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Research Article

A Non-Markovian Queue with Second Optional Service, Disasters, Working Breakdowns and Working Vacation

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Abstract. Inside this research article, we look at a non-Markovian queue (M/G/1) with second optional service, disaster events, working breakdowns and working vacation. There are two servers in the system: a primary and a backup. *First Essential Service* (FES) is delivered by a primary (backup) server to all arriving customers. A customer may choose the *Second Optional Service* (SOS) after his initial service is finished. When a disaster happens, all customers are made to evacuate the system, as well as the primary server crashes. The primary server is dispatched to repair at the first sign of a breakdown and the repair period begins right away. While a primary server is being repaired, a backup server is servicing customers at such a reduced rate. If a system is inactive while it is operating, the primary server will just go on vacation. The primary server, which is on working vacation mode and serving at a reduced rate, subsequently serves the new customers. The disasters have no effect on working vacation period. The supplementary variable methodology is used to determine the *probability generating function* (PGF) of the number of customers throughout typical peak times, working breakdown periods and working vacation periods, as well as specific metrics of effectiveness. Some statistical outcomes are shown at the end.

Keywords. M/G/1 queue, SOS, Disasters, Working breakdowns and working vacation

Mathematics Subject Classification (2020). 60K25, 90B22

1. Introduction

Throughout many queueing situations, all arriving customer expects FES and only a subset of them may need a SOS from the primary (backup) server. An M/G/1 queueing system with a second optional service was researched by Madan [14]. Many authors, including Kalyanaraman and Murugan [8], Thangaraj and Vanitha [22], and Santhi and Murugan [18], explored the single server vacation queueing models with second optional service.

The concept of disasters occurring at random compels everyone to leave at the same moment. Towsley and Tripathi [24] were the first to present queueing models with disasters. 'Mass exodus' [3], 'queue flushing' [24], 'catastrophes' [11] and 'stochastic clearing' [27] are all terms used to describe disasters. Jain and Sigman [5] extended the M/G/1 queueing model to include disasters. Sridharan and Jayasree [20], Lee *et al*. [12], and Sudhesh [21] are only a few of the researchers mentioned.

In breakdown queueing model, A great deal of thought goes into the queuing model with an unstable server. In the majority of studies, the server is supposed to have experienced a difficulty and is instantaneously sent off to repair. More researchers put forward queueing models with server breakdown. An M/G/1 queueing system with a repairable server was proposed in Cao and Cheng [2]. Thiruvengadam [23] discussed about the queueing concept in terms of breakdowns. Thangaraj and Vanitha [22] investigated single server queueing models with unpredictable breakdowns. Throughout a working breakdown, the server stops to serve entirely then instead of serving at a reduced rate. A first queueing model with working breakdown was developed in Kalidass and Kasthuri [7]. Following that, Kim and Lee [9], Liou [13], Yang and Wu [26] studied non-Markovian queueing models with working breakdowns.

To use the vacation queueing strategy, the server suspends all operations for the duration of the vacation. Working vacation is a term that describes when a server operates at a reduced rate throughout a vacation period. Servi and Finn [19] developed the working vacation queueing approach, which was then generalised by Kim *et al*. [10], and Wu and Takagi [25]. Following that, several authors added working vacation Baba [1], Parveen and Begum [17], and Murugan and Santhi [16] to their queueing models.

Within that research, we investigate at "A non-Markovian modelling approach (M/G/1) including second optional service, disasters, working breakdowns and working vacation". A primary server and a backup server make up the system. FES is delivered by a primary (backup) server to all arriving customers. A customer may choose the SOS after his initial service is finished. As a reaction of the disasters, all existing customers just vanish, prompting the primary server to crash. The primary server is dispatched to repair in the event of a crash and instantly, the repair period started. As during repair period, a backup server serves arriving customers at such a reduced rate; whether there are customers inside the system, the backup server suddenly stops serving them, and the system is rebooted at its regular service rate by the primary server. During working vacation period, a primary server serves arriving customers at such a reduced rate. During vacations and downtime, the server charges a variable tariff.

We are trying to follow that how this study is structured. Overview of the model is summarised in Section 2. In Section 3, we derive the PGF of the number of customers in the system using the Kolmogorov equations, server states and supplementary variable technique of remaining service time when the primary server is in typical peak times, working breakdown and working vacation, respectively. Finally, we looked at some specific cases. In Section 4, the number of customers in the system and the customer's waiting time in the system are obtained as metrics of effectiveness. In Section 5, we perform a statistical outcomes of the model.

2. Overview of a Model

Consider a single server queue whose arrival is governed by a Poisson process with a λ arrival rate. The FIFO (*first in, first out*) rule of service discipline governs the system in the research. When the server seems to be in a typical peak times when there are no customers inside the system, the server will take a fixed-length vacation. If a customer arrives throughout a vacation, the primary server provides two types of services: *First Essential Service* (FES) and *Second Optional Service* (SOS) would then serve them at such a reduced rates μ_{wv_1} (>0) and μ_{wv_2} (>0), which we call it as working vacation period. Let η be the vacation completion rate and let the FES time S_{wv_1} be exponentially distributed whose *probability density function* (PDF) is $s_{wv_1}(x)$ and its *Laplace Stieltjes Transform* (LST) is $S^*_{wv_1}(\theta)$. After completion of ES the customer may opt for the SOS with probability p or the customer may leave the system without taking the SOS with probability $q(p+q=1)$. The SOS time S_{wv_2} be exponentially distributed whose *probability density function* (PDF) is $s_{wv_2}(x)$ and its *Laplace Stieltjes Transform* (LST) is $S^*_{wv_2}(\theta)$. At the vacation completion epoch the service to the last customer is lost and the service is restarted in the typical peak times with different distribution.

