Communications in Mathematics and Applications
Volume 3 (2012), Number 3, pp. 293–301

http://www.rgnpublications.com

On Approximative Atomic Decompositions in Banach Spaces

S.K. Kaushik and S.K. Sharma

Abstract. Approximative atomic decomposition for Banach spaces has been defined. A characterization for approximative atomic decompositions has been obtained. Also, it has been proved that a Banach space E has an approximative atomic decomposition if and only if it possesses bounded approximation property. Further, sufficient conditions for the existence of approximative atomic decompositions in separable Banach spaces have been obtained. Finally, as an application of approximative atomic decompositions, it has been proved that if E and F are Banach spaces having bounded approximation property, then $E \times F$ also has bounded approximative property.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [3], while addressing some deep problems in non-harmonic fourier series. Feichtinger and Gröchenig [6] generalized frames for Banach spaces and called them atomic decompositions.

Atomic decompositions have played a key role in the development of wavelet theory and Gabor theory. Frazier and Jawerth [7] had constructed wavelet atomic decompositions for Besov spaces which they called *ϕ-transform*. Feichtinger [5] constructed Gabor atomic decompositions for the modulation spaces which are Banach spaces similar in many respects to Besov spaces, defined by smoothness and decay conditions. Walnut [13] improved many results by extending them for the case of weighted L^2 spaces. Infact, he showed that Gabor atomic decompositions in $L^2_w(\mathbb{R})$ need not be Hilbert frames. Feichtinger and Gröchenig [6] developed a general theory which applies to an extremely broad class of function spaces and group representations.

In [8], Gröchenig studied atomic decompositions as a generalization of frames. In the present paper, we shall extend this study further and introduce approximative atomic decompositions for Banach spaces. A characterization for

---

2000 *Mathematics Subject Classification.* 42C15, 42A38.
Key words and phrases. Atomic decomposition; Banach frame.

approximative atomic decompositions has been given. Also, it has been proved that a Banach space E has an approximative atomic decomposition if and only if it possesses bounded approximation property. Further, sufficient conditions for the existence of approximative atomic decompositions in separable Banach spaces have been obtained. Finally, as an application of approximative atomic decompositions it has been proved that if E and F are Banach spaces having bounded approximation property, then $E \times F$ with ℓ_1-norm also has bounded approximation property.

2. Preliminaries

Throughout the paper, E will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* and E^{**}, respectively, the first and second conjugate spaces of E, $B(E)$ the Banach space of all continuous linear mappings on E, $L(E,E)$ the Banach space of all linear mappings on E, E_d an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$, $[f_n]$ the closed linear span of $\{f_n\}$ and $\widetilde{[f_n]}$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*,E)$-topology. A sequence $\{f_n\} \subset E^*$ is said to be complete if $[f_n] = E^*$ and total if $\{x \in E : f_n(x) = 0,\ n \in \mathbb{N}\} = \{0\}$.

Definition 2.1 ([6])**.** Let E be a Banach space and let E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in E and let $\{f_n\}$ be a sequence in E^*. Then, the pair $(\{f_n\},\{x_n\})$ is called an atomic decomposition for E with respect to E_d, if

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$
(b) there exist constants A, B with $0 < A \le B < \infty$ such that

$$A\|x\|_E \le \|\{f_n(x)\}\|_{E_d} \le B\|x\|_E, \quad \text{for all } x \in E$$

(c) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.

The positive constants A, B are called *atomic bounds* for the atomic decomposition $(\{f_n\},\{x_n\})$.

Definition 2.2 ([12])**.** A Banach space E is said to have a bounded approximation property if there exists $\lambda \ge 1$ such that the identity operator $I_E : E \to E$ can be approximated, uniformly on every compact subset of E, by linear operators of finite rank, of norm $\le \lambda$, that is, if their exists a constant $\lambda \ge 1$ with the property: for every compact subset $Q \subset E$ and for every $\epsilon > 0$ there exists an endomorphism $u = u_{Q,\epsilon} \in L(E,E)$ of finite rank, of norm $\|u\| \le \lambda$, such that $\|u(x) - x\| < \epsilon$ $(x \in Q)$.

