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Research Article

# Some Results on Relatively Prime Edge Labeled Graph

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**Abstract.** Prime labeling and relatively prime edge labeling have the same idea for labeling the general graph $G$. Prime labeling labels the vertices of the general graph in such a way that adjacent vertices receive relatively prime labels. Similarly, relatively prime edge labeling, labels the edges in a way that the adjacent edges have relatively prime labels. Also, there are graphs that do not have relatively prime edge labeling. Hence the concept of relatively prime index is introduced, which finds the minimum number of edges to be removed from $G$ to make it a relatively prime edge labeled graph. The main purpose of the current study is to discuss some results on the topic of relatively prime edge labeled graphs and relatively prime index.

**Keywords.** Prime graph, Coprime graph, Relatively prime edge labeled graph, Relatively prime index

**Mathematics Subject Classification (2020).** 05C78, 05C85



## 1. Introduction

The labeling concept plays a significant role in recent years. Many academicians have introduced different types of labeling (Gallian [3]). Rosa [9] was the first to introduce the concept of labeling, named as $\alpha$-labeling in 1967. Later, Entringer suggested the concept of prime graph (Gallian [3]), which is studied by Tout *et al*. in 1982 [11]. In recent years, a variety of prime graphs (Kanetkar [6]) has been studied by many researchers such as the edge vertex prime graph (Jagadesh and Babujee [4]), $K$-prime graph (Vaidya and Prajapati [12]), edge-prime graph (Shiua *et al*. [10]), *SD*-prime graph (Lau and Shiu [7]), and so on. Asplund and Fox [1] introduced the concept of coprime labeling. Also, from the ideas attained from the above literature review, some new graph labeling techniques namely *Relatively Prime Edge Labeling* (*RPEL*) and coprime edge labeling are introduced by Janani and Ramachandran [5]. In prime labeling (Fu and Huang [2], and Lee [1]), vertices are labeled in such a way that, any two adjacent vertices have relatively prime labels. Similarly, in *RPEL* any two adjacent edges are labeled with relatively prime labels. Coprime labeling maintains the same criterion as prime labeling with adjacent vertices using any set of distinct positive integers. A minimum coprime number $p\tau(G)$, is the least value $k$ that $G$ has coprime labeling (Mostafa and Ghorbani [8]). There are graphs that do not possess a relatively prime edge labeling, hence have coprime edge labeling. The concept of a prime index came by removing a certain number of edges from a coprime edge labeled graph to make it a relatively prime edge labeled graph.

## 2. Preliminaries

The graph $G = (V,E)$ discussed in this paper is simple. In this section, some basic definitions needed for this study is discussed.

Roger Entringer proposed the conjecture that every tree is prime, but until now, this conjecture remains unsolved. Shiua *et al*. [2] solved this conjecture for trees up to order 15.

When a graph fails to be prime graph, the minimal coprime graph exists (Asplund and Fox [1]). In particular, a coprime labeling of $G$ is a labeling of the vertices of $G$ with distinct integers from the set $\{1,2,\dots,k\}$, for some $k \geq n$, in such a way that the labels of any two adjacent vertices are relatively prime. Also, the minimum coprime number $p\tau(G)$ to be the minimum value of $k$ for which $G$ has a coprime labeling[1]. If $p\tau(G) = n$, then the labeling is said to be a prime labeling. In [1], Asplund and Fox found the coprime number of complete graph and wheel to be $p_{n-1}$ and $n+2$, respectively. In Lee[1], the minimum coprime number of the sum of path and cycles were found.

Relatively prime edge labeling was defined by Janani and Ramachandran [5], which labels the edges of $G$ with 1 to $q$, with every pair of adjacent edges have relatively prime numbers.

