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Research Article

On (4, 2)-Labeling of Certain Graphs

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Abstract. The $(4,2)$-labeling of a graph G is a function $f : V(G) \to \mathbb{Z}^+$ such that $|f(x) - f(y)| \geq 4$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 2$ if $d(x,y) = 2$, for any $x, y \in V(G)$. In this paper, we label different types of graphs such as paths, cycles, complete and complete bipartite graphs, star graphs and ladder graphs to study the bounds of the span λ of these graphs.

Keywords. Graph labeling, Path, Cycle, Complete bipartite graph, Star graph, Complete graph, Ladder graph

Mathematics Subject Classification (2020). 05C78

1. Introduction

Graph labeling has found application in various fields like astronomy, coding theory, database management, circuit design and radio frequency assignment among others. A graphical model of the frequency assignment problem was introduced in the year 1980 by Hale [3] with the vertices of the graph denoting stations and the edges denoting their proximity.

In the year 1992, Griggs and Yeh [2] introduced the $L(2,1)$-labeling of a graph G as a function $f : V(G) \to \mathbb{Z}^+$ such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$, for any $x, y \in V(G)$. One can find literature on $L(2,1)$-labeling of graphs in [6–9].

The $L(0,1)$-labeling of a graph G is a function $f : V(G) \to \mathbb{Z}^+$ such that $|f(x) - f(y)| \geq 0$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$, for any $x, y \in V(G)$. One can find literature on $L(0,1)$-labeling of graphs in [5, 10–12].

In the year 2016, Ghosh and Pal [1] introduced the $L(3,1)$-labeling of a graph G as a function $f : V(G) \to \mathbb{Z}^+$ such that $|f(x) - f(y)| \geq 3$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, for any $x, y \in V(G)$.

In this paper, we introduce $L(4,2)$-labeling of a graph G as a function $f : V(G) \to \mathbb{Z}^+$ such that $|f(x) - f(y)| \geq 4$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 2$ if $d(x, y) = 2$, for any $x, y \in V(G)$. Here we apply $L(4,2)$-labeling technique to label paths, cycles, complete graphs, complete bipartite graphs, star graphs and ladder graphs.

Definition 1.1. Let G be a graph having vertex set V and edge set E. A function $f : V(G) \to \mathbb{Z}^+$ is said to admit a $(4,2)$-labeling of G if for all $u, v \in V$, $|f(x) - f(y)| \geq 4$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 2$ if $d(x, y) = 2$.

Definition 1.2 ([5])**.** The difference between the largest and the smallest values of f, for every possible value of f, is called the span of the labeling and is denoted by λ.

Definition 1.3 ([3])**.** A path is a trail where all the vertices (except the starting and the terminating vertices) are distinct. A path having n vertices and $n-1$ edges is denoted by P_n.

Definition 1.4 ([3])**.** A simple graph G with n vertices and n edges is said to be a cycle graph if all its edges form a cycle of length n. A cycle graph of length n is denoted by C_n.

Definition 1.5 ([3])**.** A graph G is said to be a complete graph if all its vertices are adjacent to each other. A complete graph on n vertices is denoted by K_n.

Definition 1.6 ([3])**.** A graph G is said to be a complete bipartite graph if its vertices can be partitioned into two subsets V_1 and V_2 such that each vertex of V_1 is adjacent to each vertex of V_2, but no two vertices on the same subset are adjacent. A complete bipartite graph with $|V_1| = m$ and $|V_1| = n$ is denoted by $K_{m,n}$.

Definition 1.7 ([3])**.** A star graph on n vertices, denoted by S_n, is a graph with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1.

Definition 1.8 ([3])**.** The ladder graph L_n is a planar, undirected graph obtained as the Cartesian product of two path graphs, one of which has only one edge. A ladder graph contains $2n$ vertices and $3n-2$ edges.

2. Main Results

In this section, we label some special classes of graphs and obtain the span of the $(4,2)$-labeling of these graphs. We begin this section with the $(4,2)$-labeling of paths.

Proposition 2.1. $\lambda(P_2) = 4$.

