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Research Article

Some Aspects of Radial Graphs Under Boolean Operations

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Abstract. Two vertices of a graph G are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph G, denoted by $R(G)$, has the vertex set as in G and two vertices are adjacent in $R(G)$ if and only if they are radial to each other in G. If G is disconnected, then the two vertices are adjacent in $R(G)$ if they belong to different components of G. The main objective of this paper is to determine the radial graphs of some families of product graphs.

Keywords. Radial graph, Conjunction, Rejection, Cartesian product

Mathematics Subject Classification (2020). 05C12

1. Introduction

In 1967, Harary and Wilcox [6] and Abay-Asmerom *et al.* [1] defined various boolean operations in graphs. Also, El-Kholy *et al.* [4] described the new operations in graphs. Let G be an undirected graph with vertex set V and edge set E. The *distance* $d(u,v)$ between a pair of vertices u and v is the length of the shortest path connecting them. The *eccentricity* of a vertex of a connected graph G is the maximum distance from the vertex to any other vertex. The *radius* of G, denoted by $r(G)$, is the minimum eccentricity of the vertices of the graph. The *diameter* $d(G)$ of G is the maximum eccentricity of the vertices of the graph. A graph for which $r(G) = d(G)$ is called a *self-centered graph*. Kathiresan and Marimuthu [7] introduced the concept of radial graphs and Avadaiyappan and Bhuvaneswari [3] developed the concepts of radial graphs. Let F_{12} and F_{22} denote the set of all connected graphs of G for which $r(G) = 1$, $d(G) = 2$ and $r(G) = d(G) = 2$, respectively (Kathiresan and Marimuthu [8]). The *eccentric graph* (Akiyama *et al.* [2]) based on G, denoted by G_e, is the graph whose vertex set is $V(G)$ and two vertices u and v are adjacent in G_e if and only if $d(u,v) = \min\{e(u), e(v)\}$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Then, the *Cartesian product* [6] of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph having the vertex set $V_1 \times V_2$ and whose edge set consists of all elements $\{(u_1,u_2),(v_1,v_2)\}$ where either $u_1 = v_1$ and $(u_2,v_2) \in E_2$ or $u_2 = v_2$ and $(u_1,v_1) \in E_1$.

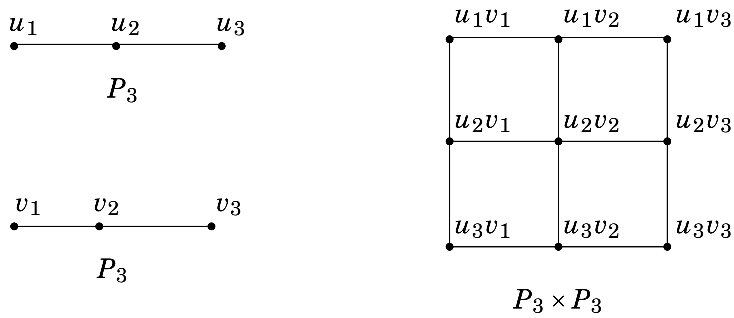

Figure 1

The *conjunction* [6] of G_1 and G_2, denoted by $G_1 \wedge G_2$, is a graph having the vertex set $V_1 \times V_2$ and whose edge set consists of all elements $\{(u_1,u_2),(v_1,v_2)\}$ where $(u_1,v_1) \in E_1$ and $(u_2,v_2) \in E_2$. The *composition* or *lexicographic product* [6] of G_1 and G_2, denoted by $G_1\ [G_2]$, is a graph having the vertex set $V_1 \times V_2$ and whose edge set consists of all elements $\{(u_1,u_2),(v_1,v_2)\}$ where either $(u_1,v_1) \in E_1$ or $[u_1 = v_1$ and $(u_2,v_2) \in E_2]$.

The *disjunction* [6] of G_1 and G_2, denoted by $G_1 \square G_2$, is a graph having the vertex set $V_1 \times V_2$ and whose edge set consists of all elements $\{(u_1,u_2),(v_1,v_2)\}$ where $(u_1,v_1) \in E_1$ or $(u_2,v_2) \in E_2$ or both. The *rejection* [6] of G_1 and G_2, denoted by G_1/G_2, is a graph having the vertex set $V_1 \times V_2$ and whose edge set consists of all elements $\{(u_1,u_2),(v_1,v_2)\}$ where $(u_1,v_1) \notin E_1$ and $(u_2,v_2) \notin E_2$. The *corona* of G_1 and G_2, denoted by $G_1 \circ G_2$, is a graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2.

