Communications in Mathematics and Applications
Vol. 14, No. 2, pp. 775–790, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i2.2101

http://www.rgnpublications.com

Research Article

Results on the Uniqueness of Difference Polynomials With Difference Operator of Entire Functions Sharing Small Function

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Received: October 28, 2022 **Accepted:** February 23, 2023

Abstract. In this article, we consider difference polynomials with difference operators, weakly weighted sharing, and relaxed weighted sharing, we investigate the uniqueness problem of difference polynomials. We use $\Phi = (Q(f(z))L(\Delta_c f))^{(k)}$ and $\Psi = (Q(g(z))L(\Delta_c g))^{(k)}$. Accordingly, we have proved three uniqueness results, which extends and improves the results due to G. Haldar (Uniqueness of entire functions concerning differential-difference polynomials sharing small functions, *arXiv*:2103.09889v1 [math.CV], (2021)).

Keywords. Uniqueness, Weakly weighted sharing, Relaxed weighted sharing, Entire functions, Small function and difference polynomial with difference operator

Mathematics Subject Classification (2020). Primary 30D35

1. Introduction, Definitions and Main Results

For this article, we deal with the uniqueness of difference polynomials of meromorphic functions sharing small function with finite order. Let f and g be a two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, the zero of $f-a$ and $g-a$ have the same locations as well as same multiplicities, we say that f and g share the value a CM (*Counting Multiplicities*). If we do not consider the multiplicities, then f and g are said to share the value a IM (*Ignoring Multiplicities*). We adopt the standard notations of the Nevanlinna theory of meromorphic functions (see Hayman [6], Yang and Yi [18], and Lahiri [11]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the characteristic function f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ outside of an exceptional set of finite measure. We say that $\alpha(z)$ is a small function of f, if $\alpha(z)$ is a meromorphic function satisfying $T(r,\alpha(z))=S(r,f)$.

We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where a-point is counted according to its multiplicity. Also, we denote by $E_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not exceeding k. We define shift and difference operators of $f(z)$ by $f(z+\eta)$ and $\Delta_\eta f(z)=f(z+\eta)-f(z)$, respectively.

The q-th order difference operator $\Delta_\eta^q f(z)$ is defined by $\Delta_\eta^q f(z)=\Delta_\eta^{q-1}(\Delta_\eta f(z))$, where $q(\geq 2)\in\mathbb{N}$ and $\eta\in\mathbb{C}\{0\}$, while the difference polynomial of difference operator is given by $L(\Delta_\eta f)=\sum_{i=1}^{q} a_i\Delta_\eta^i f$, where a_i $(i=1,2,\dots,q)$ are non-zero constants.

We can also deduce that,

$$\Delta_\eta^q f=\sum_{i=1}^{q}\binom{i}{q}f(z+(q-i)\eta). \tag{1.1}$$

Let $Q(z)=a_nz^n+a_nz^n+\ldots+a_0$ be a non-zero polynomial where $a_n(\neq 0),a_{n-1},\dots,a_0$ are complex constants. We denote $\Theta_1=\tau_1+\tau_2$ and $\Theta_2=\tau_1+2\tau_2$, respectively, where τ_1 is the number of simple zeros of $Q(z)$ and τ_2 is the number of multiple zeros of $Q(z)$. In addition, we need some following definitions:

Definition 1.1 ([9])**.** Let p be a positive integer and $a\in\mathbb{C}\cup\{\infty\}$.

(i) $N(r,a;f\mid\geq p)(\bar N(r,a;f\mid\geq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f\mid\leq p)(\bar N(r,a;f\mid\leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.2 ([13])**.** Let $a\in\mathbb{C}\cup\{\infty\}$ and k be a positive integer or infinity. If

$$\bar N(r,a;f\mid\leq k)-\bar N^E_{k)}(r,a;f,g)=S(r,f),$$

$$\bar N(r,a;g\mid\leq k)-\bar N^E_{k)}(r,a;f,g)=S(r,g),$$

$$\bar N(r,a;f\mid\geq k+1)-\bar N_0^{(k+1)}(r,a;f,g)=S(r,f),$$

$$\bar N(r,a;g\mid\geq k+1)-\bar N_0^{(k+1)}(r,a;f,g)=S(r,g),$$

or if $k=0$ and,

$$\bar N(r,a;f)-\bar N_0(r,a;f,g)=S(r,f),$$

$$\bar N(r,a;g)-\bar N_0(r,a;f,g)=S(r,g),$$

then we say that f and g share the value a weakly with weight k and we write f and g share "(a,k)".

Definition 1.3 ([1])**.** Let $a\in\mathbb{C}\cup\{\infty\}$ and k be a positive integer or infinity. Suppose that f and g share the value a "IM". If for $p\neq q$,

$$\sum_{p,q\leq k} N(r,a;f|=p;g|=q)=S(r).$$

Then, we say that f and g share the value a with weight k in a relaxed manner and we write f and g share $(a,k)^*$.

Let $P(z)=a_m z^m + a_{m-1}z^{m-1}+\ldots+a_0$ be a non-zero polynomial of degree m, where $a_m(\neq 0)$, $a_{m-1},\ldots,a_0(\neq 0)$ are complex constants and m is a positive integer.

Definition 1.4 ([12])**.** Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Definition 1.5 ([12])**.** Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

We define shift and difference operators of $f(z)$ by $f(z+c)$ and $\Delta_c f(z)=f(z+c)-f(z)$, respectively. Note that $\Delta_c^n f(z)=\Delta_c^{n-1}(\Delta_c f(z))$, where c is a non-zero complex number and $n\geq 2$ is a positive integer.

Definition 1.6 ([10, 11])**.** Let k be a non-negative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$, we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.

Clearly, if f,g share (a,k) then f,g share (a,p) for any integer p, $0\leq p\leq k$. Also, we note that f,g share a value a Im or CM if and only if f,g share $(a,0)$ or (a,∞), respectively.

In 1959, Hayman [7] proved following result.