**Example 2.1.** The relatively prime edge labeling of $C_3$ is illustrated below.

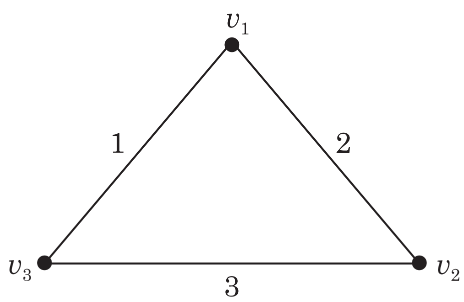


**Figure 1.** Relatively prime edge labeling of $C_3$

[1]C. Lee, Minimum coprime graph labelings, *arXiv preprint*, arXiv:1907.12670 (2019), DOI: 10.48550/arXiv.1907.12670.

Coprime edge labeling is a bijection $f: E \to \{1,2,\dots,k\}$ such that, for $k \geq q$, every pair of adjacent edges receives relatively prime labels from 1 to $k$. The minimum value of $k$, for which $G$ is coprime edge labeling is called as minimum coprime edge labeling, with minimum coprime edge number, $p\tau_E(G) = k$. If $p\tau_E(G) = q$, then the labeling of $G$ is called a relatively prime edge labeling.

**Example 2.2.** The coprime edge labeling of $K_5$ is illustrated below.

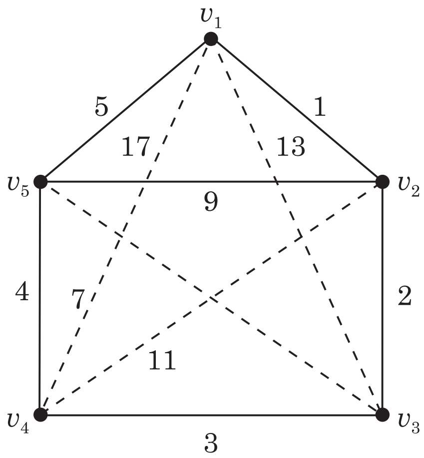


**Figure 2.** Coprime edge labeling of $K_5$

There are graphs in which the coprime edge labeled graph becomes a *RPEL* graph, by removing minimum number of edges in $G$. This leads to the following definition.
For a coprime edge labeled graph $G$, relatively prime index is the minimum number of edge removal, resulting in a *RPEL* graph, and it is denoted by, $\varepsilon_r(G)$ (Janani and Ramachandran [5]). In other words,

$$\varepsilon_r(G) = \min\{|E(H)| : H \subseteq G \text{ and } G - E(H) \text{ is a relatively prime edge labeled graph}\}|.$$

## 3. Main Results

In this section, some theorems related to *RPEL* graph and relatively prime index is discussed. Also, for certain class of graphs relatively prime index is found.

**Theorem 3.1.** *For $n > 6$, there is no relatively prime edge labeled graph $G$ whose $\Delta(G) = n-1$.*

*Proof.* Suppose for $n > 6$, there exists a *RPEL* graph $G$ whose maximum degree is $n-1$. Let $u$ be the vertex having maximum degree $n-1$. As $n > 6$, there does not exist $n-1$ labels with pairwise relatively prime. Hence, there is no *RPEL* graph $G$ with maximum degree $n-1$. □

**Corollary 3.1.** *For a graph $G$, $n > 4$ and any two vertices having degree $n-1$, there is no* RPEL *graph.*

*Proof.* Suppose, there exists a *RPEL* graph $G$ having two vertices $u$ and $v$ of maximum degree $n-1$. Let $u_1, u_2, \dots, u_{n-2}$ be the $n-2$ vertices incident on $u$ and $v_1, v_2, \dots, v_{n-2}$ be the $n-2$ vertices incident on $v$ with a common edge $uv$.

