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A-transform of Wavelet Frames

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Abstract. In this paper, we introduced the concept of A-transform $A = (a_{p,q,j,k})$ and study the action of A on $f \in L^2(\mathbb{R}^+)$ and on its wavelet coefficients. Further, we also establish the Parseval frame condition for A-transform of $f \in L^2(\mathbb{R}^+)$ whose wavelet series expansion is known.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [8] in 1952 to study some deep problems in non-harmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [7] and popularized from then on. Nice properties of frames make them useful in characterization of function spaces and other fields of applications such as filter bank theory, medicine, optics, sigma-delta quantization, signal and image processing and wireless communications. Recently the theory of frames also showed connections to theoretical problems such as the Kadison-Singer Problem. We refer [2, 12, 13] for an introduction to frame theory and its applications.

In 1982, Jean Morlet, introduced the idea of the wavelet transform and provided a new mathematical tool for time-frequency analysis. Morlet first introduced wavelets as a family of functions constructed from translations and dilations of a single function $\psi(x)$ called the *mother wavelet* and defined as

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, \, a \neq 0$$

where a represents the dilation parameter and b the translation parameter. For some very special choices of ψ and a, b, the $\psi_{a,b}$ constitute an orthonormal basis for $L^2(\mathbb{R})$. In particular, if we choose $a = 2^{-j}$, $b = k2^{-j}$, $j, k \in \mathbb{Z}$, then there exists ψ such that the functions

$$\psi_{j,k}(x) = \psi_{a,b}(x) = 2^{j/2}\psi(2^j x - k)$$

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constitute an orthonormal basis for $L^2(\mathbb{R})$ (see [6, 24]).

Wavelet systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. An important problem in practice is therefore to determine conditions for wavelet systems to be frames. Many results, including necessary conditions and sufficient conditions, have been established during last two decades. For example, in 1990, Daubechies [6] proved the first result on the necessary and sufficient conditions for the wavelet system $\psi_{a,b}(x)$ to be a frame for $L^2(\mathbb{R})$, Chui and Shi improved the result of Daubechies in [3] and gave the characterization of tight wavelet frames for an arbitrary dilation factor $a > 1$ in [4], Casazza and Christenson established a stronger version of Daubechies sufficient condition for wavelet frames in [1, 2]. Recently, Shi and Chen [25] have established a set of necessary conditions for wavelet frames and showed that these conditions are also sufficient for tight frames.

In the early nineties, a general scheme for the construction of wavelets was defined. This scheme is based on the notion of *multiresolution analysis* (MRA) introduced by Mallat [17]. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA on $\mathbb{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups [5], Lang [14, 15, 16] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathscr{C}$ by following the procedure of Daubechies [6] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [9] extended the results of Lang [14, 15, 16] on the wavelet analysis on the Cantor dyadic group $\mathscr{C}$ to the locally compact Abelian group G which is defined for an integer $p \geq 2$ and coincides with $\mathscr{C}$ when $p = 2$. The construction of dyadic compactly supported wavelets for $L^2(\mathbb{R}^+)$ have been given by Protasov and Farkov in [19] where the latter author has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ and proved necessary and sufficient conditions for scaling filters with p^n many terms $(p, n \geq 2)$ to generate a p-MRA analysis in $L^2(\mathbb{R}^+)$ (see [10]). More results in this direction can be found in [21, 22, 23] and the references therein.

Recently, Shah and Debnath [22], have constructed dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ using the Walsh-Fourier transform and have established a necessary and a sufficient conditions for the system

$$\{\psi_{j,k}(x) =: 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\}$$

to be a frame in $L^2(\mathbb{R}^+)$. The conditions obtained by Shah and Debnath are better than those of Daubechies [6], Chui and Shi [3, 4], and Casazza and Christenson [1, 2]. In this paper, we further investigate the dyadic wavelet system $\psi_{j,k}$ by A-transform, where $A = (a_{p,q,j,k})_{p,q,j,k}$ is a regular double infinite matrix and we will establish the Parseval frame condition for this system. Further, wavelet coefficients and the convergence of these coefficients are also being established.

We have organized this paper as follows. In Section 2, we state some basic notations and preliminaries related to the operations on positive half-line $\mathbb{R}^+$, Walsh functions and polynomials. In Section 3, we explore the new concept of A-transform and derive our main results.