If the working vacation ends, then the server resumes the typical peak times with typical FES rate μ_{rb_1} (>0) and SOS rate μ_{rb_2} (>0) whose typical FES time is S_{rb_1} and SOS time is S_{rb_2}. Let we denote its PDF by $s_{rb_1}(x)$ and $s_{rb_2}(x)$ and its LST by $S^*_{rb_1}(\theta)$ and $S^*_{rb_2}(\theta)$.

Only when the primary server, which is exponentially distributed with rate δ, is operational do disasters occur. When disaster happens, all customers are compelled to exit the system, and the primary server crashes. The primary server is dispatched to repair at the first sign of a breakdown and the repair period begins right away. We assume the backup server provides the repairing period at quite a reduced ES rate μ_{wb_1} (>0) and SOS rate μ_{wb_2} (>0), which we call it as working breakdown period and the repair duration is exponentially distributed with rate γ. If any customer is now in the system for the rest of a repair, the backup halts the provider as well as the primary server reboots the typical peak duration at the typical service rate. Let S_{wb_1} denote the working breakdown FES time, its PDF by $s_{wb_1}(x)$ and its LST by $S^*_{wb_1}(\theta)$ and let S_{wb_2} denote the working breakdown SOS time, its PDF by $s_{wb_2}(x)$ and its LST by $S^*_{wb_2}(\theta)$. Also, the S_{wv_1}, S_{wv_2}, S_{rb_1}, S_{rb_2}, S_{wb_1} and S_{wb_2} are considered to be mutually independent.

3. The Steady State Queue Length Distribution

Let $\mathcal{N}(t)$ represent the number of customers in the system at time t, and $\zeta(t)$ represent an indicative random variable defined by

$$\zeta(t)=\begin{cases}0, & \text{if the primary server is idle on working vacation,}\\ 1, & \text{if the primary server is idle in typical peak time,}\\ 2, & \text{if the primary server is FES in working vacation,}\\ 3, & \text{if the primary server is SOS in working vacation,}\\ 4, & \text{if the primary server is busy FES in typical peak time,}\\ 5, & \text{if the primary server is busy SOS in typical peak time,}\\ 6, & \text{if the primary server is down for repair at time } t.\end{cases}$$

Let $S^0_{wv_1}(t)$, $S^0_{wv_2}(t)$, $S^0_{rb_1}(t)$, $S^0_{rb_2}(t)$, $S^0_{wb_1}(t)$ and $S^0_{wb_2}(t)$ signify the remaining service time for the FES in working vacation, the remaining service time for the SOS in working vacation, FES in typical peak time, SOS in typical peak time, FES in repair periods and SOS in repair periods respectively, at time t.

$$\chi(t)=\begin{cases}S^0_{wv_1}(t), & \text{if } \zeta(t)=2,\\ S^0_{wv_2}(t), & \text{if } \zeta(t)=3,\\ S^0_{rb_1}(t), & \text{if } \zeta(t)=4,\\ S^0_{rb_2}(t), & \text{if } \zeta(t)=5,\\ S^0_{wb_1}(t), & \text{if } \zeta(t)=6,\\ S^0_{wb_1}(t), & \text{if } \zeta(t)=6.\end{cases}$$

The process $\{(\mathscr{N}(t),\zeta(t),\chi(t));t\geq 0\}$ is then transformed into a Markov process with the supplementary variables $S^0_{wv_1}(t)$, $S^0_{wv_2}(t)$, $S^0_{rb_1}(t)$, $S^0_{rb_2}(t)$, $S^0_{wb_1}(t)$ and $S^0_{wb_2}(t)$. The following limiting probabilities are utilised to calculate the steady state queue length PGF:

$$\begin{aligned}
\mathscr{V}_0 &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=0,\zeta(t)=0\},\\
\mathscr{B}_0 &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=0,\zeta(t)=1\},\\
\mathscr{R}_0 &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=0,\zeta(t)=6\},\\
\mathscr{V}_{n,1}(x)dx &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=n,\zeta(t)=2,x<S^0_{wv_1}(t)<x+dx\}, \quad n\geq 1,\\
\mathscr{V}_{n,2}(x)dx &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=n,\zeta(t)=3,x<S^0_{wv_2}(t)<x+dx\}, \quad n\geq 1,\\
\mathscr{B}_{n,1}(x)dx &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=n,\zeta(t)=4,x<S^0_{rb_1}(t)<x+dx\}, \quad n\geq 1,\\
\mathscr{B}_{n,2}(x)dx &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=n,\zeta(t)=5,x<S^0_{rb_2}(t)<x+dx\}, \quad n\geq 1,\\
\mathscr{R}_{n,1}(x)dx &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=n,\zeta(t)=6,x<S^0_{wb_1}(t)<x+dx\}, \quad n\geq 1,\\
\mathscr{R}_{n,2}(x)dx &= \lim_{t\to\infty}\Pr\{\mathscr{N}(t)=n,\zeta(t)=6,x<S^0_{wb_2}(t)<x+dx\}, \quad n\geq 1.
\end{aligned}$$

With the previously indicated probabilities, the Kolmogorov equations for the queue length distribution are provided.