Now label the edges $uv$ as 1, $uu_1$ as 2 and $vv_1$ as 4, then the remaining $2n-5$ edges needs to be labeled with only odd numbers. Therefore, $\frac{n}{2} \geq 2n-5$, which is not possible. Hence, there does not exist a *RPEL* graph having two vertices of degree $n-1$. □

**Illustration 3.1.** As a contrary for Corollary 3.1, for $n < 5$, Figure 3 and Figure 4 acts as a counter example.

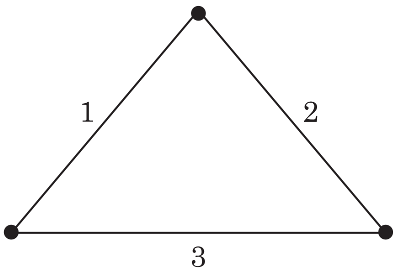


**Figure 3**

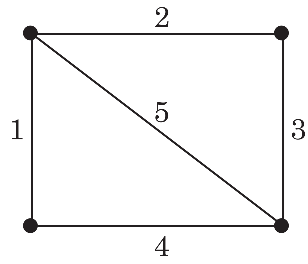


**Figure 4**

**Theorem 3.2.** *Every 2-regular graph is relatively prime edge labeled.*

*Proof.* Every 2-regular graph is a cycle. Hence the proof. □

Figure 5 is an example for 2-regular graph.

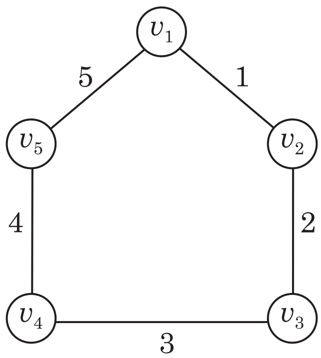


**Figure 5.** 2-Regular graph

**Theorem 3.3.** *For a 3-regular graph with $n > 3$, $\varepsilon_r(G) = \frac{n}{2} - 1$.*

*Proof.* Let $G$ be a 3-regular graph with $n$ vertices $\{v_1, v_2, \dots, v_n\}$ and $\frac{3n}{2}$ edges. Since, $G$ is a 3-regular graph then the number of vertices in $G$ will be even. The proof consists of two parts. First to prove, $G$ is not relatively prime edge labeled, then to find the prime index of $G$. Suppose $G$ is *RPEL*, then the label of every edge is shared between two vertices, and also the label of the edges incident on each vertices will be relatively prime. Let $v_1, v_2$ be the vertices that shares the common label 2, then the remaining edges incident on $v_1, v_2$ will be odd numbers. Similarly, if $v_3, v_4$ shares the common label 4, then the edges incident on $v_3, v_4$ will also be odd numbers. By proceeding in this way, then the number of odd numbers exceeds the number of edges, which is not possible. Hence G is not relatively prime edge labeled.

Next, to find the minimum number of edge removal for making $G$ to be a *RPEL* graph. We know that, every 3-regular graph has a cycle of length $n$, that is, $v_1v_2\dots v_n$. Label the edges of $C_n$ as $L(v_iv_{i+1}) = i$, $L(v_1v_{\frac{n}{2}+1}) = n+1$ for $i = 1,2,\dots,n$ and $v_{n+1} = v_1$, where the edges incident on each vertex are relatively prime. Clearly, $n+1$ is an odd number. Also, it is not possible to label $n+2$ to any of the edges as each vertex already contains a label with even number. Thus $\frac{3n}{2} - n - 1 = \frac{n}{2} - 1$ edges need to be removed to make $G$ as a *RPEL* graph. □

**Illustration 3.2.** Figure 6 shows the illustration of Theorem 3.3.

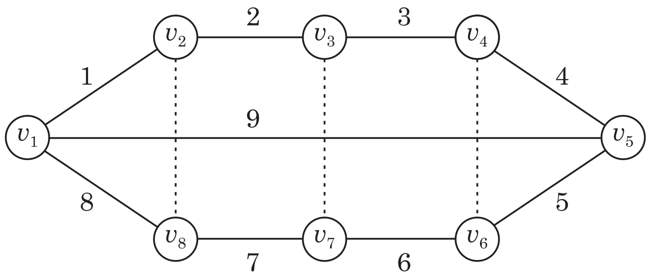


**Figure 6.** 3-Regular graph with 8 vertices

**Theorem 3.4.** *For a graph $G = (p,q)$ which contains a Hamiltonian circuit of length $k$, then $\varepsilon_r(G) \le q-k$.*