The following result which is referred in this paper is listed in the form of a lemma

Lemma 2.3 ([11, 12]). *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

3. **Main Results**

We begin with the following definition of approximative atomic decomposition

Definition 3.1. Let E be a Banach space and let E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \subset E^*$, where $\{m_n\}$ is an increasing sequence of positive integers. Then, the pair $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{x_n\})$ is called an approximative atomic decomposition for E with respect to E_d, if

(a) $\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \in E_d$, for all $x \in E$

(b) there exist constants A, B with $0 < A \le B < \infty$ such that

$$A\|x\|_E \le \|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}\|_{E_d} \le B\|x\|_E, \quad \text{for all } \; x \in E$$

(c) $x = \lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x)x_i$, for all $x \in E$.

One may observe that if $(\{f_n\}, \{x_n\})$ is an atomic decomposition for E with respect to E_d, then, for $h_{n,i} = f_i$, $i = 1, 2, \ldots, n$; $n \in \mathbb{N}$, $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,n \\ n\in\mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to some associated Banach space E_{d_0}. Indeed, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} h_{n,i}(x)x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i = x\,. \tag{3.1}$$

Also, if $h_{n,i}(x) = 0$, for all $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$ then by (3.1), $x = 0$. Therefore, by Lemma 2.3, there exists an associated Banach space $E_{d_0} = \{\{h_{n,i}(x)\}_{\substack{i=1,\ldots,n \\ n\in\mathbb{N}}} : x \in E\}$ with the norm given by $\|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,n \\ n\in\mathbb{N}}}\|_{E_{d_0}} = \|x\|_E$, $x \in E$, such that $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,n \\ n\in\mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to E_{d_0}. However, one may note that $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,n \\ n\in\mathbb{N}}}, \{x_n\})$, as such, need not be an atomic decomposition for E with respect to E_{d_0}.

In view of the above discussion, we have the following observations/queries:

(I) If E has an atomic decomposition with respect to E_d, then E also has an approximative atomic decomposition with respect to some E_{d_0}.

(II) What is the relation between E_d and E_{d_0}?

(III) Is the converse of observation (I) true?

The answer to query (III) is affirmative (Remark 3.4) and query (II) is a matter of investigation and is open.

Regarding the existence of approximative atomic decompositions, we give the following characterization

Theorem 3.2. *A Banach space E has an approximative atomic decomposition if and only if there exists a sequence $\{v_n\} \subset B(E)$ of non-zero endomorphism of finite rank such that $x = \sum_{i=1}^{\infty} v_i(x)$, $x \in E$ and* $\sup \left\| \sum_{i=1}^{n} v_i \right\| \leq \lambda$, *for some $\lambda > 0$.*

Proof. Let $\{x_n\} \subset E$ and $\{h_{n,i}\} \subset E^*$ be the sequences such that $(\{h_{n,i}\}_{\substack{i=1,2,\dots,m_n \\ n \in \mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to associated Banach space E_d, where $\{m_n\}$ is an increasing sequence of positive integers. Define

$$u_n(x) = \sum_{i=1}^{m_n} h_{n,i}(x) x_i, \quad x \in E, n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, u_n is a well defined continuous linear mapping on E with $\dim u_n(E) < \infty$ such that $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Also, by using principle of uniform boundedness, $\sup_{1 \leq n < \infty} \|u_n\| < \infty$. Without any loss of generality, we may assume that $u_1 \neq 0$ and $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$. Define

$$v_1 = u_1, \quad v_{2n} = v_{2n+1} = \frac{1}{2}(u_{n+1} - u_n), \quad n \in \mathbb{N}.$$

Then $\{v_n\}$ is a sequence of non-zero endomorphisms of finite rank in $B(E)$ such that

$$\begin{aligned} \sum_{i=1}^{n} v_i(x) &= u_1(x) + \frac{1}{2}\{(u_2(x) - u_1(x)) + (u_2(x) - u_1(x))\} \\ &\quad + \frac{1}{2}\{(u_3(x) - u_2(x)) + (u_3(x) - u_2(x))\} + \cdots \\ &= u_n(x), \quad x \in E. \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) = \lim_{n\to\infty} u_n(x) = x, \quad x \in E.$$

Also

$$\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i \right\| = \sup_{1 \leq n < \infty} \|u_n\| < \infty.$$