Theorem A ([7])**.** *Let f be a transcendental entire function and let $n(\geq 1)$ be an integer. Then $f^n f'=1$ has infinitely many solutions.*

In 2022, Haldar [5] proved the following result:

Theorem B ([5])**.** *Let f and g be two-transcendental entire functions of finite order, $P\not\equiv 0$ be a polynomial. Let c be a non-zero complex constant, and n be a positive integer such that $2\deg(P)<n+1$. Let l be a non-negative integer such that $f(z)^n L_c(f)-P(z)$ and $g(z)^n L_c(g)-P(z)$ share $(0,l)$ and $g(z)$, $g(z+c)$ share 0 CM. If $n\geq 4$ and $f(z)^n L_c(f)/P(z)$ is a Mobius transformation of $g(z)^n L_c(g)/P(z)$, or one of the following conditions holds:*

(i) *$l\geq 2$ and $n\geq 5$;*

(ii) *$l\geq 1$ and $n\geq 6$;*

(iii) *$l=0$ and $n\geq 11$, then one of the following conclusions can be realized:*

- (a) *$f=tg$, where t is a constant satisfying $t^{n+1}=1$;*
- (b) *when $c_0=0$, $f=e^U$ and $g=te^{-U}$, where $P(z)$ reduces to a non-zero constant d,t is a constant such that $t^{n+1}=d^2$ and U is a non-constant polynomial;*
- (c) *when $c_0\neq 0$, $f=c_1e^{az}$, $g=c_2e^{-az}$, where a,c_1,c_2 and d are non-zero constants satisfying $(c_1c_2)^{n+1}(e^{ac}+c_0)(e^{-ac}+c_0)=d^2$.*

Theorem C ([4]). *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n \geq 2k+m+6$. If $(f(z)^nP(f(z))\mathcal{L}_c(f))^{(k)}$ and $(g(z)^nP(g(z))\mathcal{L}_c(g))^{(k)}$ share "$(\alpha(z),2)$", then one of the following two conclusions can be realized.*

(a) *$f(z)\equiv tg(z)$, where t is a constant such that $t^d=1$, $d=\gcd(\lambda_0,\lambda_1,\dots,\lambda_m)$, where λ_j's are defined by* $\lambda_j = \begin{cases} n+1+j, & \text{if } a_j \neq 0, \\ n+1+m, & \text{if } a_j = 0, \end{cases}$ *where $j=0,1,\dots,m$.*

(b) *f and g satisfy the algebraic equation $R(\omega_1,\omega_2)=0$, where $R(\omega_1,\omega_2)$ is given by $R(\omega_1,\omega_2)=\omega_1^nP(\omega_1)\mathcal{L}_c(\omega_1)-\omega_2^nP(\omega_2)\mathcal{L}_c(\omega_2)$.*

Theorem D ([4]). *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant, n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n\geq 3k+2m+8$. If $(f(z)^nP(f(z))\mathcal{L}_c(f))^{(k)}$ and $(g(z)^nP(g(z))\mathcal{L}_c(g))^{(k)}$ share $(\alpha(z),2)^*$, then the conclusions of Theorem* C *holds.*

Theorem E ([4]). *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant, n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n\geq 9+(7k+5m)/2$. If $E_{2)}(\alpha(z)(f(z)^nP(f(z))\mathcal{L}_c(f))^{(k)}) = E_{2)}(\alpha(z),(g(z)^nP(g(z))\mathcal{L}_c(g))^{(k)})$, then the conclusions of Theorem* C *holds.*

Question 1.1. *What can be said about the uniqueness of f and g if we consider the difference polynomial with difference operator of the form $Q(f)L(\Delta_cf)$ and $Q(g)L(\Delta_cg)$ in Theorems* C *and* D*?*

In this article, we paid our attention to above question and proved the following three results that improve and extend Theorems C and D, respectively. Indeed, the following theorems are the main results of the paper.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose η be a non-zero complex constant, n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n\geq q+\Theta_1+k\tau_2+1$. If $(Q(f(z))L(\Delta_\eta(f)))^{(k)}$ and $(Q(g(z))L(\Delta_\eta(g))^{(k)}$ share "$(\alpha(z),2)$", then one of the following two conclusions can be realized.*

(a) *$f(z)\equiv hg(z)$, where h is a constant such that $h^d=1$,*

$$d=\gcd(m+1,\dots,m+1-i,\dots,1),\quad a_{m-i}\neq 0$$

for $i=0,1,\dots,m$.

(b) *f and g satisfy the algebraic equation $R(\omega_1,\omega_2)=0$, where $R(\omega_1,\omega_2)$ is given by $R(\omega_1,\omega_2)=Q(\omega_1)L(\Delta_\eta(\omega_1))-Q(\omega_2)L(\Delta_\eta(\omega_2))$.*

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose η be a non-zero complex constant, n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n\geq q+\Theta_1+\Theta_2+2k\tau_2+1$. If $(Q(f(z))L(\Delta_\eta(f)))^{(k)}$ and $(Q(g(z))L(\Delta_\eta(g)))^{(k)}$ share $(\alpha(z),2)^*$, then the conclusions of Theorem* 1.1 *holds.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)$ ($\not\equiv 0,\infty$) be a small function with respect to both $f(z)$ and $g(z)$. Suppose η be a non-zero complex constant, n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n \geq q+1+(5\Theta_1+(5k+1)\tau_2)/2$. If $E_{2)}(\alpha(z)(Q(f(z))L(\Delta_\eta(f)))^{(k)}) = E_{2)}(\alpha(z),(Q(g(z))L(\Delta_\eta(g)))^{(k)})$, then the conclusions of Theorem 1.1 holds.*

Corollary 1.1. *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose η be a non-zero complex constant, n, $k(\geq 0)$, $m(\geq k+1)$ are integers such that $n \geq q+\Theta_1+k\tau_2+1$. If $(Q(f(z))L(\mathcal{L}_c(f)))^{(k)}$ and $(Q(g(z))L(\mathcal{L}_c(g)))^{(k)}$ share "$(\alpha(z),2)$", then the conclusions of Theorem 1.1 holds.*

Corollary 1.2. *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose η be a non-zero complex constant, $n,k(\geq 0)$, $m(\geq k+1)$ are integers such that $n \geq q+\Theta_1+\Theta_2+2k\tau_2+1$. If $(Q(f(z))L(\mathcal{L}_\eta(f)))^{(k)}$ and $(Q(g(z))L(\mathcal{L}_\eta(g)))^{(k)}$ share $(\alpha(z),2)^*$, then the conclusions of Theorem 1.1 holds.*

Corollary 1.3. *Let $f(z)$ and $g(z)$ be two-transcendental entire functions of finite order $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose η be a non-zero complex constant, $n,k(\geq 0)$, $m(\geq k+1)$ are integers such that $n \geq q+1+(5\Theta_1+(5k+1)\tau_2)/2$. If $E_{2)}(\alpha(z)(Q(f(z))L(\mathcal{L}_\eta(f)))^{(k)}) = E_{2)}(\alpha(z),(Q(g(z))L(\mathcal{L}_\eta(g)))^{(k)})$, then the conclusions of Theorem 1.1 holds.*

2. Some Lemmas

In this section, we present some lemmas which will be needed in the proof of our results. Henceforth, we denote by H the function defined by

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1 ([21]). *Let $f(z)$ be a non-constant meromorphic function and $Q(f) = a_n f^n + a_{n-1}f^{n-1} + \ldots + a_0$, where $a_0(\neq 0), a_{n-1}, \ldots, a_0$ are complex constant. Then*

$$T(r,Q(f)) = nT(r,f) + S(r,f).$$

Lemma 2.2 ([2]). *Let $f(z)$ be a non-constant meromorphic function of finite order and let η be a non-zero complex constant. Then*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right) + m\left(r,\frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2.3 ([16]). *Let $f(z)$ be a non-constant meromorphic function of finite order and let η be a non-zero complex constant. Then*

$$T(r,L(\Delta_\eta(f))) = qT(r,f) + S(r,f).$$

Lemma 2.4 ([3]). *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\bar{N}(r,\infty;f)+N(r,0;f|<k)+k\bar{N}(r,0;f|\geq k)+S(r,f).$$