*Proof.* Suppose, $\varepsilon_r(G) > q-k$, where $k$ is the length of the Hamiltonian circuit and $q$ is the number of edges. Let $v_1, v_2, \dots, v_p$ be the $p$ vertices. As $G$ contains a Hamiltonian circuit of length $k$, say $v_1, v_2, \dots, v_k$ then label the edges of the Hamiltonian circuit in such a way that, $L(v_iv_{i+1}) = i$ for $i = 1,2,\dots,k$ and $v_{k+1} = v_1$. Since the prime index is greater than $q-k$, which is the contradiction to the above labeling. Hence the maximum number of edges to be removed from $G$ is less than or equal to $q-k$. □

**Corollary 3.2.** *For a complete graph $K_4$, $\varepsilon_r(K_4) \le 2$.*

*Proof.* By Theorem 3.4, Figure 7 shows that $K_4$ contains a Hamiltonian circuit of length 4 and the number of edges in $K_4$ is 6. Hence $\varepsilon_r(G) \le 6-4 = 2$.

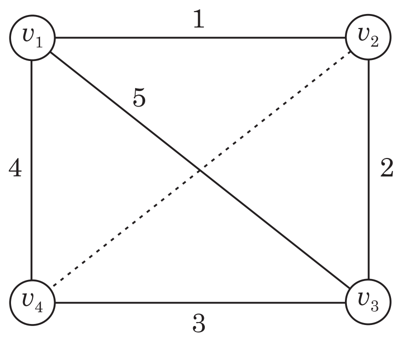


**Figure 7.** Complete graph with 4 vertices

□

As a next part, the relatively prime index of complete bipartite graph is found. For that, it is necessary to find whether complete bipartite graph is either relatively prime edge labeled or not. The next theorem helps to find that.

**Theorem 3.5.** *For $t > 2$, the complete bipartite graph $K_{2t}$ is not a relatively prime edge labeled graph.*

*Proof.* Let the number of vertices and edges in a complete bipartite graph $G = K_{2,t}$ is $t+2$ and $2t$. Suppose $G$ is a relatively prime edge labeled graph. Consider the vertices in the two partite as $\{u_1, u_2$ and $v_1, v_2, \dots, v_t\}$. By the definition of complete bipartite graph, each $u_1$ is adjacent with every $v_i$ and $u_1v_i$ is labeled with odd numbers. So that the label of the edges incident of $u_1$ is relatively prime. Similarly, each $u_2v_i$ is labeled with odd numbers. But it is not possible. Since the number of odd numbers will be $t$. Hence the proof. □

**Theorem 3.6.** *For a graph $G = K_{2,t}$, $t > 2$ then, $\varepsilon_r(G) = 2t-5$.*

*Proof.* By Theorem 3.5, $G$ is not *RPEL* graph. Now, it is enough to find the minimum number of edges to be removed from $G$ to make $G$ as a *RPEL*. The number of vertices and edges in $G = K_{2,t}$ is $t+2$ and $2t$. Label the edges of $G$ in such a way that, $L(u_1v_1) = 1$, $L(u_1v_2) = 2$, $L(u_1v_3) = 3$, $L(u_1v_4) = 5$ and $L(u_2v_1) = 4$. Also, the label 6 cannot be labeled in any of the edges incident on $u_1$ and $u_2$. Thus, it is needed to remove $2t-5$ edges from $G$ to make it as a relatively prime edge labeled graph. Hence, $\varepsilon_r(G) = 2t-5$. □

A graph $G$ is said to be self-complementary if the graph is isomorphic to its complement. The next theorem explains for a self-complementary graph to be relatively prime edge labeled.

**Theorem 3.7.** *Let $G$ be a self-complementary graph of order $n$, then $G$ is relatively prime edge labeled graph iff $n = 4, 5$.*

*Proof.* Let $G$ be a self-complementary graph of order $n$.