Proof. For a path P_2 with vertices v_0 and v_1, if we label v_0 by 0, then other vertex must be labeled by at least 4 and so $\lambda(P_2) = 4$. □

Lemma 2.1. *For any subgraph H of G, $\lambda(H) \leq \lambda(G)$.*

Proof. Let $f : V(G) \to \{0,1,2,\dots,k\}$ and let $\lambda(G) = k$. Then the function $g : V(H) \to \{0,1,2,\dots,k\}$ defined by $g(v) = f(v)$, $\forall\ v \in V(H)$, is a labeling of the vertex set of H that uses no label greater than k. Therefore, $\lambda(H) \leq k = \lambda(G)$. □

Proposition 2.2. (a) $\lambda(P_3) = 6$.

(b) $\lambda(P_4) = 6$.

Proof. (a) Since $\lambda(P_2) = 4$, using Lemma 2.1, $\lambda(P_3) \geq 4 = \lambda(P_2)$.

Let P_3 be a path having vertices v_0, v_1 and v_2 such that v_0 is adjacent to v_1 and v_1 is adjacent to v_2. Since $d(v_0,v_1) = d(v_1,v_2) = 1$ and $d(v_0,v_2) = 2$, so there are three possibilities of labeling these vertices:

(i) Let $v_0 = a$. Then $v_1 = a+4$ and $v_2 = a+8$. Therefore $\lambda(P_3) \leq 8$.

(ii) Let $v_0 = a$. Then $v_2 = a+2$ and $v_1 = a+6$. Therefore $\lambda(P_3) \leq 6$.

(iii) Let $v_1 = a$. Then $v_0 = a+4$ and $v_2 = a+8$. Therefore $\lambda(P_3) \leq 8$.

In view of these possibilities, $\lambda(P_3) = 6$.

(b) Since P_3 is a subgraph of P_4, so by Lemma 2.1, $\lambda(P_4) \geq 6 = \lambda(P_3)$. Let P_4 be a path with vertices, v_0, v_1, v_2 and v_3 such that v_0 is adjacent to v_1, v_1 is adjacent to v_2 and v_2 is adjacent to v_3. One of the possible labeling options for these four vertices is given below:

$$v_0 = 4,\ v_1 = 0,\ v_2 = 6 \text{ and } v_3 = 2.$$

So $\lambda(P_4) \leq 6$. Consequently, $\lambda(P_4) = 6$. □

Proposition 2.3. $\lambda(P_n) = 8$, $\forall\ n \geq 5$.

Proof. Let $n = 5$. Since P_4 is a subgraph of P_5, so by Lemma 2.1, $\lambda(P_5) \geq 6 = \lambda(P_4)$. Let v_0–v_1–v_2–v_3–v_4 be the vertices of P_5. As in Proposition 2.2(b), one of the possible labeling options of P_5 is given below:

$$v_0 = 4,\ v_1 = 0,\ v_2 = 6,\ v_3 = 2 \text{ and } v_4 = 8.$$

Thus $\lambda(P_5) = 8$.

For $n > 5$, the same set of labels can be repeated all over again $(4,0,6,2,8,4,0,6,2,8,4,0,6,\dots)$. Thus $\lambda(P_n) = 8$, $\forall\ n \geq 5$. □

Example 2.1. Figure 1 shows the (4,2)-labeling of the paths P_2, P_3, P_4, P_5 and P_6.

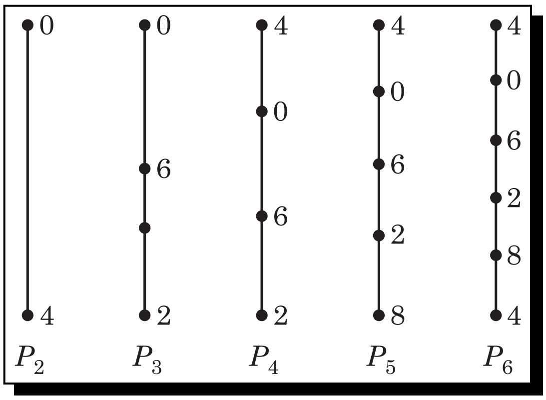

Figure 1. (4,2)-labeling of the paths P_2, P_3, P_4, P_5 and P_6

Proposition 2.4. *For any cycle* C_n, $\lambda(C_n) = 8$, $\forall\ n \geq 3$.

Proof. For $n < 5$, the result is easy to verify. For $n > 5$, the cycle C_n contains the path P_5 as a subgraph. Therefore, by Lemma 2.1, $\lambda(C_n) \geq (P_5) = 8$.
Let $u_0, u_1, u_2, \dots, u_{n-1}$ be the vertices of the cycle C_n.