Lemma 2.5. *Let $\Phi=Q(f(z))L(\Delta_\eta f)$ where $f(z)$ is an entire function of finite order and $f(z),f(z+\eta)$ share 0 CM. Then*

$$T(r,\Phi)=(n+q+1)T(r,f)+S(r,f).$$

Proof. Keeping in view of Lemmas 2.2 and 2.3, we have

$$\begin{aligned}T(r,f^{n+1})&=T(r,Q(f)L(\Delta_\eta f))\\&\leq T(r,\Phi)+T\left(r,\frac{L(\Delta_\eta f)}{f(z)}\right)+S(r,f)\\&\leq T(r,\Phi)+T\left(r,\frac{f(z)}{L(\Delta_\eta f)}\right)+S(r,f)\\&\leq T(r,\Phi)+N\left(r,\infty;\frac{f(z)}{L(\Delta_\eta f)}\right)+m\left(r,\infty;\frac{f(z)}{L(\Delta_\eta f)}\right)+S(r,f)\\&\leq T(r,\Phi)+qT(r,f)+S(r,f),\end{aligned}$$

i.e.,

$$T(r,\Phi)\leq(n+q+1)T(r,f)+S(r,f).$$ □

Lemma 2.6 ([19]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then*

$$N\left(r,\infty;\frac{f}{g}\right)-N\left(r,\infty;\frac{g}{f}\right)=N(r,\infty;f)+N(r,0;g)-N(r,\infty;g)+N(r,0;f).$$

Lemma 2.7. *Let $f(z)$ be a transcendental entire function of finite order, $\eta\in\mathbb{C}-\{0\}$ be finite complex constant and $n\in\mathbb{N}$. Let $\Phi(z)=Q(f(z))L(\Delta_\eta(f))$, where $L(\Delta_\eta f)\not\equiv 0$. Then*

$$(n+q+1)T(r,f)\leq T(r,\Phi)-N(r,0;\Delta_\eta(f))+S(r,f).$$

Proof. Using Lemmas 2.2, 2.6 and the First Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we obtain

$$\begin{aligned}m(r,Q(f))&=m\left(r,\frac{Q(f)\Phi}{L(\Delta_\eta f)}\right)\\&\leq m(r,\Phi)+m\left(r,\frac{Q(f)}{L(\Delta_\eta f)}\right)+S(r,f)\\&\leq m(r,\Phi)+m\left(r,\frac{Q(f)}{L(\Delta_\eta f)}\right)-N\left(r,\infty;\frac{Q(f)}{L(\Delta_\eta f)}\right)+S(r,f)\\&\leq m(r,\Phi)+m\left(r,\frac{L(\Delta_\eta f)}{Q(f)}\right)-N\left(r,\infty;\frac{Q(f)}{L(\Delta_\eta f)}\right)+S(r,f)\\&\leq m(r,F)+N\left(r,\infty;\frac{L(\Delta_\eta f)}{Q(f)}\right)+m\left(r,\frac{L(\Delta_\eta f)}{Q(f)}\right)-N\left(r,\infty;\frac{Q(f)}{L(\Delta_\eta f)}\right)+S(r,f)\\&\leq m(r,\Phi)+N(r,0;f)-N(r,0;L(\Delta_\eta f))+S(r,f)\end{aligned}$$

i.e.,

$$m(r,Q(f)) \le T(r,\Phi)+T(r,f)-N(r,0;L(\Delta_\eta f))+S(r,f).$$

By Lemma 2.3, we obtain

$$qT(r,f)=m(r,Q(f)) \le T(r,\Phi)+T(r,f)-N(r,0;\Delta_\eta f)+S(r,f),$$

i.e.,

$$(n+q+1)T(r,f)=T(r,\Phi)-N(r,0;\Delta_\eta f)+S(r,f).$$ □

Lemma 2.8 ([1]). *Let F and G be two non-constant meromorphic functions that share $(1,2)^*$. Then*

$$\begin{aligned}&\bar{N}_L(r,1;F)+\bar{N}_{F\ge3}(r,1;G|=1)\\ &\quad \le \bar{N}(r,0;F)+\bar{N}(r,\infty;F)-\sum_{p=3}^{\infty}\bar{N}\left(r,0;\frac{F'}{F}\Big|\ge p\right)-\bar{N}^2(r,0;F')+S(r),\end{aligned}$$

where by $\bar{N}^2(r,0;F')$ is the counting function of those zeros of F' which are not the zeros of $F(F-1)$, where each simple zero is counted once and all other zeros are counted two times.

Lemma 2.9 ([4]). *Let F and G be two non-constant meromorphic functions such that $E_{2)}(1,F)=E_{2)}(1,G)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty;H) \le\ & \bar{N}(r,0;F|\ge2)+\bar{N}(r,0;G|\ge2)+\bar{N}_L(r,1;F)+\bar{N}_L(r,1;G)+\bar{N}(r,\infty;F|\ge2)\\ &+\bar{N}(r,\infty;G|\ge2)+\bar{N}_{F\ge3}(r,\infty;F|G\ne1)+\bar{N}_{G\ge3}(r,\infty;G|F\ne1)+\bar{N}_0(r,0;F')\\ &+\bar{N}_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.10 ([1]). *If f,g be share "$(1,1)$" and $H\not\equiv 0$, then*

$$N(r,1;f|\le1) \le N(r,0;H)+S(r,f) \le N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 2.11 ([1]). *If f,g be two non-constant meromorphic functions such that $E_{2)}(1,f)=E_{2)}(1,g)$ and $H\not\equiv 0$. Then*

$$N(r,1;f|\le1) \le N(r,0;H) \le N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 2.12 ([1]). *If f,g be share $(1,1)^*$ and $H\not\equiv 0$, then*

$$N^E(r,1;f,g|\le1) \le N(r,0;H) \le N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 2.13 ([1]). *If f,g be share $(1,1)^*$ and $H\not\equiv 0$, then*

$$\begin{aligned}N(r,\infty;H) \le\ & \bar{N}(r,0;f|\ge2)+\bar{N}(r,0;g|\ge2)+\bar{N}(r,\infty;f|\ge2)+\bar{N}_*(r,1;f,g)\\ &+\bar{N}(r,\infty;g|\ge2)+\bar{N}_0(r,0;f')+\bar{N}_0(r,0;g')+S(r,f)+S(r,g),\end{aligned}$$

where $\bar{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\bar{N}_0(r,0;g')$ is similarly defined.