*Case* 1: For $n = 4$.
Let $v_1, v_2, v_3$ and $v_4$ be the vertices of $G$. Also, the maximum number of edges with 4 vertices is 6. Now considering a path of length 3 as $G$, then the remaining 3 edges which forms a path of length 3 is in $\overline{G}$. We know that, every path is relatively prime edge labeled graph as shown in Figure 8. Hence, $G$ is relatively prime edge labeled graph.

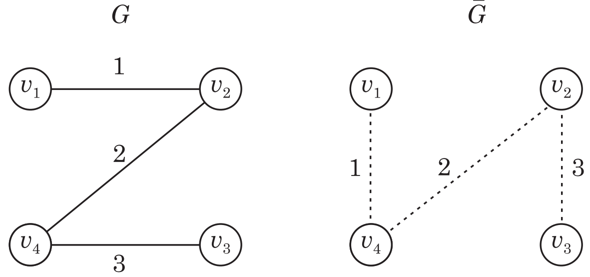


**Figure 8**

*Case* 2: For $n = 5$.
Let $v_1, v_2, v_3, v_4$ and $v_5$ be the vertices of $G$. Also, the maximum number of edges with 5 vertices is 10. Now considering a cycle $\{v_1v_2v_3v_4v_5v_1\}$ of length 5 as $G$, then the remaining 5 edges forms a cycle $\{v_1v_3v_5v_2v_4v_1\}$ of length 5 is in $\overline{G}$. We know that, every cycle is relatively prime edge labeled graph as in Figure 9. Hence, $G$ is relatively prime edge labeled graph.

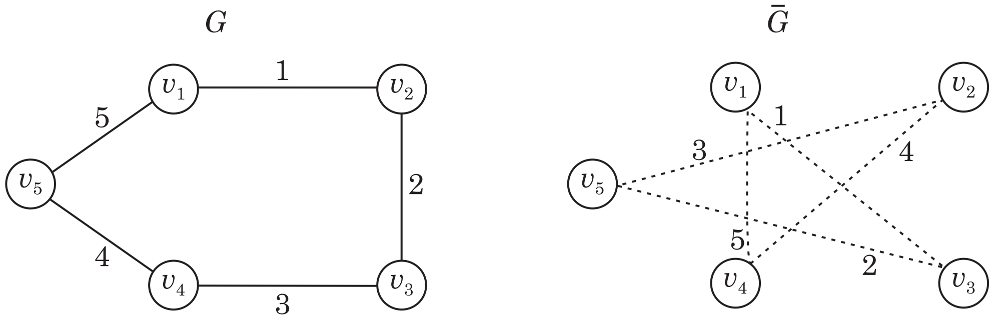


**Figure 9**

□

The edge independence number of $G$, denoted by $\alpha'(G)$, is the maximum of the cardinalities of all edge independent sets of $G$. The next theorem shows the condition for a general graph $G$ to be a *RPEL* graph.

**Theorem 3.8.** *For a graph G with order 8, $|E(G)| \leq 9$ and $\alpha'(G) \geq 4$, then G is relatively prime edge labeled graph.*

*Proof.* Let $V(G) = \{v_1, v_2, v_3, \ldots, v_8\}$ and $E(G) = \{e_1, e_2, e_3, \ldots, e_9\}$ be the vertices and edges of $G$. Assume that $\{e_1, e_2, e_3, e_4\}$ be the edge independent set of $G$. Labeling of the graph $G$ can be done in such a way that, $\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\ 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 9 \end{pmatrix}$ and assuming that $e_3, e_6$ and $e_9$ does not incident on a single vertex each other. Thus, the above labeling is the relatively prime edge labeling. □

**Theorem 3.9.** *If $G$ is a* RPEL *graph then $\alpha'(G) \geq \frac{E(G)}{2}$.*

*Proof.* Let $|E(G)| = k$. We shall prove this theorem by contradiction.

*Case* 1: For even $k$, suppose $\alpha'(G) < \frac{k}{2}$.
Consider, $\alpha'(G) = \{e_1, e_2, \ldots, e_{\frac{k}{2}-1}\}$ then, label the edges of $\alpha'(G)$ with $2, 4, \ldots, k-2$. As $G$ is relatively prime edge labeled graph, then the remaining edges $\frac{k}{2}+1$ have the label $1, 3, 5, 7, \ldots, k$. Also, since $k$ is even the edge having the label $k$ is incident with any one the edge in $\alpha'(G)$, which is not possible. Since the labels of $\alpha'(G)$ are even numbers.