2. Notations and preliminaries

Let $\mathbb{R}^+ = [0, +\infty)$ be the positive half-line, $\mathbb{Z}^+$ and $\mathbb{N}$ be the sets of positive integers and natural numbers, respectively. Designate C_0, the space of all double sequences in $\mathbb{R}^+$ which converges to zero. Let us denote the integer and the fractional parts of a number $x \in \mathbb{R}^+$ by $[x]$ and $\{x\}$ respectively. Then, for each $x \in \mathbb{R}^+$ and any $j \in \mathbb{N}$, the values $x_j, x_{-j} \in \{0, 1\}$ are defined as follows:

$$x_j = [2^j x](mod\, 2), \quad x_{-j} = [2^{1-j} x](mod\, 2). \tag{2.1}$$

For each $x \in \mathbb{R}^+$, these numbers are the digits of the binary expansion

$$x = [x] + \{x\} = \sum_{j<0} x_j 2^{-j-1} + \sum_{j>0} x_j 2^{-j}.$$

It is clear that

$$[x] = \sum_{j=1}^{\infty} x_{-j} 2^{j-1}, \quad \{x\} = \sum_{j=1}^{\infty} x_j 2^{-j}$$

and there exists $k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0$ for all $j > k$.

The binary addition on $\mathbb{R}^+$ is defined by the formula

$$x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j}$$

where x_j, y_j are defined in (2.1). Moreover, we note that $x \oplus y = x \ominus y$ as $x \ominus y = 0$ where $\ominus$ denotes the substitution *modulo* 2 on $\mathbb{R}^+$.

For $x \in [0, 1)$, let $w_1(x)$ be given by

$$w_1(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1). \end{cases}$$

The extension of the function w_1 to $\mathbb{R}^+$ is denoted by the equality $w_1(x+1) = w_1(x)$ for all $x \in \mathbb{R}^+$. Then, the generalized Walsh functions $\{w_n(x) : n \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1, \;\; w_n(x) = \prod_{j=0}^{k} \left(w_1(2^j x) \right)^{\mu_j}, \quad n \in \mathbb{N}, \; x \in \mathbb{R}^+,$$

where $n = \sum\limits_{j=0}^{k} \mu_j 2^j$, $\mu_j \in \{0, 1\}$, $\mu_k = 1$, $k = k(n)$.

Note that the Walsh functions almost behaves like characters with respect to dyadic addition, namely

$$w_n(x \oplus y) = w_n(x) w_n(y), \quad n \in \mathbb{N}, \; x, y \in [0, 1). \tag{2.2}$$

Thus, for each fixed y, equality (2.2) is valid for all $x \in \mathbb{R}^+$ except countably many of them.

For $x, y \in \mathbb{R}^+$, let

$$\chi(x,y) = (-1)^{\sigma(x,y)}, \quad \text{where } \sigma(x,y) = \sum_{j=1}^{\infty}(x_j y_{-j} + x_{-j} y_j) \tag{2.3}$$

and x_j, y_j are given by (2.1). Note that $\chi(x, 2^{-n}m) = \chi(2^{-n}x, m) = w_m(2^{-n}x)$, for all $x \in [0, 2^{-n})$ and $m, n \in \mathbb{Z}^+$. It is shown in [11] that both the systems $\{\chi(\alpha,\cdot)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot,\alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1)$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\overline{\chi(x,\xi)}\,dx,$$

where $\chi(x,\xi)$ is given by (2.3). The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [11, 20]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}.$$

For any function $\psi \in L^2(\mathbb{R}^+)$, we consider the system of functions $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^+}$ in $L^2(\mathbb{R}^+)$ as

$$\{\psi_{j,k}(x) =: 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+, x \in \mathbb{R}^+\}. \tag{2.4}$$

By taking Walsh-Fourier transform to (2.4), we obtain

$$\hat{\psi}_{j,k}(\xi) = 2^{-j/2}\hat{\psi}(2^{-j}\xi)w_k(2^{-j}\xi).$$

Therefore, by Plancheral theorem, we have

$$c_{j,k} = \langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}^+} f(x)\overline{\psi_{j,k}}(x)dx, \quad f \in L^2(\mathbb{R}^+). \tag{2.5}$$