$$0=-(\lambda+\eta)\mathscr{V}_0+q\mathscr{V}_{1,1}(0)+q\mathscr{B}_{1,1}(0)+\mathscr{B}_{1,2}(0)+\mathscr{V}_{1,2}(0), \tag{1}$$

$$-\frac{d}{dx}\mathscr{V}_{1,1}(x)=-(\lambda+\eta)\mathscr{V}_{1,1}(x)+\lambda\mathscr{V}_0 s_{wv_1}(x)+q\mathscr{V}_{2,1}(0)s_{wv_1}(x)+\mathscr{V}_{2,2}(0)s_{wv_1}(x), \tag{2}$$

$$-\frac{d}{dx}\mathscr{V}_{n,1}(x)=-(\lambda+\eta)\mathscr{V}_{n,1}(x)+\lambda\mathscr{V}_{n-1,1}(x)+q\mathscr{V}_{n+1,1}(0)s_{wv_1}(x)+\mathscr{V}_{n+1,2}(0)s_{wv_1}(x),\quad n\geq 2, \tag{3}$$

$$-\frac{d}{dx}\mathscr{V}_{1,2}(x)=-(\lambda+\eta)\mathscr{V}_{1,2}(x)+p\mathscr{V}_{1,1}(0)s_{wv_2}(x), \tag{4}$$

$$-\frac{d}{dx}\mathscr{V}_{n,2}(x)=-(\lambda+\eta)\mathscr{V}_{n,2}(x)+p\mathscr{V}_{n,1}(0)s_{wv_2}(x)+\mathscr{V}_{n-1,2}(x),\quad n\geq 2, \tag{5}$$

$$0=-\lambda\mathscr{B}_0+\eta\mathscr{V}_0+\gamma\mathscr{R}_0, \tag{6}$$

$$\begin{aligned}-\frac{d}{dx}\mathscr{B}_{1,1}(x)&=-(\lambda+\delta)\mathscr{B}_{1,1}(x)+\lambda\mathscr{B}_0s_{rb_1}(x)+q\mathscr{B}_{2,1}(0)s_{rb_1}(x)+\mathscr{B}_{2,2}(0)s_{rb_1}(x)\\&\quad+\eta s_{rb_1}(x)\int_0^\infty\mathscr{V}_{1,1}(y)dy+\gamma s_{rb_1}(x)\int_0^\infty\mathscr{R}_{1,1}(y)dy,\end{aligned} \tag{7}$$

$$\begin{aligned}-\frac{d}{dx}\mathscr{B}_{n,1}(x)&=-(\lambda+\delta)\mathscr{B}_{n,1}(x)+\lambda\mathscr{B}_{n-1,1}(x)+q\mathscr{B}_{n+1,1}(0)s_{rb_1}(x)+\mathscr{B}_{n+1,2}(0)s_{rb_1}(x)\\&\quad+\eta s_{rb_1}(x)\int_0^\infty\mathscr{V}_{n,1}(y)dy+\gamma s_{rb_1}(x)\int_0^\infty\mathscr{R}_{n,1}(y)dy,\quad n\geq 2,\end{aligned} \tag{8}$$

$$\begin{aligned}-\frac{d}{dx}\mathscr{B}_{1,2}(x)&=-(\lambda+\delta)\mathscr{B}_{1,2}(x)+p\mathscr{B}_{1,1}(0)s_{rb_2}(x)+\eta s_{rb_2}(x)\int_0^\infty\mathscr{V}_{1,2}(y)dy\\&\quad+\gamma s_{rb_2}(x)\int_0^\infty\mathscr{R}_{1,2}(y)dy,\end{aligned} \tag{9}$$

$$\begin{aligned}-\frac{d}{dx}\mathscr{B}_{n,2}(x)&=-(\lambda+\delta)\mathscr{B}_{n,2}(x)+p\mathscr{B}_{n,1}(0)s_{rb_2}(x)+\lambda\mathscr{B}_{n-1,2}(x)+\eta s_{rb_2}(x)\int_0^\infty\mathscr{V}_{n,2}(y)dy\\&\quad+\gamma s_{rb_2}(x)\int_0^\infty\mathscr{R}_{n,2}(y)dy,\quad n\geq 2,\end{aligned} \tag{10}$$

$$0=-(\lambda+\gamma)\mathscr{R}_0+q\mathscr{R}_{1,1}(0)+\mathscr{R}_{1,2}(0)+\delta\sum_{n=1}^\infty\mathscr{B}_{n,1}+\delta\sum_{n=1}^\infty\mathscr{B}_{n,2}, \tag{11}$$

$$-\frac{d}{dx}\mathscr{R}_{1,1}(x)=-(\lambda+\gamma)\mathscr{R}_{1,1}(x)+\lambda\mathscr{R}_0s_{wb_1}(x)+q\mathscr{R}_{2,1}(0)s_{wb_1}(x)+\mathscr{R}_{2,2}(0)s_{wb_1}(x), \tag{12}$$

$$-\frac{d}{dx}\mathscr{R}_{n,1}(x)=-(\lambda+\gamma)\mathscr{R}_{n,1}(x)+\lambda\mathscr{R}_{n-1,1}(x)+q\mathscr{R}_{n+1,1}(0)s_{wb_1}(x)+\mathscr{R}_{n+1,2}(0)s_{wb_1}(x),\ n\geq 2, \tag{13}$$

$$-\frac{d}{dx}\mathscr{R}_{1,2}(x)=-(\lambda+\gamma)\mathscr{R}_{1,2}(x)+p\mathscr{R}_{1,1}(0)s_{wb_2}(x), \tag{14}$$

$$-\frac{d}{dx}\mathscr{R}_{n,2}(x)=-(\lambda+\gamma)\mathscr{R}_{n,2}(x)+\lambda\mathscr{R}_{n-1,2}(x)+p\mathscr{R}_{n,1}(0)s_{wb_2}(x),\quad n\geq 2, \tag{15}$$

where

$$\mathscr{V}_{n,i}=\int_0^\infty\mathscr{V}_{n,i}(y)dy;\ \mathscr{B}_{n,i}=\int_0^\infty\mathscr{B}_{n,i}(y)dy;\ \mathscr{R}_{n,i}=\int_0^\infty\mathscr{R}_{n,i}(y)dy,\quad i=1,2.$$