Conversely, taking $u_n = \sum_{i=1}^{n} v_i$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Since for each $n \in \mathbb{N}$, $u_n(E)$ is finite dimensional, there exist a sequence $\{y_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E^* such that

$$u_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E, \quad n \in \mathbb{N},$$

where $\{m_n\}$ is an increasing sequence of positive integers with $m_0 = 0$. Define $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n\\ n\in\mathbb{N}}} \subset E^*$ by

$$x_i = y_{n,i}, \quad i = m_{n-1}+1,\ldots,m_n,\ n \in \mathbb{N} \quad \text{and}$$

$$h_{n,i} = \begin{cases} 0, & \text{if } i = 1,2,\ldots,m_{n-1} \\ g_{n,i} & \text{if } i = m_{n-1}+1,\ldots m_n\,. \end{cases}$$

Then, for each $x \in E$, $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x)x_i = \lim_{n\to\infty} u_n(x) = x. \tag{3.2}$$

Let $x \in E$ be such that $h_{n,i}(x) = 0$, for all $i = 1,\ldots,m_n$ and $n \in \mathbb{N}$, then by (3.2), $x = 0$. Therefore, by Lemma 2.3, there exists an associated Banach space $E_d = \{\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n\\ n\in\mathbb{N}}} : x \in E\}$ with norm given by $\|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n\\ n\in\mathbb{N}}}\|_{E_d} = \|x\|_E$, $x \in E$.

Hence, $(\{h_{n,i}\}_{\substack{i=1,\ldots,m_n\\ n\in\mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to E_d. □

Note. *For the converse part of Theorem* 3.2*, we do not require the assumption that* $\sup\limits_{1\le n<\infty} \left\|\sum\limits_{i=1}^{n} v_n\right\| < \infty$.

Let $E = \ell^\infty$. For each $n \in \mathbb{N}$, define $v_n : E \to E$ by

$$v_n(x) = \xi_n e_n, \qquad x = \{\xi_n\}_{i=1}^{\infty} \in \ell^\infty$$

Then $\|v_n(x)\| = \|\xi_n e_n\| = |\xi_n| \le \|x\|$, $n \in \mathbb{N}$. Therefore $v_n \in B(E)$, $n \in \mathbb{N}$. Further, note that $\dim v_n(E) = \dim(\overline{\text{span}\{e_n\}}) = 1$ and $\sup \left\|\sum\limits_{i=1}^{n} v_i\right\| \le 1$. But E is non-separable. Hence E does not have an approximative atomic decomposition.

In view of the above discussion, we have the following problem

Problem. *Does a Banach space E has an approximative atomic decomposition (or atomic decomposition) if there exists a sequence* $\{v_n\} \subset B(E)$ *such that* $x = \sum\limits_{i=1}^{\infty} v_i(x)$, $x \in E$?

The next result gives the relation between approximative atomic decomposition and bounded approximation property.

Theorem 3.3. *A Banach space E has approximative atomic decomposition if and only if E has bounded approximation property.*

Proof. Let $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n\\ n\in\mathbb{N}}}, \{x_n\})$ be an approximative atomic decomposition for E with respect to E_d, where $\{m_n\}$ is an increasing sequence of positive integers. Therefore, proceeding as in Theorem 3.2, there exists a sequence of continuous linear mappings $\{u_n\}$ with $\dim u_n(E) < \infty$ such that $\lim\limits_{n\to\infty} u_n(x) = x$, $x \in E$ and $\sup\limits_{1\le n<\infty} \|u_n\| < \infty$. Let Q be any compact subset of E. Let $\epsilon > 0$ be arbitrary and $\{y_i\}_{i=1}^{r}$ be a finite $\frac{\epsilon}{2(1+\lambda)}$-net for Q, where $\lambda = \sup\limits_{1\le n<\infty} \|u_n\| < \infty$. Then, for any $x \in Q$, there exists some j such that

$$\|x - y_j\| < \frac{\epsilon}{2(1+\lambda)}, \quad 1 \le j \le r\,.$$

Since for each $i = 1, 2, \ldots r$, $y_i \in E$, there exists a positive integer n_i such that

$$\|y_i - u_n(y_i)\| < \frac{\epsilon}{2}, \text{ for all } \; n \ge n_i, \text{ and } \; i = 1, 2, \ldots, r\,.$$