The *join* [5] of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. The *tensor product* [9] $G_1 \otimes G_2$ of graphs G_1 and G_2 is a graph such that the vertex set of $G_1 \otimes G_2$ is the Cartesian product $V(G_1) \times V(G_2)$ and vertices (g,h) and (g',h') are adjacent in $G_1 \otimes G_2$ if and only if g is adjacent to g' in G_1 and h is adjacent to h' in G_2.

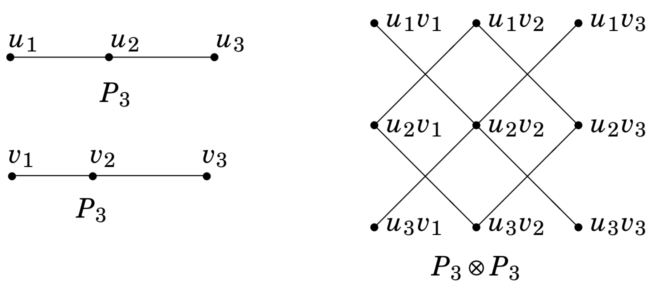

Figure 2

This paper determines the radial graphs for some families of Boolean operation graphs.

Result 1.1 ([7]). *For a graph of order n, $R(G) = K_n$ if and only if either G or $\overline{G}$ is K_n.*

2. Main Results

Theorem 2.1. *For any two integers $m,n \geq 3$, $R(C_m \times C_n) = R(C_m) \wedge R(C_n)$.*

Proof. Let u and v be vertices in C_m which are radial to each other and x,y be two vertices in C_n which are radial to each other. Then $uv \in E(R(C_m))$ and $xy \in E(R(C_n))$, that is, $(u,x),(v,y) \in E(R(C_m) \wedge R(C_n))$. Also, (u,x) and (v,y) are radial to each other in $C_m \times C_n$. Thus, $(u,x),(v,y) \in E(R(C_m \times C_n))$. Hence $R(C_m \times C_n) = R(C_m) \wedge R(C_n)$. □

Corollary 2.2. *For any two integers $m,n \geq 3$, $R(P_m \times P_n) = R(P_m) \wedge R(P_n)$.*

Theorem 2.3. *For any integers m and n, $R(K_m \times K_n) = \overline{K_m \times K_n} = R(K_m) \wedge R(K_n) = K_m \wedge K_n$.*

Proof. Let $u_1,u_2,\dots,u_m$ be the vertices of K_m and $v_1,v_2,\dots,v_n$ be the vertices of K_n. Then $(u_i,v_j) \in K_m \times K_n$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Since $u_iu_k \in E(K_m)$, for all $k \neq i$ and $v_jv_l \in E(K_n)$ for all $l \neq j$, $d((u_i,v_j),(u_k,v_l)) \leq 2$. Therefore, $K_m \times K_n \in F_{22}$. Hence $R(K_m \times K_n) = \overline{K_m \times K_n}$. In $K_m \times K_n, d((u_i,v_j),(u_k,v_l)) = 2$ if and only if $i \neq k$ and $j \neq l$, that is, (u_i,v_j) and (u_k,v_l) are adjacent in $R(K_m \times K_n)$ if and only if $i \neq k$ and $j \neq l$. On the other hand, $R(K_m) \wedge R(K_n) = K_m \wedge K_n$. In $K_m \wedge K_n, (u_i,v_j)$ and (u_k,v_l) are adjacent if and only if $u_iu_k \in E(K_m)$ and $v_jv_l \in E(K_n)$ if and only if $i \neq k$ and $j \neq l$. Hence $R(K_m \times K_n) = R(K_m) \wedge R(K_n) = K_m \wedge K_n$. □

Theorem 2.4. *For any two integers $m,n \geq 3$, $R(C_m[K_n]) = R(C_m) \vee \overline{R(K_n)}$.*