For $i=1,2$ the following LSTs and PGFs are used to solve (1)-(15):

$$\mathscr{V}^*_{n,i}(\theta)=\int_0^\infty e^{-\theta x}\mathscr{V}_{n,i}(x)dx;\quad \mathscr{B}^*_{n,i}(\theta)=\int_0^\infty e^{-\theta x}\mathscr{B}_{n,i}(x)dx;\quad \mathscr{R}^*_{n,i}(\theta)=\int_0^\infty e^{-\theta x}\mathscr{R}_{n,i}(x)dx$$

$$\mathscr{V}^*_i(z,\theta)=\sum_{n=1}^\infty\mathscr{V}^*_{n,i}(\theta)z^n;\qquad \mathscr{B}^*_i(z,\theta)=\sum_{n=1}^\infty\mathscr{B}^*_{n,i}(\theta)z^n;\qquad \mathscr{R}^*_i(z,\theta)=\sum_{n=1}^\infty\mathscr{R}^*_{n,i}(\theta)z^n$$

$$\mathscr{V}_i(z,0)=\sum_{n=1}^\infty\mathscr{V}_{n,i}(0)z^n;\qquad \mathscr{B}_i(z,0)=\sum_{n=1}^\infty\mathscr{B}_{n,i}(0)z^n;\qquad \mathscr{R}_i(z,0)=\sum_{n=1}^\infty\mathscr{R}_{n,i}(0)z^n.$$

The following is the normalizing condition based on the notations above:

$$\mathscr{V}_0+\mathscr{B}_0+\mathscr{R}_0+\mathscr{V}_1^*(1,0)+\mathscr{V}_2^*(1,0)+\mathscr{B}_1^*(1,0)+\mathscr{B}_2^*(1,0)+\mathscr{R}_1^*(1,0)+\mathscr{R}_2^*(1,0)=1. \tag{16}$$

We acquire the LSTs from equations (2), (3), (4), (5), (7), (8), (9), (10), (12), (13), (14) and (15).

$$-(\theta\mathscr{V}_{1,1}^*(\theta)-\mathscr{V}_{1,1}(0))=-(\lambda+\eta)\mathscr{V}_{1,1}^*(\theta)+\lambda\mathscr{V}_0S_{wv_1}^*(\theta)+q\mathscr{V}_{2,1}(0)S_{wv_1}^*(\theta)+\mathscr{V}_{2,2}(0)S_{wv_1}^*(\theta), \tag{17}$$

$$-(\theta\mathscr{V}_{n,1}^*(\theta)-\mathscr{V}_{n,1}(0))=-(\lambda+\eta)\mathscr{V}_{n,1}^*(\theta)+\lambda\mathscr{V}_{n-1,1}^*(\theta)+q\mathscr{V}_{n+1,1}(0)S_{wv_1}^*(\theta)+\mathscr{V}_{n+1,2}(0)S_{wv_1}^*(\theta),\quad n\geq 2, \tag{18}$$

$$-(\theta\mathscr{V}_{1,2}^*(\theta)-\mathscr{V}_{1,2}(0))=-(\lambda+\eta)\mathscr{V}_{1,2}^*(\theta)+p\mathscr{V}_{1,1}(0)S_{wv_2}^*(\theta), \tag{19}$$

$$-(\theta\mathscr{V}_{n,2}^*(\theta)-\mathscr{V}_{n,2}(0))=-(\lambda+\eta)\mathscr{V}_{n,2}^*(\theta)+p\mathscr{V}_{n,1}(0)S_{wv_2}^*(\theta)+\lambda\mathscr{V}_{n-1,2}^*(\theta), \tag{20}$$

$$-(\theta\mathscr{B}_{1,1}^*(\theta)-\mathscr{B}_{1,1}(0))=-(\lambda+\delta)\mathscr{B}_{1,1}^*(\theta)+\lambda\mathscr{B}_0S_{rb_1}^*(\theta)+q\mathscr{B}_{2,1}(0)S_{rb_1}^*(\theta)+\mathscr{B}_{2,2}(0)S_{rb_1}^*(\theta)+\eta\mathscr{V}_{1,1}S_{rb_1}^*(\theta)+\gamma\mathscr{R}_{1,1}S_{rb_1}^*(\theta), \tag{21}$$

$$-(\theta\mathscr{B}_{n,1}^*(\theta)-\mathscr{B}_{n,1}(0))=-(\lambda+\delta)\mathscr{B}_{n,1}^*(\theta)+\lambda\mathscr{B}_{n-1,1}^*(\theta)+q\mathscr{B}_{n+1,1}(0)S_{rb_1}^*(\theta)+\mathscr{B}_{n+1,2}(0)S_{rb_1}^*(\theta)+\eta\mathscr{V}_{n,1}S_{rb_1}^*(\theta)+\gamma\mathscr{R}_{n,1}S_{rb_1}^*(\theta),\quad n\geq 2, \tag{22}$$

$$-(\theta\mathscr{B}_{1,2}^*(\theta)-\mathscr{B}_{1,2}(0))=-(\lambda+\delta)\mathscr{B}_{1,2}^*(\theta)+p\mathscr{B}_{1,1}(0)S_{rb_2}^*(\theta)+\eta\mathscr{V}_{1,2}S_{rb_2}^*(\theta)+\gamma\mathscr{R}_{1,2}S_{rb_2}^*(\theta), \tag{23}$$

$$-(\theta\mathscr{B}_{n,2}^*(\theta)-\mathscr{B}_{n,2}(0))=-(\lambda+\delta)\mathscr{B}_{n,2}^*(\theta)+p\mathscr{B}_{n,1}(0)S_{rb_2}^*(\theta)+\lambda\mathscr{B}_{n-1,2}^*(\theta)+\eta\mathscr{V}_{n,2}S_{rb_2}^*(\theta)+\gamma\mathscr{R}_{n,2}S_{rb_2}^*(\theta),\quad n\geq 2, \tag{24}$$

$$-(\theta\mathscr{R}_{1,1}^*(\theta)-\mathscr{R}_{1,1}(0))=-(\lambda+\gamma)\mathscr{R}_{1,1}^*(\theta)+\lambda\mathscr{R}_0S_{wb_1}^*(\theta)+q\mathscr{R}_{2,1}(0)S_{wb_1}^*(\theta)+\mathscr{R}_{2,2}(0)S_{wb_1}^*(\theta), \tag{25}$$