We look at the following three cases:

Case 1: $n \equiv 0 \pmod 3$.
For $u_i \in V(C_n)$, the vertices of the cycle C_n can be labeled by the following function:

$$f(u_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 4, & \text{if } i \equiv 1 \pmod 3, \\ 8, & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Case 2: $n \equiv 1 \pmod 3$.
For $u_i \in V(C_n)$, the vertices of the cycle C_n can be labeled by the following function:

$$f(u_i) = \begin{cases} 0, & \text{if } i = n-(3k+1), \text{ where } 1 \leq k \leq \frac{n-1}{3}, \\ 4, & \text{if } i = n-3k, \text{ where } 2 \leq k \leq \frac{n-1}{3}, \\ 8, & \text{if } i = n-1 \text{ or } i = n-(3k+2), \text{ where } 1 \leq k \leq \left\lfloor \frac{n-2}{3} \right\rfloor, \\ 2, & \text{if } i = n-2, \\ 6, & \text{if } i = n-3. \end{cases}$$

Case 3: $n \equiv 2 \pmod 3$.
For $u_i \in V(C_n)$, the vertices of the cycle C_n can be labeled by the following function:

$$f(u_i) = \begin{cases} 0, & \text{if } i = n-(3k+2), \text{ where } 1 \leq k \leq \frac{n-2}{3}, \\ 4, & \text{if } i = n-(3k+1), \text{ where } 2 \leq k \leq \frac{n-2}{3}, \\ 8, & \text{if } i = n-3k, \text{ where } 1 \leq k \leq \frac{n-2}{3}, \\ 2, & \text{if } i = n-2, \\ 6, & \text{if } i = n-1. \end{cases}$$

In view of these three cases, we find that $\lambda(C_n) = 8$. □

Example 2.2. Figure 2 shows the (4,2)-labeling of the cycles C_7, C_9 and C_{10}.

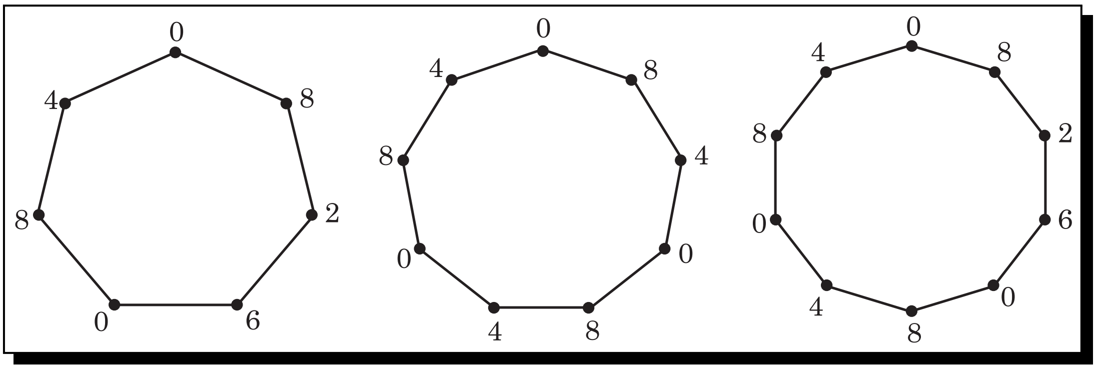

Figure 2. (4,2)-labeling of the cycles C_7, C_9 and C_{10}

Proposition 2.5. *For any complete graph* K_n, $\lambda(K_n) = 4(n-1)$.

Proof. Each $v_i \in K_n$ can be labeled by the function $f : V(K_n) \to \{0,4,8,\dots,4(n-1)\}$ defined by $f(v_i) = 4i$, for $0 \le i \le n-1$. Clearly, $\lambda(K_n) = 4(n-1)$. □

Proposition 2.6. *For any complete bipartite graph* $K_{m,n}$, $\lambda(K_{m,n}) = 2(m+n)$.

Proof. Let V_1 and V_2 be the two vertex sets of $K_{m,n}$ such that $|V_1| = m$ and $|V_2| = n$. Since $d(u,v) = 2$, $\forall\ u,v \in V_1$, we can label the vertices of V_1 with $a, a+2, a+4, \dots, a+2(m-1)$.

Similarly, the vertices of V_2 can be labeled with $a+2(m-1)+4, a+2(m-1)+4+2, a+2(m-1)+4+4, \dots, a+2(m-1)+4+2(n-1)$. Taking $a = 0$ gives us the minimum integers. Therefore, $\lambda(K_{m,n}) = 2(m-1)+4+2(n-1) = 2(m+n)$. □

Example 2.3. Figure 3 shows the (4,2)-labeling of the complete bipartite graph $K_{3,4}$.