Lemma 2.14 ([1]). *Let $E_{2)}(1,f)=E_{2)}(1,g)$. Then*

$$\bar{N}_{f\ge3}(r,1;f|g\ne1) \le \frac{1}{2}\bar{N}(r,0;f)+\frac{1}{2}\bar{N}(r,\infty;f)-\frac{1}{2}\sum_{p=3}^{\infty}\bar{N}\left(r,0;\frac{f'}{F}\Big|\ge p\right)-\frac{1}{2}\bar{N}_0^2(r,0;f')+S(r).$$

Lemma 2.15 ([20]). *Let f be a non-constant meromorphic function, and p,k be positive integers. Then*

$$\begin{aligned}N_p(r,0;f^{(k)}) &\leq T(r,f^{(k)})-T(r,f)+N_{p+k}(r,0;f)+S(r,f)\\ &\leq k\bar{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f).\end{aligned}$$

3. Proofs of the Theorems

Proof of Theorem 1.1. Let $\Phi=\frac{\Phi_*^{(k)}}{\alpha(z)}$ and $\Psi=\frac{\Psi_*^{(k)}}{\alpha(z)}$, where $\Phi_*=Q(f(z))L(\Delta_\eta(f))$ and $\Psi_*=Q(g(z))L(\Delta_\eta(g))$. Then Φ and Ψ are two-transcendental meromorphic functions that share "(1,2)" except the zeros and poles of $\alpha(z)$. We consider the following two cases:

Case 1: Suppose $H\not\equiv 0$. Since Φ and Ψ share "(1,2)", it follows that Φ and Ψ share $(1,1)^*$. Keeping in view of Lemmas 2.10 and 2.13, we see that

$$\begin{aligned}\bar{N}(r,1;\Phi)&=N(r,1;\Phi|\leq 1)+\bar{N}(r,1;\Phi|\geq 2)\\ &\leq N(r,\infty;H)+\bar{N}(r,1;\Phi|\geq 2)\\ &\leq \bar{N}(r,0;\Phi|\geq 2)+\bar{N}(r,0;\Psi|\geq 2)+\bar{N}_*(r,1;\Phi,\Psi)+\bar{N}(r,1;\Phi|\geq 2)+\bar{N}_0(r,1;\Phi')\\ &\quad+\bar{N}_0(r,1;\Psi')+S(r,\Phi)+S(r,\Psi).\end{aligned}\tag{3.1}$$

Since Φ and Ψ share "(1,2)", we must have $\bar{N}_{\Phi\geq 2}(r,1;\Phi|\Psi\neq 1)=S(r,\Phi)$ and $\bar{N}(r,1;\Phi|\geq 2,\Psi|=1)=S(r,\Phi)$. Therefore, keeping in view of the above equation and Lemma 2.4, we get

$$\begin{aligned}&\bar{N}_0(r,0;\Psi')+\bar{N}(r,0;\Phi|\geq 2)+\bar{N}_*(r,1;\Phi,\Psi)\\ &\leq \bar{N}_0(r,0;\Psi')+\bar{N}(r,0;\Phi|\geq 3)+\bar{N}_{\Phi\geq 2}(r,1;\Phi|\Psi\neq 1)+\bar{N}(r,1;\Phi|\geq 2,\Psi|=1)\\ &\quad+\bar{N}(r,1;\Phi|\geq 2,\Psi|\geq 2)+S(r,\Psi)\\ &\leq \bar{N}_0(r,0;\Psi')+\bar{N}(r,0;\Phi|\geq 3)+\bar{N}(r,1;\Psi|\geq 2)+S(r,\Phi)+S(r,\Psi)\\ &\leq \bar{N}(r,0;'|\Psi\neq 0)\\ &\leq \bar{N}_0(r,0;\Psi)+S(r,\Psi).\end{aligned}\tag{3.2}$$

Hence using (3.1), (3.2), Lemmas 2.2, 2.7 and 2.15, we get from the Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]) that

$$\begin{aligned}(n+q+1)T(r,f)&\leq T(r,\Phi_*)-N(r,0;L(\Delta_\eta f))+S(r,f)\\ &\leq T(r,\Phi)+N_{k+2}(r,0;\Phi_*)-N_2(r,0;\Phi)-N(r,0;L(\Delta_\eta f))+S(r,f)\\ &\leq \bar{N}(r,0;\Phi)+\bar{N}(r,1;\Phi)+\bar{N}(r,\infty;\Phi)+N_{k+2}(r,0;\Phi_*)-N_2(r,0;\Phi)\\ &\quad-N(r,0;L(\Delta_\eta f))-\bar{N}_0(r,0;\Phi')+S(r,f)\\ &\leq \bar{N}(r,0;\Phi)+\bar{N}(r,0;\Phi|\geq 2)+\bar{N}(r,0;\Psi|\geq 2)+\bar{N}(r,1;\Phi|\geq 2)+\bar{N}_L(r,1;\Phi)\\ &\quad+\bar{N}_L(r,1;\Psi)+\bar{N}_0(r,0;\Psi')+N_{k+2}(r,0;\Phi_*)+N_2(r,0;\Phi)-N(r,0;L(\Delta_\eta f))\\ &\leq N_{k+2}(r,0;\Phi_*)+N_2(r,0;\Psi)-N(r,0;L(\Delta_\eta f))+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,0;Q(f)L(\Delta_\eta f))+N_{k+2}(r,0;Q(g)L(\Delta_\eta g))-N(r,0;L(\Delta_\eta f))\\ &\quad+S(r,f)+S(r,g)\\ &\leq(\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+T(r,L(\Delta_\eta f))+S(r,f)+S(r,g),\end{aligned}$$

i.e.,

$$(n+q+1)T(r,f)\leq(\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+m(r,L(\Delta_\eta f))+S(r,f)+S(r,g)$$

$$\leq (\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+m\left(r,\frac{L(\Delta_\eta g)}{g}\right)+m(r,g)+S(r,f)+S(r,g)$$
$$\leq (\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+T(r,g)+S(r,f)+S(r,g). \tag{3.3}$$

Similarly, we get

$$(n+q+1)T(r,g) \leq (\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+T(r,f)+S(r,f)+S(r,g). \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$(n+q-\tau_1-k\tau_2-2\tau_2)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which is a contradicts with $n \geq q+\Theta_1+k\tau_2+1$, where $\Theta_1=\tau_1+\tau_2$.

Case 2: Suppose $H \equiv 0$. Then, by integration we get

$$\Phi=\frac{A\Psi+B}{C\Psi+D}, \tag{3.5}$$

where A,B,C,D are complex constant satisfying $AD-BC \neq 0$.