Proof. Let $u_0,u_1,u_2,\dots,u_{m-1}$ be the vertices of C_m and $v_0,v_1,v_2,\dots,v_{n-1}$ be the vertices of K_n. Then (u_i,v_j) are the vertices of $C_m\,[K_n]$, for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Case (i): m is odd.
For each u_i, the radial vertices in C_m are $u_{i+\lfloor \frac{m}{2} \rfloor}$ and $u_{i-\lfloor \frac{m}{2} \rfloor}$ where the subscript is taken over addition modulo n. Therefore, $r(C_m[K_n]) = r(C_m) = \lfloor \frac{m}{2} \rfloor$. In $C_m[K_n]$, for any j, the radial vertices of (u_i,v_j) are (u_k,v_l) where $k = i \pm \lfloor \frac{m}{2} \rfloor$ and $0 \leq l \leq n-1$, $0 \leq i \leq m-1$. Thus, any two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(C_m[K_n])$ if and only if $k = i \pm \lfloor \frac{m}{2} \rfloor$. On the other hand, since $R(K_n) = K_n$, we have $\overline{R(K_n)} = \overline{K_n}$, a totally disconnected graph. Hence any two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(C_m) \vee \overline{R(K_n)}$ if and only if u_i and u_k are adjacent in $R(C_m)$ if and only if $k = i \pm \lfloor \frac{m}{2} \rfloor$ where $0 \leq i \leq m-1$.

Case (ii): m is even.
For each u_i, the radial vertices in C_m are $u_{i+\frac{m}{2}}$ where the subscript is taken over addition modulo n. Therefore, $r(C_m[K_n]) = r(C_m) = \frac{m}{2}$. For any j, the radial vertices of (u_i,v_j) in $C_m[K_n]$ are (u_k,v_l) where $k = i + \frac{m}{2}$ and $0 \leq l \leq n-1$, $0 \leq i \leq m-1$. Hence any two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(C_m[K_n])$ if and only if $k = i + \frac{m}{2}$. On the other hand, since $\overline{R(K_n)} = \overline{K_n}$, a totally disconnected graph. Thus, any two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(C_m) \vee \overline{R(K_n)}$ if and only if u_i and u_k are adjacent in $R(C_m)$ if and only if $k = i + \frac{m}{2}$ where $0 \leq i \leq m-1$. □

Theorem 2.5. *For any two integers m and n, $R(P_m[K_n]) = R(P_m) \vee \overline{R(K_n)}$.*

Proof. Let $u_0,u_1,u_2,\dots,u_{m-1}$ be the vertices of P_m and $v_0,v_1,v_2,\dots,v_{n-1}$ be the vertices of K_n.
Case (i): m is odd.
In P_m, each u_i is radial to $u_{i+\frac{m-1}{2}}$ for $0 \le i \le \frac{m-3}{2}$ and radial to u_0 and u_{m-1} for $i=\frac{m-1}{2}$ and radial to $u_{i-\frac{m-1}{2}}$ for $\frac{m+1}{2} \le i \le m-1$. Therefore, $r(P_m)=\frac{m-1}{2}=r(P_m[K_n])$. In $P_m[K_n]$, for any j and $0 \le i \le \frac{m-3}{2}$, (u_i,v_j) is radial to (u_k,v_l) where $k=i+\frac{m-1}{2}$, $0 \le l \le n-1$. For $i=\frac{m-1}{2}$, (u_i,v_j) is radial to (u_k,v_l) where $k=0$ and $k=m-1$, $0 \le l \le n-1$, and for $\frac{m+1}{2} \le i \le m-1$, (u_i,v_j) is radial to (u_k,v_l), where $k=i-\frac{m-1}{2}$, $0 \le l \le n-1$. Hence in $R(P_m[K_n])$, two vertices (u_i,v_j) and (u_k,v_l) are adjacent if and only if u_i and u_k are radial vertices in P_m.
Case (ii): m is even.
In P_m, each u_i is radial to $u_{i+\frac{m}{2}}$ for $0 \le i \le \frac{m}{2}-1$ and radial to $u_{i-\frac{m}{2}}$ for $\frac{m}{2} \le i \le m-1$. Thus, $r(P_m)=\frac{m}{2}=r(P_m[K_n])$. In $P_m[K_n]$, for any j and $0 \le i \le \frac{m}{2}-1$, (u_i,v_j) is radial to (u_k,v_l) where $k=i+\frac{m}{2}$, $0 \le l \le n-1$, and for $\frac{m}{2} \le i \le m-1$, (u_i,v_j) is radial to (u_k,v_l) where $k=i-\frac{m}{2}$, $0 \le l \le n-1$. Thus, two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(P_m[K_n])$ if and only if u_i and u_k are radial vertices in P_m.