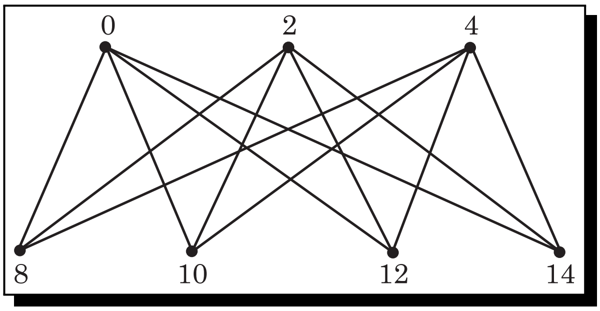

Figure 3. (4,2)-labeling of the complete bipartite graph $K_{3,4}$

Proposition 2.7. *For any star graph graph* S_n, $\lambda(S_n) = 2+2n$.

Proof. Since the star graph S_n is a complete bipartite $K_{m,n}$ with $m = 1$, thus by Proposition 2.6, $\lambda(S_n) = 2(1+n) = 2+2n$. □

Proposition 2.8. *For the ladder graph* L_n, $\lambda(L_n) = 10$, *for all* $n \ge 2$.

Proof. Let $V = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le n\}$ be the vertex set of L_n. Since L_n contains the cycle C_4 and $\lambda(C_4) = 8$, thus by Lemma 2.1, $\lambda(L_n) \ge 8 = \lambda(C_4)$.

We define a function $f : V(L_n) \to \mathbb{Z}^+$ such that for $i \equiv 1 \pmod 3$,

$$f(u_i) = 0, \quad f(u_{i+1}) = 8, \quad f(u_{i+2}) = 4$$

and

$$f(v_i) = 6, \quad f(v_{i+1}) = 2, \quad f(v_{i+2}) = 10.$$

We now proceed to claim that the edges of L_n conform to the adjacency rules for $L(4,2)$-labeling:

(1) $|f(u_i) - f(u_{i+1})| \ge 4$, for all $i \equiv 1 \pmod 3$,

(2) $|f(u_{i+1}) - f(u_{i+2})| \ge 4$, for all $i \equiv 1 \pmod 3$,

(3) $|f(v_i) - f(v_{i+1})| \ge 4$, for all $i \equiv 1 \pmod 3$,

(4) $|f(v_{i+1}) - f(v_{i+2})| \ge 4$, for all $i \equiv 1 \pmod 3$,

(5) $|f(u_i) - f(v_{i+1})| \ge 2$, for all $i \equiv 1 \pmod 3$,

(6) $|f(u_{i+1}) - f(v_i)| \ge 4$, for all $i \equiv 1 \pmod 3$.

From (1)-(6), it can be seen that L_n admits a (4,2)-labeling and $\lambda(L_n) = 10$. □

Example 2.4. Figure 4 shows the (4,2)-labeling of the ladder graph L_9.

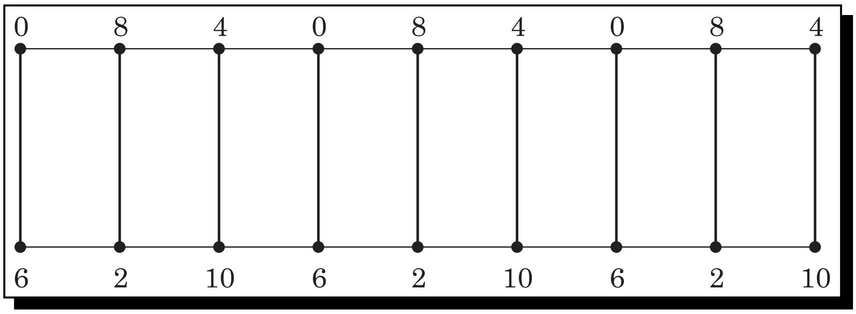

Figure 4. (4,2)-labeling of the ladder graph L_9

3. Conclusion and Future Scope

The (4,2)-labeling of different classes of graphs including paths, cycles, complete and complete bipartite graphs, star graphs and ladder graphs have been studied to investigate the bounds of the span λ of these graphs. Labeling other classes of graphs and studying their bounds leaves ample scope for future research on this topic.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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