Subcase 2.1: Suppose $AC \neq 0$. Then $\Phi-\frac{A}{C}=\frac{-(AD-BC)}{C(C\Psi+D)} \neq 0$. So Φ omits the value $\frac{A}{C}$. Therefore, by Lemma 2.7 and the Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we get

$$\begin{aligned}(n+q+1)T(r,f) &\leq T(r,Q(f(z))L(\Delta_\eta(f)))-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq T(r,\Phi_*))-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq T(r,\Phi)+N_{k+1}(r,0;\Phi_*)-\bar{N}(r,0;\Psi)-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq \bar{N}(r,0;\Phi)+\bar{N}(r,\infty;\Phi)+\bar{N}\left(r,\frac{A}{C};\Phi\right)+N_{k+1}(r,0;\Phi_*)-N(r,0;\Phi)\\
&\quad -N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq N_{k+1}(r,0;Q(f(z))L(\Delta_\eta(f)))-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq (\tau_1+(k+1)\tau_2+1)T(r,f)+S(r,f)\\
&\leq (\Theta_1+k\tau_2+1)T(r,f)+S(r,f),\end{aligned}$$

which is a contradicts with $n \geq q+\Theta_1+k\tau_2+1$, where $\Theta_1=\tau_1+\tau_2$.

Subcase 2.2: Suppose $AC=0$. Since $AD-BC \neq 0$, a and C both can not be simultaneously zero.

Subcase 2.2.1: Let $A \neq 0$ and $C=0$. Then (3.5) becomes $\Phi=A_1\Psi+B_1$, where $A_1=\frac{A}{D}$ and $B_1=\frac{B}{D}$. If f has no 1-point, then by Lemma 2.7 and the Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we get

$$\begin{aligned}(n+q+1)T(r,f) &\leq T(r,Q(f(z))L(\Delta_\eta(f))-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq T(r,\Phi_*)-N(r,0;L(\Delta_\eta(f))+S(r,f)\\
&\leq T(r,\Phi)+N_{k+1}(r,0;\Phi_*)-\bar{N}(r,0;\Phi)-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq \bar{N}(r,0;\Phi)+\bar{N}(r,\infty;\Phi)+\bar{N}(r,1;\Phi)+N_{k+1}(r,0;\Phi_*)-N(r,0;\Phi)\\
&\quad -N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq N_{k+1}(r,0;Q(f(z))L(\Delta_\eta(f)))-N(r,0;L(\Delta_\eta(f)))+S(r,f)\\
&\leq (\tau_1+(k+1)\tau_2+1)T(r,f)+S(r,f)\end{aligned}$$

$$\leq (\Theta_1 + k\tau_2 + 1)T(r,f) + S(r,f),$$

which is a contradiction since $n \geq q + \Theta_1 + k\tau_2 + 1$, where $\Theta_1 = \tau_1 + \tau_2$. Let f has some 1-point. Then $A_1 + B_1 = 1$. Therefore, $\Phi = A_1\Psi + 1 - A_1$. If $A_1 \neq 1$, then using Lemmas 2.7, 2.5, 2.15 and the Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we get

$$\begin{aligned}(n+q+1)T(r,g) &\leq T(r, Q(g(z))L(\Delta_\eta(g))) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\
&\leq T(r,\Psi_*)) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\
&\leq T(r,\Psi) + N_{k+1}(r,0;\Psi_*) - \bar{N}(r,0;\Psi) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\
&\leq \bar{N}(r,0;\Psi) + \bar{N}(r,\infty;\Psi) + \bar{N}\left(r, \frac{1-A_1}{A_1};\Psi\right)\\
&\quad + N_{k+1}(r,0;\Psi_*) - N(r,0;\Psi) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\
&\leq N_{k+1}(r,0;\Psi_*) + \bar{N}(r,0;\Phi) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\
&\leq N_{k+1}(r,0;\Phi_*) + N_{k+1}(r,0;\Psi_*) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\
&\leq (\tau_1+(k+1)\tau_2)T(r,f) + T(r,L(\Delta_\eta f)) + (\tau_1+(k+1)\tau_2)T(r,g) + S(r,f) + S(r,g)\\
&\leq (\tau_1 + (k+1)\tau_2 + 1)T(r,g) + S(r,g),\end{aligned}$$

i.e.,

$$(n+q+1)T(r,g) \leq (\Theta_1 + k\tau_2 + 1)T(r,g) + S(r,g),$$

which is a contradiction since $n \geq q + \Theta_1 + k\tau_2 + 1$, where $\Theta_1 = \tau_1 + \tau_2$. Hence $A_1 = 1$, and therefore we have $\Phi = \Psi$, i.e.,

$$(Q(f)L(\Delta_\eta f))^{(k)} \equiv (Q(g)L(\Delta_\eta g))^{(k)}.$$

Integrating k times, we get

$$Q(f)L(\Delta_\eta f) \equiv Q(g)L(\Delta_\eta g) + p(z), \tag{3.6}$$

where $p(z)$ is a polynomial of degree at most $k-1$. Suppose $p(z) \not\equiv 0$. Then from (3.6), we have

$$\frac{Q(f)L(\Delta_\eta f)}{p(z)} \equiv \frac{Q(g)L(\Delta_\eta g)}{p(z)} + 1. \tag{3.7}$$

Now, using Lemmas 2.2, 2.6 and the Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we get

$$\begin{aligned}(n+q+1)T(r,f) &\leq T(r, Q(f(z))L(\Delta_\eta(f))) - N(r,0;L(\Delta_\eta(f))) + S(r,f)\\
&\leq T(r, Q(f(z))L(\Delta_\eta(f))/p(z)) - N(r,0;L(\Delta_\eta(f))) + S(r,f)\\
&\leq \bar{N}\left(r,0;\frac{Q(f(z))L(\Delta_\eta(f))}{p}\right) + \bar{N}\left(r,\infty;\frac{Q(f(z))L(\Delta_\eta(f))}{p}\right)\\
&\quad + \bar{N}\left(r,1;\frac{Q(f(z))L(\Delta_\eta(f))}{p}\right) - N(r,0;L(\Delta_\eta(f))) + S(r,f)\\
&\leq \bar{N}(r,0;Q(f)) + \bar{N}\left(r,0;\frac{Q(g(z))L(\Delta_\eta(g))}{p}\right) + S(r,f)\\
&\leq \bar{N}(r,0;Q(f)) + \bar{N}(r,0;Q(g)) + \bar{N}(r,0;L(\Delta_\eta(f))) + S(r,f)\\
&\leq (\tau_1+(k+1)\tau_2)T(r,f) + (\tau_1+(k+1)\tau_2)T(r,g) + T(r,L(\Delta_\eta f)) + S(r,f) + S(r,g)\end{aligned}$$

$$\leq (\tau_1+(k+1)\tau_2)T(r,f)+(\tau_1+(k+1)\tau_2)T(r,g)+m\left(r,\frac{L(\Delta_\eta g)}{g}\right)$$
$$+m(r,g)+S(r,f)+S(r,g)$$
$$\leq (\tau_1+(k+1)\tau_2)T(r,f)+(\tau_1+(k+1)\tau_2+1)T(r,g)+S(r,f)+S(r,g). \quad (3.8)$$

