mathcal{D}_{\lambda}(P^*,Q^{c^*})-\mathcal{D}_{\lambda}(P^{c^*},Q^*)$$

$$=\left[\frac{2^{\lambda}}{mn\log 2}\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\right.\right.\right.$$

$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})))+(1-\lambda)(1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))))}\right)\Bigg]\Bigg]$$

$$+\left[\sum_{i=1}^{n}\left[\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\right.\right.$$

$$+\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))}{(\lambda 1-\widetilde{\theta}_{Q^*}(\alpha_{ij})-\widetilde{\eta}_{Q^*}(\alpha_{ij}))+(1-\lambda)1-\widetilde{\eta}_{P^*}(\alpha_{ij})-\widetilde{\theta}_{P^*}(\alpha_{ij})))}\right)\Bigg]\Bigg]\Bigg]$$

$$-\left[\frac{2^\lambda}{mn\log 2}\left[\sum_{i=1}^{n}\left[\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\right.\right.\right.$$

$$+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij})))+(1-\lambda)(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))))}\right)\Bigg]\Bigg]$$

$$+\left[\sum_{i=1}^{n}\left[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)\right.\right.$$

$$+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)$$

$$+(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))\log\left(\frac{1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))}{(\lambda(1-\widetilde{\eta}_{Q^*}(\alpha_{ij})-\widetilde{\theta}_{Q^*}(\alpha_{ij}))+(1-\lambda)1-\widetilde{\theta}_{P^*}(\alpha_{ij})-\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\Bigg]$$

$$=0.$$ □

Theorem 3.4. *Let P^* and Q^* be intuitionistic fuzzy matrices and P^{c^*} and Q^{c^*} be compliment of matrices then*

$$\mathcal{D}_\lambda(P^*,Q^*)+\mathcal{D}_\lambda(P^*,Q^{c^*})=\mathcal{D}_\lambda(P^{c^*},Q^{c^*})+\mathcal{D}_\lambda(P^{c^*},Q^*).$$

Proof. From Theorems 3.2 and 3.4, we get

$$\mathcal{D}_\lambda(P^*,Q^*)=\mathcal{D}_\lambda(P^{c^*},Q^{c^*}), \tag{3.4}$$

$$\mathcal{D}_\lambda(P^*,Q^{c^*})=\mathcal{D}_\lambda(P^{c^*},Q^*). \tag{3.5}$$

Adding the equation (3.4) and (3.5), we get the result

$$\mathcal{D}_\lambda(P^*,Q^*)+\mathcal{D}_\lambda(P^*,Q^{c^*})=\mathcal{D}_\lambda(P^{c^*},Q^{c^*})+\mathcal{D}_\lambda(P^{c^*},Q^*).$$ □

4. Application

Divergence measure is a one of the useful information measures which measures the degree of discrimination between two sets. Here, the proposed generalized divergence measure is used in *multi-criterion decision making* (MCDM) problem related to the field of share marketing. In share market, there are many factors on which the decision of investor depends such as (time horizon, volatility, revenue and earnings growth, risk factor and return on investment). The proposed generalized divergence measure is helpful in making decision.

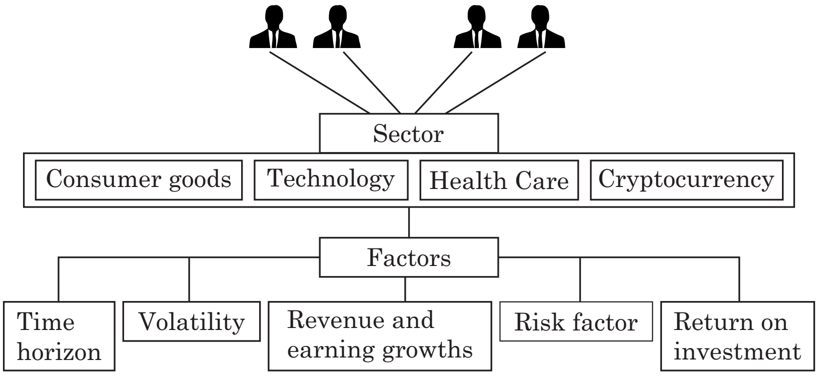


Figure 1. An outline for making decision

Algorithm

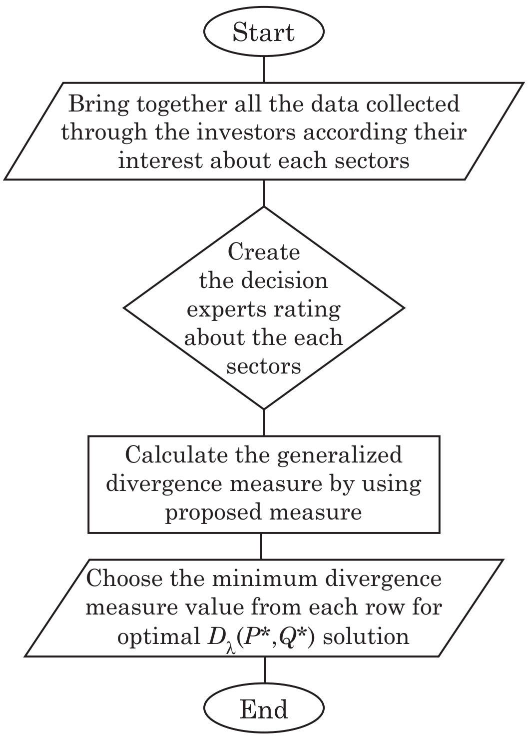
In this section, we write an algorithm for solving a multi-criterion decision making problem for intuitionistic fuzzy matrix. This contains four step to solve the problem by applying the proposed generalized divergence measure and choose an optimal solution.