$$-(\theta\mathscr{R}_{n,1}^*(\theta)-\mathscr{R}_{n,1}(0))=-(\lambda+\gamma)\mathscr{R}_{n,1}^*(\theta)+\lambda\mathscr{R}_{n-1,1}^*(\theta)+q\mathscr{R}_{n+1,1}(0)S_{wb_1}^*(\theta)+\mathscr{R}_{n+1,2}(0)S_{wb_1}^*(\theta),\quad n\geq 2, \tag{26}$$

$$-(\theta\mathscr{R}_{1,2}^*(\theta)-\mathscr{R}_{1,2}(0))=-(\lambda+\gamma)\mathscr{R}_{1,2}^*(\theta)+p\mathscr{R}_{1,1}(0)S_{wb_2}^*(\theta), \tag{27}$$

$$-(\theta\mathscr{R}_{n,2}^*(\theta)-\mathscr{R}_{n,2}(0))=-(\lambda+\gamma)\mathscr{R}_{n,2}^*(\theta)+p\mathscr{R}_{n,1}(0)S_{wb_2}^*(\theta)+\lambda\mathscr{R}_{n-1,2}^*(\theta). \tag{28}$$

We get (29)-(31) by substituting $\theta=0$ into (17)-(28) and then summing over n from 1 to ∞.

$$q\mathscr{V}_{1,1}(0)+\mathscr{V}_{1,2}(0)=-\eta(\mathscr{V}_1^*(1,0)+\mathscr{V}_2^*(1,0))+\lambda\mathscr{V}_0, \tag{29}$$

$$q\mathscr{B}_{1,1}(0)+\mathscr{B}_{1,2}(0)=-\delta(\mathscr{B}_1^*(1,0)+\mathscr{B}_2^*(1,0))+\eta(\mathscr{V}_1^*(1,0)+\mathscr{V}_2^*(1,0))+\gamma(\mathscr{R}_1^*(1,0)+\mathscr{R}_2^*(1,0))+\lambda\mathscr{B}_0, \tag{30}$$

$$q\mathscr{R}_{1,1}(0)+\mathscr{R}_{1,2}(0)=-\gamma(\mathscr{R}_1^*(1,0)+\mathscr{R}_2^*(1,0))+\lambda\mathscr{R}_0. \tag{31}$$

From (11) and (31), we get

$$\delta(\mathscr{B}_1^*(1,0)+\mathscr{B}_2^*(1,0))=\gamma(\mathscr{R}_0+\mathscr{R}_1^*(1,0)+\mathscr{R}_2^*(1,0)). \tag{32}$$

We obtain (33)-(38) by multiplying by z^n into (17)-(28) as well as summing over n from 1 to ∞.

$$(\theta-\eta-\lambda+\lambda z)\mathscr{V}_1^*(z,\theta)=z^{-1}\mathscr{V}_1(z,0)(z-qS_{wv_1}^*(\theta))-S_{wv_1}^*(\theta)(\lambda\mathscr{V}_0z-q\mathscr{V}_{1,1}(0)-\mathscr{V}_{1,2}(0)+z^{-1}\mathscr{V}_2(z,0)), \tag{33}$$

$$(\theta-\eta-\lambda+\lambda z)\mathscr{V}_2^*(z,\theta)=\mathscr{V}_2(z,0)-p\mathscr{V}_1(z,0)S_{wv_2}^*(\theta), \tag{34}$$

$$(\theta-\delta-\lambda+\lambda z)\mathscr{B}_1^*(z,\theta)=z^{-1}\mathscr{B}_1(z,0)(z-qS_{rb_1}^*(\theta))-S_{rb_1}^*(\theta)(\eta\mathscr{V}_1^*(z,0)+\gamma\mathscr{R}_1^*(z,0)+\lambda\mathscr{B}_0z+z^{-1}\mathscr{B}_2(z,0)-q\mathscr{B}_{1,1}(0)-\mathscr{B}_{1,2}(0)), \tag{35}$$

$$(\theta-\delta-\lambda+\lambda z)\mathscr{B}_2^*(z,\theta)=\mathscr{B}_2(z,0)-pS_{rb_2}^*(\theta)\mathscr{V}_1(z,0)-\eta S_{rb_2}^*(\theta)\mathscr{V}_2^*(z,0)-\gamma S_{rb_2}^*(\theta)\mathscr{R}_2^*(z,0), \tag{36}$$

$$(\theta-\gamma-\lambda+\lambda z)\mathscr{R}_1^*(z,\theta)=z^{-1}\mathscr{R}_1(z,0)(z-qS_{wb_1}^*(\theta))-S_{wb_1}^*(\theta)(\lambda\mathscr{R}_0z-q\mathscr{R}_{1,1}(0)-\mathscr{R}_{1,2}(0)+z^{-1}\mathscr{R}_2(z,0)), \tag{37}$$

$$(\theta-\gamma-\lambda+\lambda z)\mathscr{R}_2^*(z,\theta)=\mathscr{R}_2(z,0)-p\mathscr{R}_1(z,0)S_{wb_2}^*(\theta). \tag{38}$$