Similarly, we get

$$(n+q+1)T(r,g)\leq (\tau_1+(k+1)\tau_2)T(r,g)+(\tau_1+(k+1)\tau_2+1)T(r,f)+S(r,f)+S(r,g). \quad (3.9)$$

Combining (3.8) and (3.9), we get

$$(n+q-\Theta_1-k\tau_2)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

which contradicts to the fact that $n\geq q+\Theta_1+k\tau_2+1$, where $\Theta_1=\tau_1+\tau_2$. Hence $p(z)\equiv 0$, and thus from (3.6)

$$Q(f)L(\Delta_\eta f)\equiv Q(g)L(\Delta_\eta g),$$

i.e.,

$$(a_mf^m+a_{m-1}f^{m-1}+\ldots+a_1f+a_0)(f(z+\eta)-f(z))$$
$$\equiv (a_mg^m+a_{m-1}g^{m-1}+\ldots+a_1g+a_0)(g(z+\eta)-g(z)). \quad (3.10)$$

Let $h=\frac{f}{g}$.
If h is a constant then substituting $f=gh$ and $f(z+\eta)=g(z+\eta)h(z+\eta)$ in (3.10), we deduce

$$a_mg^m[(h^mh(z+\eta)-1)g(z+\eta)-(h^mh(z)-1)g(z)]$$
$$+a_{m-1}g^{m-1}[(h^{m-1}h(z+\eta)-1)g(z+\eta)-(h^{m-1}h(z)-1)g(z)]+\ldots$$
$$+a_0[(h(z+\eta)-1)g(z+\eta)-(h(z)-1)g(z)]=0,$$

which implies $h^d=1$, where

$$d=\gcd(m+1,\ldots,m+1-i,\ldots,1),\quad a_{m-i}\neq 0,$$

for $i=0,1,\ldots,m$.
Thus $f(z)=tg(z)$ for a constant t such that $t^d=1$, where $d=\gcd(m+1,\ldots,m+1-i,\ldots,1)$, $a_{m-i}\neq 0$ for $i=0,1,\ldots,m$ which is the conclusion in *Case* 2 in [17, Proof of Theorem 11].
If h is not a constant then $f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f,g)\equiv 0$, where

$$R(\omega_1,\omega_2)=(a_m\omega_1^m+a_{m-1}\omega_1^{m-1}+\ldots+a_0)[\omega_1(z+\eta)-\omega_1(z)]$$
$$-(a_m\omega_2^m+a_{m-1}\omega_2^{m-1}+\ldots+a_0)[\omega_2(z+\eta)-\omega_2(z)].$$

Subcase 2.2.2: Let $A=0$ and $C\neq 0$. Then (3.5) becomes

$$\Phi=\frac{1}{A_2\Psi+B_2}, \quad (3.11)$$

where $A_2=\frac{C}{D}$ and $B_2=\frac{D}{B}$. If Φ has no 1-point, then by a similar argument as done in *Subcase* 2.2.1, we can get a contradiction. Let Φ has some 1-point. Then $A_2+B_2=1$. If $A_2\neq 1$, then (3.11) can be written as

$$\Phi=\frac{1}{A_2\Psi+1-A_2}. \quad (3.12)$$

Since Φ is entire and $A_2 \neq 0$, Ψ omits the value $(1-A_2/A_2)$. Therefore, by Lemma 2.7 and the Second Fundamental Theorem (Yang and Yi [18]), we get

$$\begin{aligned}(n+q+1)T(r,g) &\le T(r,Q(g(z))L(\Delta_\eta(g))) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\ &\le T(r,\Psi_*)) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\ &\le T(r,\Psi) + N_{k+1}(r,0;\Psi_*) - \bar{N}(r,0;\Psi) - N(r,0;L(\Delta_\eta(g))) + S(r,g)\\ &\le \bar{N}(r,0;\Psi) + \bar{N}(r,\infty;\Psi) + \bar{N}\left(r,\frac{1-A_2}{A_2};\Psi\right) + N_{k+1}(r,0;\Psi_*) - N(r,0;\Psi)\\ &\quad - N(r,0;\Delta_\eta(g)) + S(r,g)\\ &\le N_{k+1}(r,0;P(f)\Delta_\eta f) - N(r,0;\Delta_\eta(g)) + S(r,g)\\ &\le (\tau_1 + (k+1)\tau_2 + 1)T(r,g) + S(r,g)\\ &\le (\Theta_1 + k\tau_2 + 1)T(r,g) + S(r,g),\end{aligned}$$

which is a contradiction since $n \ge q + \Theta_1 + k\tau_2 + 1$, where $\Theta_1 = \tau_1 + \tau_2$. Hence $A_2 = 1$. So, from (3.12), we get $\Phi\Psi \equiv 1$, i.e.,

$$(Q(f)L(\Delta_\eta f))^{(k)}(Q(g)L(\Delta_\eta g))^{(k)} \equiv \alpha^2(z). \tag{3.13}$$

Let $u_1, u_2, \dots, u_t$, $1 \le t \le m$ be the distinct zeros of $P(z)$. Since $m \ge k+1$, $a_0 \neq 0$ and f is entire, it is easily seen from (3.13) that f has atleast two finite Picard exceptional values, which is not possible. Hence the proof is complete. □

Proof of Theorem 1.2. Let Φ and Ψ be defined as in Theorem 1.1. Then Φ and Ψ are two-transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$. We consider the following two cases:

Case 1: Suppose $H \not\equiv 0$. Since Φ and Ψ share $(1,2)^*$, it follows that Φ and Ψ share $(1,1)^*$. Also, we note that $\bar{N}(r,1;\Phi|=1,\Psi|=0) = S(r,\Phi) + S(r,\Psi)$. Keeping in view of Lemmas 2.12 and 2.13, we see that

$$\begin{aligned}\bar{N}(r,1;\Phi) &= \bar{N}(r,1;\Phi|\le 1) + \bar{N}(r,1;\Phi|\ge 2)\\ &\le \bar{N}(r,1;\Phi|=1,\Psi|=0) + \bar{N}^E(r,1;\Phi,\Psi|\le 1) + \bar{N}(r,1;\Phi|\ge 2)\\ &\le N(r,\infty;H) + \bar{N}(r,1;\Phi|\ge 2) + S(r,\Phi) + S(r,\Psi)\\ &\le \bar{N}(r,0;\Phi|\ge 2) + \bar{N}(r,0;\Psi|\ge 2) + \bar{N}_*(r,1;\Phi,\Psi)\\ &\quad + \bar{N}(r,1;\Phi|\ge 2) + \bar{N}_0(r,1;\Phi') + \bar{N}_0(r,1;\Psi') + S(r,\Phi) + S(r,\Psi).\end{aligned} \tag{3.14}$$