Step 1: Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ and $\mathcal{F} = (f_1, f_2, \dots, f_n)$ be the set of choices and criteria, respectively. Assume that the characteristics of the choice $\mathcal{C}_k$ in terms of criteria $\mathcal{F}_k$.

Step 2: Further, we arrange all the information in the form of intuitionistic fuzzy matrices.

Step 3: Calculate the generalized divergence measures between the ideal matrices and the matrices corresponding to criterion-choices.

Step 4: Arrange the value of divergence measure in decreasing order and choose the option correspondence to minimum value and this choice is known as optimum solution.



Flow Chart 1. Algorithm for solving MCDM problems

Illustration

Let us consider, there are four investors $\tau = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, $\varphi = \{\varphi_1 =$ consumer goods, $\varphi_2 =$ health care, $\varphi_3 =$ technology, $\varphi_4 =$ cryptocurrency$\}$ are sectors, respectively. The sets of factors on $\mathcal{D} = \{\partial_1^* =$ time horizon, $\partial_2^* =$ volatility, $\partial_3^* =$ revenue and earning growths, $\partial_4^* =$ risk factor, $\partial_5^* =$ return on investment.

Step 1: Organise the gathering information in the form of intuitionistic fuzzy matrix.

The Matrix Representation of Investors Corresponding to Factors

$$\begin{bmatrix} [0.81,0.10,0.09] & [0.52,0.13,0.35] & [0.40,0.60,0.00] & [0.64,0.20,0.16] & [0.20,0.70,0.10] \\ [0.90,0.10,0.00] & [0.61,0.22,0.17] & [0.80,0.10,0.10] & [0.72,0.10,0.08] & [0.00,0.90,0.10] \\ [0.60,0.28,0.12] & [0.50,0.20,0.30] & [0.80,0.20,0.00] & [0.80,0.10,0.10] & [0.90,0.10,0.00] \\ [0.65,0.26,0.09] & [0.70,0.10,0.20] & [0.70,0.00,0.30] & [0.20,0.70,0.10] & [0.69,0.12,0.19] \end{bmatrix}$$

The Matrix Representation of Factors Corresponding to Sectors

$$\begin{bmatrix} [0.90,0.10,0.00] & [0.55,0.32,0.13] & [0.72,0.22,0.06] & [0.70,0.30,0.00] \\ [0.90,0.10,0.00] & [0.60,0.30,0.10] & [0.82,0.10,0.08] & [0.35,0.65,0.00] \\ [0.80,0.10,0.10] & [0.77,0.22,0.01] & [0.77,0.10,0.13] & [0.80,0.00,0.10] \\ [0.32,0.68,0.00] & [0.90,0.00,0.10] & [0.10,0.80,0.10] & [0.20,0.50,0.30] \\ [0.20,0.78,0.02] & [0.90,0.10,0.00] & [0.80,0.00,0.20] & [0.70,0.10,0.20] \end{bmatrix}$$

Step 2: The proposed generalized logarithmic divergence measure for intuitionistic fuzzy matrix is used to calculate the discrimination between the sectors and the investors based on some factor.

Now, we find the divergence measure between sets (τ,φ) and (φ,∂^*) by using proposed measure for $\lambda = 0.2$, $\lambda = 0.4$, $\lambda = 0.6$, $\lambda = 0.8$

$$\begin{aligned}\mathcal{D}_\lambda(P^*,Q^*) = \Bigg[\frac{2^\lambda}{mn\log 2}\Bigg[\sum_{i=1}^{n}\Bigg[&\widetilde{\eta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{Q^*}(\alpha_{ij}))}\right)\\ &+\widetilde{\theta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{Q^*}(\alpha_{ij}))}\right)\\ &+\widetilde{\vartheta}_{P^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{P^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{P^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{Q^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\\ &+\Bigg[\sum_{i=1}^{n}\Bigg[\widetilde{\eta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\eta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\eta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\eta}_{P^*}(\alpha_{ij}))}\right)\\ &+\widetilde{\theta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\theta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\theta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\theta}_{P^*}(\alpha_{ij}))}\right)\\ &+\widetilde{\vartheta}_{Q^*}(\alpha_{ij})\log\left(\frac{\widetilde{\vartheta}_{Q^*}(\alpha_{ij})}{(\lambda\widetilde{\vartheta}_{Q^*}(\alpha_{ij})+(1-\lambda)\widetilde{\vartheta}_{P^*}(\alpha_{ij}))}\right)\Bigg]\Bigg]\Bigg].\end{aligned}$$