We recall that the system (2.4) is a wavelet frame for $L^2(\mathbb{R}^+)$ if there exist constants C and D, $0 < C \le D < \infty$ such that, for every $f \in L^2(\mathbb{R}^+)$

$$C\|f\|^2 \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\langle f, \psi_{j,k}\rangle\right|^2 \le D\|f\|^2. \tag{2.6}$$

The constants C and D are known respectively as lower and upper frame bounds. A frame is called tight frame if the lower and upper frame bounds are equal, $C = D$. A frame is a Parseval frame if $C = D = 1$ and in this case, every function $f \in L^2(\mathbb{R}^+)$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+} c_{j,k}\psi_{j,k}(x) \tag{2.7}$$

where $c_{j,k} = \langle f, \psi_{j,k}\rangle$ are given by (2.5), usually known as wavelet coefficients of the given wavelet series (2.7).

Let $A = (a_{p,q,j,k})$ be a double infinite matrix of real numbers. Then, *A-transform* of double sequence $\{x_{j,k}\}_{(j,k)\in\mathbb{Z}^+\times\mathbb{Z}^+}$ is defined as

$$\sum_{j\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+} a_{p,q,j,k}x_{j,k}. \tag{2.8}$$

This definition is due to Moricz and Rhoades [18]. It is shown in [18] that the necessary and sufficient condition for a matrix A to be regular is

(i) $\lim\limits_{p,q\to+\infty}\sum\limits_{j\in\mathbb{Z}^+}\sum\limits_{k\in\mathbb{Z}^+} a_{p,q,j,k} = 1$

(ii) $\lim\limits_{p,q\to+\infty}\sum\limits_{j\in\mathbb{Z}^+} |a_{p,q,j,k}| = 0$, for every $k\in\mathbb{Z}^+$

(iii) $\lim\limits_{p,q\to+\infty}\sum\limits_{k\in\mathbb{Z}^+} |a_{p,q,j,k}| = 0$, for every $j\in\mathbb{Z}^+$

(iv) $||A|| = \sup_{p,q\in\mathbb{N}}\sum\limits_{j\in\mathbb{Z}^+}\sum\limits_{k\in\mathbb{Z}^+} |a_{p,q,j,k}| < \infty.$

Either of conditions (ii) and (iii) implies that

$$\lim_{p,q\to\infty} a_{p,q,j,k} = 0. \tag{2.9}$$

3. Main results

Theorem 3.1. *Let $A = (a_{p,q,j,k})$ be a double regular matrix whose elements are of the form $a_{p,q,j,k} = \langle\psi_{p,q},\psi_{j,k}\rangle$ and if*

(i) $\sum\limits_{j\in\mathbb{Z}^+}\sum\limits_{k\in\mathbb{Z}^+}\psi_{j,k}\int_{\mathbb{R}^+} f(y)\overline{\psi_{j,k}(y)}dy = 1$

(ii) $\lim\limits_{p,q\to\infty}\psi_{p,q}(x) = \psi(x).$

Then, A-transform of the sequence of wavelet coefficients $\{c_{j,k}\}$ belong to C_0.

Proof. Since the elements of a double non-negative infinite matrix are of the type $\langle\psi_{j,k},\psi_{p,q}\rangle$ and the wavelet coefficients $c_{j,k}$ are given by Eq. (2.5). Thus, we have

$$\begin{aligned} a_{p,q,j,k}c_{j,k} &= \langle\psi_{j,k},\psi_{p,q}\rangle\langle f,\psi_{j,k}\rangle \\ &= \int_{\mathbb{R}^+}\psi_{j,k}(x)\overline{\psi_{p,q}(x)}dx\int_{\mathbb{R}^+} f(x)\overline{\psi_{j,k}(x)}dx \\ &= \int_{\mathbb{R}^+} f(x)\overline{\psi_{p,q}(x)}dx\int_{\mathbb{R}^+}\psi_{j,k}(x)\overline{\psi_{j,k}(x)}dx \end{aligned}$$

Therefore, we can write

$$\sum_{j\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+} a_{p,q,j,k}c_{j,k} = \sum_{j\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\psi_{p,q}(x)\overline{\psi_{j,k}(x)}f(y)\overline{\psi_{j,k}(y)}dx\,dy$$