Since Φ and Ψ share $(1,2)^*$, we must have $\bar{N}_{F\ge 2}(r,1;\Phi|\Psi \neq 1) = S(r,\Phi) + S(r,\Psi)$, $\bar{N}(r,1;\Phi|=2, \Psi=1) = S(r,\Phi) + S(r,\Psi)$. Therefore, using Lemma 2.8, we get

$$\begin{aligned}\bar{N}(r,1;\Phi|\ge 2) &\le \bar{N}_{F\ge 2}(r,1;\Phi|\Psi \neq 1) + \bar{N}(r,1;\Phi|\ge 2,\Psi=1) + \bar{N}(r,1;\Phi|\ge 2,\Psi\ge 2)\\ &\le \bar{N}_{F\ge 2}(r,1;\Phi|\Psi \neq 1) + \bar{N}(r,1;\Phi|=2,\Psi=1) + \bar{N}_{F\ge 3}(r,1;\Psi=1)\\ &\quad + \bar{N}(r,1;\Psi|\ge 2) + S(r,\Phi) + S(r,\Psi)\\ &\le \bar{N}(r,0;\Phi) + \bar{N}(r,1;\Psi\ge 2) + S(r,\Phi) + S(r,\Psi).\end{aligned} \tag{3.15}$$

Again using (3.15) and Lemma 2.4, we get

$$\begin{aligned}&\bar{N}(r,0;\Psi') + \bar{N}(r,1;\Phi|\ge 2) + \bar{N}_*(r,1;\Phi,\Psi)\\ &\quad \le \bar{N}(r,0;\Psi') + \bar{N}(r,1;\Psi|\ge 2) + \bar{N}(r,1;\Psi|\ge 3) + \bar{N}(r,0;\Phi) + S(r,\Psi)\end{aligned}$$

$$\leq \bar{N}(r,0;\Psi') + N(r,1;\Psi) - \bar{N}(r,1;\Psi) + \bar{N}(r,0;\Phi) + S(r,\Phi) + S(r,\Psi)$$
$$\leq \bar{N}(r,0;\Psi'|\Psi \neq 0) + \bar{N}(r,0;\Phi) + S(r,\Phi) + S(r,\Psi)$$
$$\leq \bar{N}(r,0;\Phi) + \bar{N}(r,0;\Psi) + S(r,\Phi) + S(r,\Psi). \tag{3.16}$$

Hence using (3.14), (3.16), Lemmas 2.2 and 2.7, the Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we get

$$\begin{aligned}
(n+q+1)T(r,f) &\leq T(r,\Phi_*) - N(r,0;L(\Delta_\eta f)) + S(r,f)\\
&\leq T(r,\Phi) + N_{k+2}(r,0;\Phi_*) - N_2(r,0;\Phi) - N(r,0;L(\Delta_\eta f)) + S(r,f)\\
&\leq \bar{N}(r,0;\Phi) + \bar{N}(r,1;\Phi) + \bar{N}(r,\infty;\Phi) + \bar{N}_{k+2}(r,0;\Phi_*) - \bar{N}_2(r,0;\Phi)\\
&\quad - N(r,0;L(\Delta_\eta f)) - \bar{N}_0(r,0;\Phi') + S(r,f)\\
&\leq N_2(r,0;\Phi) + N_2(r,0;\Psi) + \bar{N}_0(r,0;\Phi) + N_{k+2}(r,0;\Phi_*) - N_2(r,0;\Phi)\\
&\quad - N(r,0;L(\Delta_\eta f)) + S(r,f) + S(r,g)\\
&\leq N_{k+2}(r,0;\Phi_*) + N_{k+2}(r,0;\Psi_*) + N_{k+1}(r,0;\Phi_*)\\
&\quad - N(r,0;L(\Delta_\eta f)) + S(r,f) + S(r,g)\\
&\leq (\tau_1 + (k+2)\tau_2)(T(r,f) + T(r,g)) + (\tau_1 + (k+1)\tau_2)(T(r,f))\\
&\quad + T(r,L(\Delta_\eta f)) + T(r,L(\Delta_\eta g)) + S(r,f) + S(r,g)\\
&\leq (\tau_1 + (k+2)\tau_2)(T(r,f) + T(r,g)) + (\tau_1 + (k+1)\tau_2)(T(r,f)) + m\left(r,\frac{L(\Delta_\eta f)}{f}\right)\\
&\quad + m(r,f) + m\left(r,\frac{L(\Delta_\eta g)}{g}\right) + m(r,g) + T(r,L(\Delta_\eta g)) + S(r,f) + S(r,g)\\
&\leq (\tau_1 + (k+2)\tau_2 + 1)(T(r,f) + T(r,g)) + (\tau_1 + (k+1)\tau_2)(T(r,f))\\
&\quad + S(r,f) + S(r,g).
\end{aligned} \tag{3.17}$$

Similarly, we get

$$\begin{aligned}
(n+q+1)T(r,g) &\leq (\tau_1 + (k+2)\tau_2 + 1)(T(r,f) + T(r,g)) + (\tau_1 + (k+1)\tau_2)(T(r,g))\\
&\quad + S(r,f) + S(r,g).
\end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we get

$$(n + q - 2\tau_1 - 2k\tau_2 - 3\tau_2)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which is a contradicts with $n \geq q + \Theta_1 + \Theta_2 + 2k\tau_2 + 1$, where $\Theta_1 = \tau_1 + \tau_2$ and $\Theta_2 = \tau_1 + 2\tau_2$.

Case 2: Suppose $H \equiv 0$. This case can be carried out similarly as done in *Case* 2 of Proof of Theorem 1.1. So, we omit the details. This proves Theorem 1.2. □

Proof of Theorem 1.3. Let Φ and Ψ be defined as in Theorem 1.1. Then Φ and Ψ are two-transcendental meromorphic functions such that $E_{2)}(1,\Phi) = E_{2)}(1,\Psi)$ except the zeros and poles of $\alpha(z)$. We consider the following two cases:

Case 1: Suppose $H \not\equiv 0$. Since $E_{2)}(1,\Phi) = E_{2)}(1,\Psi)$, it follows that $E_{1)}(1,\Phi) = E_{1)}(1,\Psi)$. Keeping in view of Lemmas 2.9, 2.11 and 2.14, we see that

$$\begin{aligned}
\bar{N}(r,1;\Phi) &= \bar{N}(r,1;\Phi| \leq 1) + \bar{N}(r,1;\Phi| \geq 2)\\
&\leq N(r,H) + \bar{N}(r,1;\Phi| = 2) + \bar{N}_{\Phi\geq 3}(r,1;\Phi|\Psi \neq 1) + \bar{N}(r,1;\Phi| \geq 3, \Psi| \geq 3)\\
&\leq N(r,\infty;H) + \bar{N}(r,1;\Psi = 2) + \bar{N}(r,1;\Psi| \geq 3) + \bar{N}_{\Phi\geq 3}(r,1;\Phi|\Psi \neq 1) + S(r,\Phi) + S(r,\Psi)
\end{aligned}$$