From Table 1, we observed that for $\lambda = 0.4$, the minimum value of divergence measure between investors and sectors on the basis of factors are as

$$\mathcal{D}_{\lambda=0.4}(\tau_1^*,\varphi_4^*) = 3.33833, \quad \mathcal{D}_{\lambda=0.4}(\tau_2^*,\varphi_4^*) = 3.74967,$$
$$\mathcal{D}_{\lambda=0.4}(\tau_3^*,\varphi_1^*) = 2.79387, \quad \mathcal{D}_{\lambda=0.4}(\tau_4^*,\varphi_2^*) = 3.21265.$$

Table 1. Logarithmic divergence measure among investors and sectors

Parameter	Sectors / Investors	φ_1	φ_2	φ_3	φ_4
$\lambda = 0.4$	τ_1	4.05640	3.75547	3.47058	**3.33833**
	τ_2	6.79794	4.11627	3.77219	**3.74967**
	τ_3	**2.79387**	3.86901	3.47363	3.15602
	τ_4	3.47292	**3.21265**	4.18658	3.51432
$\lambda = 0.6$	τ_1	4.27519	4.00627	3.92613	**3.67534**
	τ_2	4.54158	4.38982	4.26793	**4.13653**
	τ_3	**2.74838**	3.73317	3.37562	3.13152
	τ_4	3.24152	**2.97515**	3.87293	3.29414
$\lambda = 0.8$	τ_1	4.94326	4.73561	4.70371	**4.41846**
	τ_2	5.62074	5.12101	5.03409	**4.90244**
	τ_3	**3.87348**	4.94102	4.60833	4.26699
	τ_4	4.36537	**3.97091**	5.09903	4.38606

Similarly, for different parameters the minimum divergence measures are described in Table 2.

Table 2. Divergence measure values for different parameters

	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
$\mathcal{D}_\lambda(\tau_1^*,\varphi_4^*)$	3.33833	3.67534	4.41846
$\mathcal{D}_\lambda(\tau_2^*,\varphi_4^*)$	3.74967	4.13653	4.90244
$\mathcal{D}_\lambda(\tau_3^*,\varphi_1^*)$	2.79387	2.74838	3.87348
$\mathcal{D}_\lambda(\tau_4^*,\varphi_2^*)$	3.21265	2.97515	3.97091

From Table 2, we observed that the choice of optimal solution at different parameters are same. If we change the values of parameter its does not affect the choice of options. Hence, we can say that the best alternative for first investors is cryptocurrency, for second investors is also cryptocurrency, for third investors is consumer goods and for fourth investors is technology.

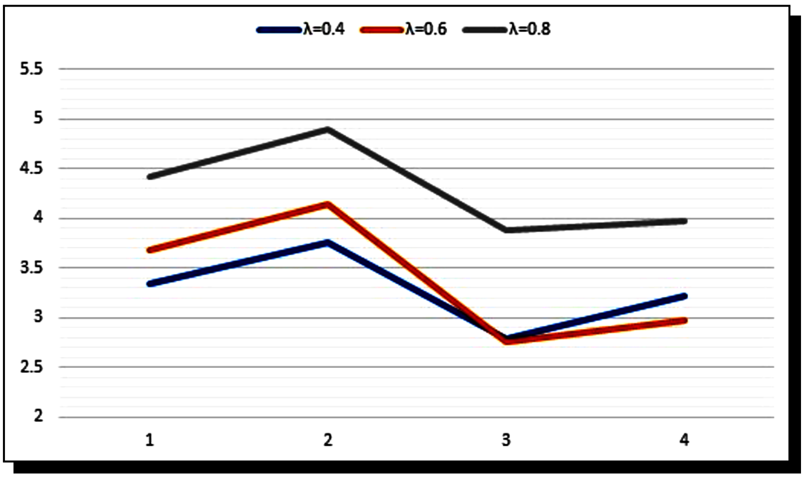


Figure 2. Graphical representation of divergence values for different parameters

Figure 2 shows the graphical representation of divergence measures values corresponding the investors and sectors at different parameters.

5. Conclusion

A new generalized divergence measure for intuitionistic fuzzy matrices is proposed. Some operations of intuitionistic fuzzy matrices are also defined. The proposed measure satisfies all properties that are useful in dealing with any dynamic issue. The validity of proposed generalized measure are also discussed. The intuitionistic fuzzy matrix method can be used to solve a variety of real-world decision-making problems. It is proposed to use an MCDM technique based on the divergence measure to solve real-world decision-making problems with intuitionistic fuzzy information. The generalized measure gives more flexibility in multi-criterion decision making problem due to presence of several parameter. The proposed measure's application in the trade market for making optimal decisions is discussed. In future, we can extend the generalized divergence measure on interval-valued fuzzy matrix and interval-valued intuitionistic fuzzy matrix.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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