Thus

$$\begin{aligned} \|x - u_n(x)\| &\le \|x - y_j\| + \|y_j - u_n(y_j)\| + \|u_n(y_j) - u_n(x)\| \\ &< \epsilon, \quad \text{for all } \; n \ge n_i\,. \end{aligned}$$

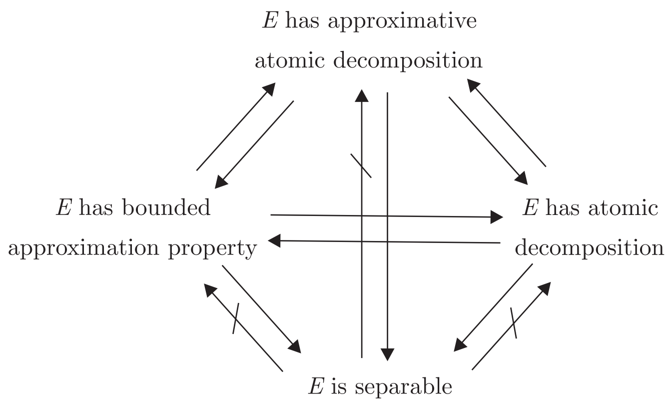
Since $x \in Q$ and $\epsilon > 0$ is an arbitrary, we have

$$\lim_{n\to\infty} \sup_{x\in Q} \|x - u_n(x)\| = 0.$$

Hence E has bounded approximation property.
Converse part of the theorem follows from Theorem 2.10 in [1] and Observation (I). □

Remark 3.4. In view of Theorem 3.3 and Theorem 2.10 in [1], we conclude that a Banach space E has an approximative atomic decomposition if and only if E has an atomic decomposition.

One may observe that if E has an approximative atomic decomposition, then E is separable. However, in view of Remark 3.4 and the example in [4], the converse need not be true. We give below a diagram depicting relationship among various concepts.



Since a separable Banach space need not have an approximative atomic decomposition, it is natural to ask for conditions under which a separable Banach space has an approximative atomic decomposition. The following results gives sufficient conditions for the same

Theorem 3.5. *Let E be a separable Banach space and $\{G_n\}$ be any sequence of subspaces of E such that for each $n \in \mathbb{N}$, $\dim G_n < \infty$, $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n$ is dense in E. If for each $n \in \mathbb{N}$, there exists a finite rank endomorphism $u_n \subset L(E,E)$ such that $u_n|_{G_n} = I_{G_n}$ and $\sup_{1\le n<\infty} \|u_n\| < \infty$, then E has an approximative atomic decomposition.*

Proof. Let $\{y_n\}$ be a sequence in E such that $[y_n] = E$. Put $G_n = [y_1, y_2, \dots, y_n]$, $n \in \mathbb{N}$. Then, $\{G_n\}$ is a sequence of subspaces of E as desired in the hypotheses. Therefore, for each $n \in \mathbb{N}$, there exists a finite rank endomorphism u_n such that $\lim_{n\to\infty} u_n(x) = x$, for all $x \in \bigcup_{n=1}^{\infty} G_n$. Since, $\sup_{1\le n<\infty} \|u_n\| < \infty$ and $\bigcup_{n=1}^{\infty} G_n$ is dense in E, $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Then, proceeding as in Theorem 3.2, one may conclude that E has an approximative atomic decomposition. □

Theorem 3.6. *A separable Banach space E has an approximative atomic decomposition if there exists a constant $\lambda \ge 1$ such that for every finite dimensional subspace G of E and every $\delta > 0$ there exists a finite rank endomorphism $u \in B(E)$ satisfying*

$$\|u\| \le \lambda \text{ and } \|u(x) - x\| < \delta\|x\|, \quad x \in G.$$

Proof. Let $\{y_n\}$ be a sequence in E such that $[y_n] = E$. Then, by hypotheses, for each $n \in \mathbb{N}$ there exists a finite rank endomorphism $u_n \in B(E)$ such that

$$\|u_n\| \le \lambda, \quad n \in \mathbb{N} \quad \text{and}$$

$$\|u_n(x) - x\| < \frac{1}{n}\|x\|, \quad x \in G_n = [y_1, y_2, \dots, y_n], \; n \in \mathbb{N}.$$

Since $G_n \subset G_{n+1}$, $n \in \mathbb{N}$ and $\|u_n\| < \lambda$, $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Also $\sup_{1\le n<\infty} \|u_n\| < \infty$.