$$\begin{aligned}&\le \bar{N}(r,0;\Phi|\ge 2)+\bar{N}(r,0;\Psi|\ge 2)+\bar{N}_L(r,1;\Phi)+\bar{N}_L(r,1;\Psi)+\bar{N}(r,1;\Phi|\ge 2)\\&\quad+2\bar{N}_{\Phi\ge 3}(r,1;\Phi|\Psi\ne 1)+\bar{N}_{\Psi\ge 3}(r,1;\Phi|\Psi\ne 1)+\bar{N}_0(r,0;\Phi')+\bar{N}_0(r,0;\Psi')\\&\quad+S(r,\Phi)+S(r,\Psi),\end{aligned}$$

i.e.,

$$\begin{aligned}\bar{N}(r,1;\Phi)&\le \bar{N}(r,0;\Phi|\ge 2)+\bar{N}(r,0;\Psi|\ge 2)+\bar{N}_L(r,1;\Phi)+\bar{N}_L(r,1;\Psi)+\bar{N}(r,1;\Phi|\ge 2)\\&\quad+2\bar{N}_{\Phi\ge 3}(r,1;\Phi|\Psi\ne 1)+\bar{N}_{\Psi\ge 3}(r,1;\Phi|\Psi\ne 1)+\bar{N}_0(r,0;\Phi')+\bar{N}_0(r,0;\Psi')\\&\quad+S(r,\Phi)+S(r,\Psi)\\&\le \bar{N}(r,0;\Phi|\ge 2)+\bar{N}(r,0;\Psi|\ge 2)+\bar{N}_L(r,1;\Phi)+\bar{N}_L(r,1;\Psi)+\bar{N}(r,1;\Phi|\ge 2)\\&\quad+\bar{N}(r,0;\Phi)+\frac{1}{2}\bar{N}(r,0;\Psi)+\bar{N}_0(r,0;\Phi')+\bar{N}_0(r,0;\Psi')+S(r,\Phi)+S(r,\Psi).\end{aligned}\tag{3.19}$$

Now using Lemma 2.4, we get

$$\begin{aligned}&\bar{N}_0(r,0;\Psi')+\bar{N}(r,1;\Psi|\ge 2)+\bar{N}_L(r,1;\Phi)+\bar{N}_L(r,1;\Psi)\\&\quad\le \bar{N}_0(r,0;\Psi')+\bar{N}(r,1;\Psi|\ge 2)+\bar{N}(r,1;\Psi|\ge 3)+S(r,\Psi)\\&\quad\le \bar{N}_0(r,0;\Psi')+N(r,1;\Psi)-\bar{N}(r,1;\Psi)+S(r,\Phi)+S(r,\Psi)\\&\quad\le N(r,0;\Psi'|\Psi\ne 0)\le \bar{N}(r,0;\Psi)+S(r,\Phi)+S(r,\Psi).\end{aligned}\tag{3.20}$$

Hence using (3.19), (3.20), Lemmas 2.2 and 2.7, Second Fundamental Theorem of Nevanlinna (Yang and Yi [18]), we get

$$\begin{aligned}(n+q+1)T(r,f)&\le T(r,\Phi_*)-N(r,0;L(\Delta_\eta f))+S(r,f)\\&\le T(r,\Phi)+N_{k+2}(r,0;\Phi_*)-N_2(r,0;\Phi)-N(r,0;L(\Delta_\eta f))+S(r,f)\\&\le \bar{N}(r,0;\Phi)+\bar{N}(r,1;\Phi)+\bar{N}(r,\infty;\Phi)+\bar{N}_{k+2}(r,0;\Phi_*)-\bar{N}(r,0;\Phi')\\&\quad-N(r,0;L(\Delta_\eta f))-\bar{N}_2(r,0;\Phi')+S(r,f)\\&\le N_2(r,0;\Phi)+N_2(r,0;\Psi)+\bar{N}_0(r,0;\Phi)+\frac{1}{2}\bar{N}(r,0;\Psi)+N_{k+2}(r,0;\Phi_*)\\&\quad-N_2(r,0;\Phi)-N(r,0;L(\Delta_\eta f))+S(r,f)+S(r,g)\\&\le N_{k+2}(r,0;\Phi_*)+N_{k+2}(r,0;\Psi_*)+N_{k+1}(r,0;\Phi_*)+\frac{1}{2}N_{k+1}(r,0;\Psi_*)\\&\quad-N(r,0;L(\Delta_\eta f))+S(r,f)+S(r,g)\\&\le(\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+(\tau_1+(k+1)\tau_2)(T(r,f))\\&\quad+T(r,L(\Delta_\eta f))+T(r,L(\Delta_\eta g))+\frac{1}{2}(\tau_1+(k+1)\tau_2)(T(r,g))\\&\quad+\frac{1}{2}T(r,L(\Delta_\eta g))+S(r,f)+S(r,g)\\&\le(\tau_1+(k+2)\tau_2)(T(r,f)+T(r,g))+(\tau_1+(k+1)\tau_2)(T(r,f))+m\left(r,\frac{L(\Delta_\eta f)}{f}\right)\\&\quad+m(r,f)+m\left(r,\frac{L(\Delta_\eta g)}{g}\right)+m(r,g)+\frac{1}{2}(\tau_1+(k+1)\tau_2)(T(r,g))\\&\quad+\frac{1}{2}m\left(r,\frac{L(\Delta_\eta g)}{g}\right)+\frac{1}{2}m(r,g)+S(r,f)+S(r,g)\end{aligned}$$

$$\leq(\tau_1+(k+2)\tau_2+1)(T(r,f)+T(r,g))+(\tau_1+(k+1)\tau_2)(T(r,f))$$
$$+\frac{1}{2}(\tau_1+(k+1)\tau_2+1)(T(r,g))+S(r,f)+S(r,g). \tag{3.21}$$

Similarly, we get

$$(n+q+1)T(r,g)\leq(\tau_1+(k+2)\tau_2+1)(T(r,f)+T(r,g))$$
$$+(\tau_1+(k+1)\tau_2)(T(r,g))+\frac{1}{2}(\tau_1+(k+1)\tau_2+1)(T(r,f))+S(r,f)+S(r,g). \tag{3.22}$$

Combining (3.21) and (3.22), we get

$$\left(n+q-\frac{5}{2}\tau_1-\frac{5}{2}k\tau_2-\frac{7}{2}\tau_2\right)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

which is not possible since $n\geq q+\frac{5}{2}\Theta_1+\frac{5k+1}{2}\tau_2+1$, where $\Theta_1=\tau_1+\tau_2$.

Case 2: Suppose $H\equiv 0$. This case can be carried out similarly as done in *Case* 2 of proof of Theorem 1.1. Thus, we omit the details. This proves Theorem 1.3. □

Acknowledgement

The authors would like to thank the referee(s) for the helpful suggestions and comments to improve the exposition of the paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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