*Case* 2: For odd $k$, suppose $\alpha'(G) < \left[\frac{k}{2}\right] = \frac{k-1}{2}$.
Consider, $\alpha'(G) = \{e_1, e_2, \ldots, e_{\frac{k-3}{2}}\}$ then, label the edges of $\alpha'(G)$ with $2, 4, \ldots, k-3$. As $G$ is *RPEL* graph, then the remaining edges $\frac{k+3}{2}$ is labeled with $1, 3, 5, 7, \ldots, k-2, k-1, k$. Also, since $k$ is

odd the edge having the label $k-1$ is even and it is incident with any one the edge in $\alpha'(G)$, which is not possible. Since the labels of $\alpha'(G)$ are even numbers. □

**Theorem 3.10.** $|E(G)| > 13$*, there does not exists a relatively prime edge labeled graph for* $\Delta(G) = \left\lceil \frac{E(G)}{2} \right\rceil + 1$.

*Proof.* Let $|E(G)| = m > 13$. We shall prove this theorem by contradiction. Suppose there exists a *RPEL* graph for $\Delta(G) = \lceil \frac{m}{2} \rceil + 1$. Let $u$ be the vertices having maximum degree $\lceil \frac{m}{2} \rceil + 1$ and the $\lceil \frac{m}{2} \rceil$ edges incident with $u$ is labeled with $\lceil \frac{m}{2} \rceil$ odd numbers and 2 is labeled with the remaining one vertex.
But since $m > 13$, there exists odd numbers 3 and 9 that are not relatively prime, which is a contradiction to our assumption that $G$ has a relatively prime edge labeled graph with $\Delta(G) = \left\lceil \frac{E(G)}{2} \right\rceil + 1$. □

Next theorem, finds a condition for a complete bipartite graph to be a relatively prime edge labeled.

**Theorem 3.11.** *For the bipartite graph* $G = K_{\frac{n}{2},\frac{n}{2}}$*, with even* $n$ *and if* $|E(G)| \le n$ *then there exist a bipartite graph* $G$ *which is relatively prime edge labeled.*

*Proof.* Let $G$ be a bipartite graph with $\frac{n}{2}$ vertices on each partite. As $|E(G)| \le n$, then consider the vertices in both partite with degree 2. Hence, the maximal graph in $G$ containing n edges will be a cycle. Thus, $G$ is a relatively prime edge labeled. □

**Corollary 3.3.** *For the bipartite graph* $G = K_{\frac{n+1}{2},\frac{n-1}{2}}$*, with odd* $n$ *and if* $|E(G)| \le n$ *then there exist a bipartite graph* $G$ *which is relatively prime edge labeled.*

*Proof.* The proof is same as Theorem 3.11. □

**Theorem 3.12.** *For every* $r \ge 3$*,* $K_{r,r}$ *is not* RPEL *graph.*

*Proof.* Let $X$ and $Y$ be the two partite of the complete bipartite graph $K_{r,r}$. The number of edges in $K_{r,r}$ is $r^2$. Each vertex in the partite will have degree $r$. Therefore, it is necessary to find $r$ pairwise relatively prime numbers for each vertex in the set containing 1 to $r^2$. But it is not possible. Hence, $K_{r,r}$ is not relatively prime edge labeled graph. □

**Theorem 3.13.** *Every induced subgraph of a complete graph* $K_n$ *is not relatively prime edge labeled graph, except for* $n = 4$.

*Proof.* We know that, every complete graph $K_n$, $n > 4$ is not relatively prime edge labeled. Also, every induced subgraph of a complete graph $K_n$ is again a complete graph with less than n vertices. Thus, every induced subgraph of a complete graph $K_n$ is not relatively prime edge labeled graph, except for $n = 4$. □

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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