Hence, proceeding as in Theorem 3.2, E has an approximative atomic decomposition. □

In [10], it has been proved that if two Banach spaces have Banach frames, then their product space also has a Banach frame. The following is a similar result regarding approximative atomic decompositions

Theorem 3.7. *Let E and F be Banach spaces with approximative atomic decompositions. Then the product space $E \times F$ also has an approximative atomic decomposition.*

Proof. Proceeding as in Theorem 3.2, for each $n \in \mathbb{N}$, u_n is a continuous linear operator on E with $\dim(u_n(E)) < \infty$ and v_n is a continuous linear operator on F with $\dim(v_n(F)) < \infty$ such that

$$\lim_{n\to\infty} u_n(x) = x,\ x \in E \quad \text{and} \quad \lim_{n\to\infty} v_n(y) = y,\ y \in F\,.$$

For each $n \in \mathbb{N}$, define $h_n : E \times F \to E \times F$ by

$$h_n(x, y) = (u_n(x), v_n(y)), \quad (x, y) \in E \times F\,.$$

Then each h_n is well defined continuous linear operator on $E \times F$ with $\dim h_n(E \times F) < \infty$ and

$$\lim_{n\to\infty} h_n(x, y) = (x, y), \quad (x, y) \in E \times F\,.$$

Therefore, again as in Theorem 3.2, $E \times F$ has an approximative atomic decomposition. □

Finally, as an application of approximative atomic decompositions, we have the following corollary

Corollary 3.8. If E and F are Banach spaces having bounded approximation property, then the product space $E \times F$ also has bounded approximation property.

Proof. Follows in view of Theorem 3.3 and Theorem 3.7. □

Acknowledgement

The research of first author is supported by the UGC vide letter No. F. No. 8-1(5)/2010(MRP/NRCB) dated 23.03.2010.

References

[1] P.G. Casazza, D. Han and D.R. Larson, *Frames for Banach spaces, The functional and harmonic analysis of wavelets and frames* (San Antonio, TX, 1999), 149-182; *Contemporary Mathematics* 247; Amer. Math. Soc., Providence, RI, 1999.

[2] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[3] R.J. Duffin and A.C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952), 341–366.

[4] P. Enflo, A counter-example to the approximation problem in Banach spaces, *Acta Math. 130* (1973), 309–317.

[5] H.G. Feichtinger, Atomic characterizations of Modulation spaces through Gabor-Type Representation, *Rocky Mountain J. Math.* **19** (1989), 113–126.

[6] H.G. Feichtinger and K. Gröchenig, A unified approach to atomic decompositions via integrable group representations, in *Proc. Conf. "Function Spaces and Applications"*, Lecture Notes Math. **1302**, Springer, Berlin — Heidelberg — New York, 1988, 52–73.

[7] M. Frazier and B. Jawerth, Deompositions of Besov spaces, *Indiana Univ. Math. J.* **34** (1985), 777–799.

[8] K. Gröchenig, Describing functions: Atomic decompositions versus frames, *Monatsh. Math.* **112** (1991), 1–41.

[9] D. Han, K. Kornelson, D. Larson and E. Weber, *Frames for Undergradutates*, AMS, Providence, 2007.

[10] P.K. Jain, S.K. Kaushik and Nisha Gupta, On near exact Banach frames in Banach spaces, *Bull. Aust. Math. Soc.* **78** (2008), 335–342.

[11] P.K. Jain, S.K. Kaushik and L.K. Vashisht, Banach frames for conjugate Banach spaces, *Zeitschrift für Analysis und ihre Anwendungen* **23**(4) (2004), 713–720.

[12] I. Singer, *Bases in Banach Spaces II*, Springer-Verlag, Berlin — Heidelberg — New York, 1981.

[13] D. Walnut, *Weyl-Heisenberg Wavelet Expansions: Existence and Stability in Weighted Spaces*, Ph.D. Thesis, University of Maryland, College Park, MD, 1989.

S.K. Kaushik, *Department of Mathematics, Kirori Mal College, University of Delhi, Delhi 110007, India*.
E-mail: shikk2003@yahoo.co.in

S.K. Sharma, *Department of Mathematics, University of Delhi, Delhi 110007, India*.
E-mail: sumitkumarsharma@gmail.com