Using condition (i) and (ii), and the fact that the dyadic wavelet ψ satisfies $\int_{\mathbb{R}^+}\psi(x)dx = 0$, we obtain

$$\lim_{p,q\to\infty}\sum_{j\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+} a_{p,q,j,k}c_{j,k} = \lim_{p,q\to\infty}\int_{\mathbb{R}^+}\psi_{p,q}(x)dx = \int_{\mathbb{R}^+}\psi(x)dx = 0. \qquad \square$$

Theorem 3.2. *Let $A = (a_{p,q,j,k})$ be a double non-negative regular matrix and if $c_{j,k}$ are the wavelet coefficients associated with the wavelet expansion (2.7). Then, the frame condition for A-transform of $f \in L^2(\mathbb{R}^+)$ is given by*

$$C_\psi \|f\|_2^2 \le \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \le D_\psi \|f\|_2^2$$

where Af is the A-transform of $f \in L^2(\mathbb{R}^+)$ and $0 < C_\psi \le D_\psi < \infty$.

Proof. We have

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x)$$

Taking A-transform of f, we get

$$Af(x) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} \langle Af, \psi_{p,q}\rangle \psi_{p,q}(x)$$

and therefore

$$\begin{aligned} \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 &\le \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} \int_{\mathbb{R}^+} |Af(x)|^2 |\overline{\psi_{p,q}(x)}|^2 dx \\ &\le \|A\|^2 \|f\|_2^2 \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} \|\psi_{p,q}\|^2. \end{aligned}$$

Since we have assumed that A is a regular matrix and $\|\psi_{p,q}\|_2 = 1$. Thus, we have

$$\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \le D_\psi \|f\|_2^2 \tag{3.1}$$

where D_ψ is a positive constant.

Now, for any arbitrary function $f \in L^2(\mathbb{R}^+)$, we define

$$g(x) = \left[\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \right]^{-1/2} f(x).$$

Clearly

$$\langle Ag, \psi_{p,q}\rangle = \left[\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \right]^{-1/2} \langle Af, \psi_{p,q}\rangle.$$

Hence

$$\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \le 1.$$

Now, if there exists a constant $M > 0$ such that $\|Ag\|_2^2 \le M$, then

$$\left[\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \right]^{-1} \|Af\|_2^2 \le M$$

or

$$\left[\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \right]^{-1} \|f\|_2^2 \le \frac{M}{\|A\|_2^2} = C_\psi > 0$$

or

$$C_\psi \|f\|_2^2 \leq \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$C_\psi \|f\|_2^2 \leq \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle Af, \psi_{p,q}\rangle|^2 \leq D_\psi \|f\|_2^2.$$

which is the desired result. □

Theorem 3.3. *If $\{c_{j,k}\}$ are the wavelet coefficients of $f \in L^2(\mathbb{R}^+)$. Then*

$$d_{p,q} = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} a_{p,q,j,k} c_{j,k} \tag{3.3}$$

where $\{d_{p,q}\}$ is defined as the A-transform of $\{c_{j,k}\}$.

Proof. By taking A-transform of Eq. (2.5), we obtain

$$\begin{aligned}\langle Af, \psi_{j,k}\rangle &= \int_{\mathbb{R}^+} Af(x)\overline{\psi_{p,q}(x)}dx \\ &= \int_{\mathbb{R}^+} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} a_{p,q,j,k} c_{j,k} \psi_{j,k}(x)\overline{\psi_{p,q}(x)}dx\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} \langle Af, \psi_{p,q}\rangle \psi_{p,q}(x) &= \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} a_{p,q,j,k} c_{j,k} \psi_{j,k}(x)\overline{\psi_{p,q}(x)}dx \\ &= \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} d_{p,q}\psi_{p,q}(x) \int_{\mathbb{R}^+} \|\psi_{p,q}\|_2^2 \\ &= \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} d_{p,q}\psi_{p,q}(x).\end{aligned}$$

Thus

$$\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} d_{p,q}\psi_{p,q}(x) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} \langle Af, \psi_{p,q}\rangle \psi_{p,q}(x)$$

which implies that $d_{p,q} = \langle f, \psi_{p,q}\rangle$ are the wavelet coefficients of Af. □

Theorem 3.4. *Let $A = (a_{p,q,j,k})$ be a double non-negative infinite matrix whose elements are $\langle \psi_{j,k}, \psi_{p,q}\rangle$. Then*

$$\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |d_{p,q}|^2 = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}^+} |\langle f, \psi_{p,q}\rangle|^2 = \|f\|_2^2$$

where $d_{p,q} = \langle f, \psi_{p,q}\rangle$ is the A-transform of the wavelet coefficients $c_{j,k}$.