Inserting $\theta=(\eta+\lambda-\lambda z)$ into (33) and (34), $\theta=(\delta+\lambda-\lambda z)$ into (35) and (36) and $\theta=(\gamma+\lambda-\lambda z)$ into (37) and (38), we get

$$\mathscr{V}_1(z,0)=\frac{zS_{wv_1}^*(\eta+\lambda-\lambda z)(q\mathscr{V}_{1,1}(0)+\mathscr{V}_{1,2}(0)-\lambda\mathscr{V}_0z)}{qS_{wv_1}^*(\eta+\lambda-\lambda z)+pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)-z}, \tag{39}$$

$$\mathscr{V}_2(z,0)=\frac{\left[pzS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)(q\mathscr{V}_{1,1}(0)+\mathscr{V}_{1,2}(0)-\lambda\mathscr{V}_0z)\right]}{qS_{wv_1}^*(\eta+\lambda-\lambda z)+pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)-z}, \tag{40}$$

$$\mathscr{B}_1(z,0)=\frac{\left[\begin{array}{l} zS_{rb_1}^*(\delta+\lambda-\lambda z)[q\mathscr{B}_{1,1}(0)+\mathscr{B}_{1,2}(0)\\ -\eta z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{V}_2^*(z,0)-\gamma z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{R}_2^*(z,0)\\ -\eta\mathscr{V}_1^*(z,0)-\gamma\mathscr{R}_1^*(z,0)-\lambda\mathscr{B}_0z]\end{array}\right]}{qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z}, \tag{41}$$

$$\mathscr{B}_2(z,0)=\frac{\left[\begin{array}{l} S_{rb_2}^*(\delta+\lambda-\lambda z)\{pzS_{rb_1}^*(\delta+\lambda-\lambda z)(q\mathscr{B}_{1,1}(0)+\mathscr{B}_{1,2}(0)\\ -\eta z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{V}_2^*(z,0)-\gamma z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{R}_2^*(z,0)\\ -\eta\mathscr{V}_1^*(z,0)-\gamma\mathscr{R}_1^*(z,0)-\lambda\mathscr{B}_0z)-(\eta\mathscr{V}_2^*(z,0)+\gamma\mathscr{R}_2^*(z,0))\\ \times(qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z)\}\end{array}\right]}{qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z}, \tag{42}$$

$$\mathscr{R}_1(z,0)=\frac{zS_{wb_1}^*(\gamma+\lambda-\lambda z)(q\mathscr{R}_{1,1}(0)+\mathscr{R}_{1,2}(0)-\lambda\mathscr{R}_0z)}{qS_{wb_1}^*(\gamma+\lambda-\lambda z)+pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)-z}, \tag{43}$$

$$\mathscr{R}_2(z,0)=\frac{pzS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)(q\mathscr{R}_{1,1}(0)+\mathscr{R}_{1,2}(0)-\lambda\mathscr{R}_0z)}{qS_{wb_1}^*(\gamma+\lambda-\lambda z)+pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)-z}. \tag{44}$$

We get (45)-(50) by substituting (39) and (40) into (33) and (34), (41) and (42) into (35) and (36), and (43) and (44) into (37) and (38), respectively.

$$\mathscr{V}_1^*(z,\theta)=\frac{z(q\mathscr{V}_{1,1}(0)+\mathscr{V}_{1,2}(0)-\lambda\mathscr{V}_0z)(S_{wv_1}^*(\eta+\lambda-\lambda z)-S_{wv_1}^*(\theta))}{\left[\begin{array}{l}(\theta-\eta-\lambda+\lambda z)(qS_{wv_1}^*(\eta+\lambda-\lambda z)\\ +pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)-z)\end{array}\right]}, \tag{45}$$

$$\mathscr{V}_2^*(z,\theta)=\frac{\left[\begin{array}{l}pzS_{wv_1}^*(\eta+\lambda-\lambda z)(q\mathscr{V}_{1,1}(0)+\mathscr{V}_{1,2}(0)-\lambda\mathscr{V}_0z)\\ \times(S_{wv_2}^*(\eta+\lambda-\lambda z)-S_{wv_2}^*(\theta))\end{array}\right]}{\left[\begin{array}{l}(\theta-\eta-\lambda+\lambda z)(qS_{wv_1}^*(\eta+\lambda-\lambda z)\\ +pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)-z)\end{array}\right]}, \tag{46}$$

$$\mathscr{B}_1^*(z,\theta)=\frac{\left[\begin{array}{l}z(q\mathscr{B}_{1,1}(0)+\mathscr{B}_{1,2}(0)-\eta z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{V}_2^*(z,0)\\ -\gamma z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{R}_2^*(z,0)-\eta\mathscr{V}_1^*(z,0)\\ -\gamma\mathscr{R}_1^*(z,0)-\lambda\mathscr{B}_0z)(S_{rb_1}^*(\delta+\lambda-\lambda z)-S_{rb_1}^*(\theta))\end{array}\right]}{\left[\begin{array}{l}(\theta-\delta-\lambda+\lambda z)(qS_{rb_1}^*(\delta+\lambda-\lambda z)\\ +pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z)\end{array}\right]}, \tag{47}$$

$$\mathscr{B}_2^*(z,\theta)=\frac{\left[\begin{array}{l}\{pzS_{rb_1}^*(\delta+\lambda-\lambda z)(q\mathscr{B}_{1,1}(0)+\mathscr{B}_{1,2}(0)-\eta z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\\ \mathscr{V}_2^*(z,0)-\gamma z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{R}_2^*(z,0)-\eta\mathscr{V}_1^*(z,0)\\ -\gamma\mathscr{R}_1^*(z,0)-\lambda\mathscr{B}_0 z)-(\eta\mathscr{V}_2^*(z,0)+\gamma\mathscr{R}_2^*(z,0))\\ \times(qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z)\}\\ \times(S_{rb_2}^*(\delta+\lambda-\lambda z)-S_{rb_2}^*(\theta))\end{array}\right]}{\left[\begin{array}{l}(\theta-\delta-\lambda+\lambda z)(qS_{rb_1}^*(\delta+\lambda-\lambda z)\\ +pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z)\end{array}\right]},\tag{48}$$