Therefore, in both cases, two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(P_m[K_n])$ if and only if u_i and u_k are radial vertices in P_m. On the other hand, since $\overline{R(K_n)}$ is a totally disconnected graph, any two vertices (u_i,v_j) and (u_k,v_l) are adjacent in $R(P_m)\vee\overline{R(K_n)}$ if and only if u_i and u_k are adjacent in $R(P_m)$ if and only if u_i and u_k are radial vertices in P_m. □

Theorem 2.6. *For any graph G with $r(G)\ge 2$, $R(K_m[G])=K_m/G$.*

Proof. Let $u_1,u_2,\dots,u_m$ be the vertices of K_m and $v_1,v_2,\dots,v_n$ be the vertices of G and let (u_i,v_j) and (u_k,v_l) be any two vertices in $K_m[G]$.
Case (i): $i \ne k$.
Since u_i is adjacent to u_k in K_m, $d((u_i,v_j),(u_k,v_l))=1$ in $K_m[G]$ where j and l varies from $1,2,\dots,n$.
Case (ii): $i=k$.
If v_j is adjacent to v_l in G, then $d((u_i,v_j),(u_k,v_l))=1$ and if v_j is not adjacent to v_l in G, then $d((u_i,v_j),(u_k,v_l))=2$.

Thus, the eccentricity of each vertex in $K_m[G]$ is 2 and hence $K_m[G]$ is of radius 2 and diameter 2. Hence $R(K_m[G])=\overline{K_m[G]}$. Then by the definition of rejection and composition, this result follows. □

Observation 2.7. *For any graphs G and H, $r(G\circ H)=r(G)+1$ and $d(G\circ H)=d(G)+2$.*

Proof. Clearly, $r(G\circ H)=r(G)+1$. Let $d(G)=k$. Then the length of the path in G is k. Let it be $v_1,v_2,\dots,v_k$. In $G\circ H$, every vertex v_i of G is joined to each other vertex of the corresponding ith copy of H. In particular, v_{k+1} is joined to each vertex in the $(k+1)$th copy of H. Therefore, the length of the longest path in $G\circ H$ should be $k+2$. □

Theorem 2.8. *If G_1 is a self centered graph with p_1 vertices and $r(G_1)\ge 2$, then $R(G_1\circ G_2)$ is a (p_1+1)-partite graph where G_2 is any connected graph.*

Proof. Let $v_1,v_2,\dots,v_{p_1}$ be the vertices of G_1 and let $w_{i_1},w_{i_2},\dots,w_{i_{p_2}}$ be the vertices of ith copy of G_2. In $G_1\circ G_2, e(v_i)=e_{G_1}(v_i)+1$ where $1\le i\le p_1$ and $e(w_{i_j})=e_{G_1}(v_i)+2$ where $1\le i\le p_1$ and $1\le j\le p_2$. Hence $r(G_1\circ G_2)=r(G_1)+1$ and $d(G_1\circ G_2)=d(G_1)+2$. Since G_1

is self-centered, $d_{G_1}(v_i,v_j) \leq r(G_1)$ and hence $d_{G_1\circ G_2}(v_i,v_j)$ will never be equal to $r(G_1)+1$. Therefore, $(v_i,v_j) \notin E(R(G_1\circ G_2))$. In $G_1\circ G_2, d(w_{i_j},w_{i_k}) \leq 2$ and hence $(w_{i_j},w_{i_k}) \notin E(R(G_1\circ G_2))$. Hence $\{v_1,v_2,\dots,v_{p_1}\}$, $\{w_{i_1},w_{i_2},w_{i_3},\dots,w_{i_{p_2}} : 1 \leq i \leq p_1\}$ are the partitions of $R(G_1\circ G_2)$. Since $r(G_1\circ G_2)$ is finite, $R(G_1\circ G_2)$ is connected and hence the result follows. □

Theorem 2.9. *If $G_1 \in F_{12}$, $\overline{G_1}$ is a m-partite graph and $\overline{G_2}$ is a n-partite graph, then $R(G_1\circ G_2)$ is a $(m+n)$-partite graph.*