Proof. We have

$$\begin{aligned}
\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}|d_{p,q}|^2 &= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}|a_{p,q,j,k}c_{j,k}|^2\\
&= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}|\langle f,\psi_{p,q}\rangle|^2\\
&= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}|\langle \hat f,\hat\psi_{p,q}\rangle|^2\\
&= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}\left|2^{-p/2}\int_{\mathbb{R}^+}\hat f(\xi)\overline{\hat\psi(2^{-p}\xi)}w_q(2^{-p}\xi)d\xi\right|^2\\
&= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}\left|2^{-p/2}\int_0^{2^p}\left[\sum_{\ell\in\mathbb{Z}^+}F_{p,\ell}(\xi)\right]w_q(2^{-p}\xi)d\xi\right|^2 \qquad (3.4)
\end{aligned}$$

where $F_{p,\ell}(\xi)=\hat f(\xi\oplus 2^p\ell)\overline{\hat\psi(2^{-p}\xi\oplus\ell)}$. Now, for each $p\in\mathbb{Z}$, we define

$$F_p(\xi)=\sum_{\ell\in\mathbb{Z}^+}F_{p,\ell}(\xi). \qquad (3.5)$$

Clearly $F_p(\xi\oplus 2^p)=F_p(\xi)$, for all $\xi\in\mathbb{R}^+$ and therefore it can be expanded in Walsh series as

$$F_p(\xi)=\sum_{q\in\mathbb{Z}^+}g_q(F_p)w_q(2^{-p}\xi),\quad \xi\in[0,2^p),$$

where $g_q(F_p)=2^{-p}\int_0^{2^p}F_p(\xi)w_q(2^{-p}\xi)d\xi$. Moreover, by Parsevals formula, we have

$$\sum_{q\in\mathbb{Z}^+}|g_q(F_p)|^2=2^{-p}\int_0^{2^p}|F_p(\xi)|^2d\xi. \qquad (3.6)$$

By in-cooperating (3.5) and (3.6) in (3.4) and using the fact that $\sum_{p\in\mathbb{Z}}|\hat\psi(2^{-p}\xi)|^2=1$ $a.e$, we get

$$\begin{aligned}
\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}|d_{p,q}|^2 &= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}^+}2^{-p}\left|\int_0^{2^p}F_p(\xi)w_q(2^{-p}\xi)d\xi\right|^2\\
&= \sum_{p\in\mathbb{Z}}2^{-p}\sum_{q\in\mathbb{Z}^+}|g_q(F_p)|^2\\
&= \sum_{p\in\mathbb{Z}}\int_0^{2^p}|F_p(\xi)|^2d\xi\\
&= \sum_{p\in\mathbb{Z}}\int_0^{2^p}F_p(\xi)\overline{F_p(\xi)}d\xi
\end{aligned}$$

$$= \sum_{p\in\mathbb{Z}} \int_0^{2^p} \left[\sum_{\ell\in\mathbb{Z}^+} \hat{f}(\xi \oplus 2^p\ell)\overline{\hat{\psi}(2^{-p}\xi \oplus \ell)}\hat{\psi}(2^{-p}\xi \oplus \ell)\overline{\hat{f}(\xi \oplus 2^p\ell)} \right] d\xi$$

$$= \sum_{p\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}^+} \int_{2^p\ell}^{2^p(\ell+1)} \hat{f}(\xi)\overline{\hat{f}(\xi)}\hat{\psi}(2^{-p}\xi)\overline{\hat{\psi}(2^{-p}\xi)} d\xi$$

$$= \int_{\mathbb{R}^+} |\hat{f}(\xi)|^2 \sum_{p\in\mathbb{Z}} |\hat{\psi}(2^{-p}\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^+} |\hat{f}(\xi)|^2 d\xi$$

$$= \|f\|_2^2.$$

This completes the proof of the theorem. □

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