$$\mathscr{R}_1^*(z,\theta)=\frac{z(q\mathscr{R}_{1,1}(0)+\mathscr{R}_{1,2}(0)-\lambda\mathscr{R}_0 z)(S_{wb_1}^*(\gamma+\lambda-\lambda z)-S_{wb_1}^*(\theta))}{\left[\begin{array}{l}(\theta-\gamma-\lambda+\lambda z)(qS_{wb_1}^*(\gamma+\lambda-\lambda z)\\ +pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)-z)\end{array}\right]},\tag{49}$$

$$\mathscr{R}_2^*(z,\theta)=\frac{\left[\begin{array}{l}pzS_{wb_1}^*(\gamma+\lambda-\lambda z)(q\mathscr{R}_{1,1}(0)+\mathscr{R}_{1,2}(0)-\lambda\mathscr{R}_0 z)\\ \times(S_{wb_2}^*(\gamma+\lambda-\lambda z)-S_{wb_2}^*(\theta))\end{array}\right]}{\left[\begin{array}{l}(\theta-\gamma-\lambda+\lambda z)(qS_{wb_1}^*(\gamma+\lambda-\lambda z)\\ +pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)-z)\end{array}\right]}.\tag{50}$$

Eventually, the PGF of the system size, indicated by $P(z)$, can be determined:

$$\begin{aligned}P(z)&=\mathscr{V}_0+\mathscr{B}_0+\mathscr{R}_0+\mathscr{V}_1^*(z,0)+\mathscr{V}_2^*(z,0)+\mathscr{B}_1^*(z,0)+\mathscr{B}_2^*(z,0)+\mathscr{R}_1^*(z,0)+\mathscr{R}_2^*(z,0)\\ &=\mathscr{V}_0+\mathscr{B}_0+\mathscr{R}_0+\frac{\left[\begin{array}{l}z(\lambda\mathscr{V}_0(1-z)-\eta\mathscr{V}_1^*(1,0)-\eta\mathscr{V}_2^*(1,0))\\ (1-(qS_{wv_1}^*(\eta+\lambda-\lambda z)+pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z))\end{array}\right]}{\left[\begin{array}{l}(\eta+\lambda-\lambda z)(qS_{wv_1}^*(\eta+\lambda-\lambda z)\\ +pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)-z)\end{array}\right]}\\ &\quad+\frac{\left[\begin{array}{l}z(\lambda\mathscr{B}_0(1-z)+\eta(\mathscr{V}_1^*(1,0)-\mathscr{V}_1^*(z,0))+\gamma(\mathscr{R}_1^*(1,0)-\mathscr{R}_1^*(z,0))+\eta(\mathscr{V}_2^*(1,0)\\ -z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{V}_2^*(z,0))+\gamma(\mathscr{R}_2^*(1,0)-z^{-1}S_{rb_2}^*(\delta+\lambda-\lambda z)\mathscr{R}_2^*(z,0))\\ -\delta\mathscr{B}_1^*(1,0)-\delta\mathscr{B}_2^*(1,0))(1-(qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)))\\ -(\eta\mathscr{V}_2^*(z,0)+\gamma\mathscr{R}_2^*(z,0))(qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z)\end{array}\right]}{\left[\begin{array}{l}(\delta+\lambda-\lambda z)(qS_{rb_1}^*(\delta+\lambda-\lambda z)\\ +pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z)\end{array}\right]}\\ &\quad+\frac{\left[\begin{array}{l}z(\lambda\mathscr{R}_0(1-z)-\eta\mathscr{R}_1^*(1,0)-\gamma\mathscr{R}_2^*(1,0))\\ \times(1-(qS_{wb_1}^*(\gamma+\lambda-\lambda z)+pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z))\end{array}\right]}{\left[\begin{array}{l}(\gamma+\lambda-\lambda z)(qS_{wb_1}^*(\gamma+\lambda-\lambda z)\\ +pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)-z)\end{array}\right]}.\end{aligned}\tag{51}$$

From Rouche's theorem, $qS_{wv_1}^*(\eta+\lambda-\lambda z)+pS_{wv_1}^*(\eta+\lambda-\lambda z)S_{wv_2}^*(\eta+\lambda-\lambda z)-z=0$. For $|z|<1$, there is only one solution, represented by z_0. Assuming $z=z_0$, the denominator for $\mathscr{V}_1^*(z,0)+\mathscr{V}_2^*(z,0)$ be zero, the numerator must also be zero as well. Likewise, $qS_{rb_1}^*(\delta+\lambda-\lambda z)+pS_{rb_1}^*(\delta+\lambda-\lambda z)S_{rb_2}^*(\delta+\lambda-\lambda z)-z=0$. For $|z|<1$, there is only one solution, represented by z_1. Assuming $z=z_1$, the denominator for $\mathscr{B}_1^*(z,0)+\mathscr{B}_2^*(z,0)$ be zero, the numerator must also be zero as well. Moreover, $qS_{wb_1}^*(\gamma+\lambda-\lambda z)+pS_{wb_1}^*(\gamma+\lambda-\lambda z)S_{wb_2}^*(\gamma+\lambda-\lambda z)-z=0$. For $|z|<1$, there is only one solution, represented by z_2 . Assuming $z=z_2$, the denominator for

$\mathscr{R}_1^*(z,0)+\mathscr{R}_2^*(z,0)$ is zero, the numerator must also be zero as well. Thus, we obtain

$$\eta(\mathscr{V}_1^*(1,0)+\mathscr{V}_2^*(1,0))=\lambda\mathscr{V}_0(1-z_0), \tag{52}$$

$$\begin{aligned}\delta(\mathscr{B}_1^*(1,0)+\mathscr{B}_2^*(1,0))&=\lambda\mathscr{B}_0(1-z_1)+\lambda\mathscr{V}_0(1-z_0)+\lambda\mathscr{R}_0(1-z_2)\\&\quad-\eta(\mathscr{V}_1^*(z_1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{V}_1^*(z_1,0))\\&\quad-\gamma(\mathscr{R}_1^*(z_1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{R}_2^*(z_1,0)),\end{aligned} \tag{53}$$

$$\gamma(\mathscr{R}_1^*(1,0)+\mathscr{R}_2^*(1,0))=\lambda\mathscr{R}_0(1-z_2). \tag{54}$$