Proof. Let $G_1 \in F_{12}$. Then $r(G_1) = 1$ and $d(G_1) = 2$. Now, $r(G_1\circ G_2) = r(G_1)+1 = 2$ and $d(G_1\circ G_2) = d(G_1)+2 = 4$. Let $\overline{G_1}$ be the m-partite graph with partitions $\{a_{11},a_{12},\dots,a_{1r_1}\}$, $\{a_{21},a_{22},\dots,a_{2r_2}\},\dots,\{a_{m1},a_{m2},\dots,a_{mr_m}\}$ and $\overline{G_2}$ be the n-partite graph with partitions $\{b_{11},b_{12},\dots,b_{1s_1}\},\{b_{21},b_{22},\dots,b_{2s_2}\},\dots,\{b_{n1},b_{n2},\dots,a_{ns_n}\}$ where $r_1+r_2+\dots+r_m = p_1$ and $s_1+s_2+\dots+s_n = p_2$. Since $a_{i1},a_{i2},\dots,a_{ir_i}$ belong to the same partition in $\overline{G_1}$ for each i, they are not adjacent to each other in $\overline{G_1}$ and hence they are adjacent in G_1. Then by the definition of $G_1\circ G_2$, they are adjacent to each other in $R(G_1\circ G_2)$. As $r(G_1\circ G_2) = 2$, they are not adjacent to each other in $R(G_1\circ G_2)$. Therefore, $\{a_{i1},a_{i2},\dots,a_{ir_i}\}$, $1 \leq i \leq m$ are the m-partitions of vertices of G_1 in $G_1\circ G_2$. Since $b_{j1},b_{j2},\dots,b_{js_j}$ are in the same partition of $\overline{G_2}$ for each j, they are not adjacent in $\overline{G_2}$ and hence adjacent in G_2. Thus, they are adjacent in $G_1\circ G_2$ with labels as $b_{ij1},b_{ij2},\dots,b_{ijs_j}$ in the ith copy of G_2 where $1 \leq i \leq p_1$. These vertices are not adjacent in $R(G_1\circ G_2)$. Thus, $b_{ij1},b_{ij2},\dots,b_{ijs_j}$ for all $i = 1,2,\dots,p_1$ belong to the same partition in $R(G_1\circ G_2)$. This is true for each of the n-partitions. Hence the result follows. □

Theorem 2.10. *For any two integers $m,n > 3$, the radius of the join graph of P_m and P_n is 2.*

Proof. In P_m+P_n, every vertices of P_m is adjacent to every vertices of P_n, that is, the distance between every vertices of P_m and every vertices of P_n is 1. Also, the distance between any two vertices of P_m is 2 in P_m+P_n, and the distance between any two vertices of P_n is 2 in P_m+P_n. Hence the radius of the graph P_m+P_n is 2. □

Corollary 2.11. *For any two integers $m,n > 3$, the radius of the join graph of C_m and C_n is 2.*

Theorem 2.12. *For any two integers $m,n > 3$, $R(K_m+K_n) = K_m+K_n$.*

Proof. In K_m+K_n, the distance between every vertices of P_m and every vertices of P_n is 1, and $r(K_n) = 1$ for all n. Thus, the radius of K_m+K_n is 1, that is, K_m+K_n is a complete graph. Therefore, by the Result 1.1, $R(K_m+K_n) = K_m+K_n$. □

Theorem 2.13. *For any two graphs G_1 and G_2, and both are not complete graphs with atleast four vertices, the radius of the join graph of G_1 and G_2 is 2.*

Proof. Every vertices of G_1 is adjacent to every vertices of G_2 in G_1+G_2. Also, the distance between any two vertices of G_1 is 2 in G_1+G_2, and the distance between any two vertices of G_2 is 2 in G_1+G_2. Hence the radius of the graph G_1+G_2 is 2. □

Theorem 2.14. *For any integers m and $n \geq 3$, $P_m \otimes P_n$ is a graph of infinite radius.*

Proof. Since there is no path connecting the vertices u_iv_i and u_iv_{i+1} in $P_m \otimes P_n$, the distance between u_iv_i and u_iv_{i+1} is ∞ for any vertex u_iv_i, that is, the eccentricity of any vertex u_iv_i is ∞. Hence $P_m \otimes P_n$ is a graph of infinite radius. □

3. Conclusion

In this paper, we explored the concept of radial graphs, defining radial vertices as those pairs whose distance equals the radius of the graph, and constructed radial graphs $R(G)$ based on this definition. We examined the behavior of radial graphs in various families of product graphs, including Cartesian, direct, and strong products, each exhibiting unique properties affecting the structure of their radial graphs. This study enhances our understanding of the connectivity and structure of complex networks, with implications for network theory, communication systems, and parallel computing. The findings provide a foundation for further research into specific applications and extensions to other graph operations and transformations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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