From (16), (6) and (51), $\mathscr{R}_0$ is obtained as

$$\mathscr{R}_0=\frac{N_{R_0}}{D_{R_0}},$$

where

$$\begin{aligned}N_{R_0}&=\eta\lambda\delta\gamma(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\\&\quad\times[\eta(1-z_1)(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)\\&\quad+\lambda(1-z_0)(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)\\&\quad-\lambda(1-z_0)(B_1(z_1)(S^*_{wv_1}(\eta+\lambda-\lambda z_1)-z_1)-pz_1^{-1}B_2(z_1)\\&\quad\times S^*_{rb_2}(\delta+\lambda-\lambda z_1)S^*_{wv_1}(\eta+\lambda-\lambda z_1)(S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1))]\\&\quad\times(qS^*_{wb_1}(\gamma+\lambda-\lambda z_1)+pS^*_{wb_1}(\gamma+\lambda-\lambda z_1)S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1),\\D_{R_0}&=\eta\lambda(\delta+\gamma)(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\\&\quad\times(qS^*_{wb_1}(\gamma+\lambda-\lambda z_1)+pS^*_{wb_1}(\gamma+\lambda-\lambda z_1)S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1)\\&\quad\times[\eta(1-z_1)(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)\\&\quad+\lambda(1-z_0)(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)\\&\quad-\lambda(1-z_0)(B_1(z_1)(S^*_{wv_1}(\eta+\lambda-\lambda z_1)-z_1)-pz_1^{-1}B_2(z_1)S^*_{rb_2}(\delta+\lambda-\lambda z_1)\\&\quad\times S^*_{wv_1}(\eta+\lambda-\lambda z_1)(S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1))](\gamma+\lambda-\lambda z_2)\\&\quad+\delta\gamma\{\eta(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\\&\quad\times[\eta(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)\\&\quad\times(\gamma(qS^*_{wb_1}(\gamma+\lambda-\lambda z_1)+pS^*_{wb_1}(\gamma+\lambda-\lambda z_1)S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1)\\&\quad+\lambda(1-z_2)(A_1(z_1)(S^*_{wb_1}(\gamma+\lambda-\lambda z_1)-z_1)+pz_1^{-1}A_2(z_1)S^*_{rb_2}(\delta+\lambda-\lambda z_1)\\&\quad\times(S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1)))+\gamma(qS^*_{wb_1}(\gamma+\lambda-\lambda z_1)+pS^*_{wb_1}(\gamma+\lambda-\lambda z_1)\\&\quad\times S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1)(\lambda(1-z_0)(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)\\&\quad\times S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)-\lambda(1-z_0)(B_1(z_1)(S^*_{wv_1}(\eta+\lambda-\lambda z_1)-z_1)\\&\quad-pz_1^{-1}B_2(z_1)S^*_{rb_2}(\delta+\lambda-\lambda z_1)S^*_{wv_1}(\eta+\lambda-\lambda z_1)(S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)))]\\&\quad+\lambda(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\\&\quad\times[\lambda(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)(1-z_2)\\&\quad\times(A_1(z_1)(S^*_{wb_1}(\gamma+\lambda-\lambda z_1)-z_1)+pz_1^{-1}A_2(z_1)S^*_{rb_2}(\delta+\lambda-\lambda z_1)(S^*_{wb_2}(\gamma+\lambda-\lambda z_1)\\&\quad-z_1))+\gamma z_1(qS^*_{wv_1}(\eta+\lambda-\lambda z_1)+pS^*_{wv_1}(\eta+\lambda-\lambda z_1)S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)\end{aligned}$$

$$\times (qS^*_{wb_1}(\gamma+\lambda-\lambda z_1)+pS^*_{wb_1}(\gamma+\lambda-\lambda z_1)S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1)](\eta+\lambda-\lambda z_0)\},$$

where

$$A_1(z_1)=\frac{\gamma z_1(z_1-z_2)(S^*_{wb_1}(\gamma+\lambda-\lambda z_1)-1)}{(1-z_2)(\gamma+\lambda-\lambda z_1)(S^*_{wb_1}(\gamma+\lambda-\lambda z_1)-z_1)},$$

$$A_2(z_1)=\frac{\gamma z_1(z_1-z_2)(S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-1)}{(1-z_2)(\gamma+\lambda-\lambda z_1)(S^*_{wb_2}(\gamma+\lambda-\lambda z_1)-z_1)},$$

$$B_1(z_1)=\frac{\eta z_1(z_1-z_0)(S^*_{wv_1}(\eta+\lambda-\lambda z_1)-1)}{(1-z_0)(\eta+\lambda-\lambda z_1)(S^*_{wv_1}(\eta+\lambda-\lambda z_1)-z_1)},$$

$$B_2(z_1)=\frac{\eta z_1(z_1-z_0)(S^*_{wv_2}(\eta+\lambda-\lambda z_1)-1)}{(1-z_0)(\eta+\lambda-\lambda z_1)(S^*_{wv_2}(\eta+\lambda-\lambda z_1)-z_1)}.$$

(Kim *et al*. [10], and Kim and Lee [9])

Special Cases

Case I: In our model if we place $p=0$ and $\eta\to\infty$ then our model is remodeled as "An M/G/1 queue with disasters and working breakdowns" (Kim and Lee [9]).

Case II: In our model if we place $p=0$, $\eta\to\infty$ and $\delta=0$ then our model is remodeled as "An M/G/1 queue" (Medhi [15]).

4. Metrics of Effectiveness

Size of the Average System

Allow L_{wv}, L_{rb} and L_{wb} to represent the average system size throughout working vacation, typical busy and repair periods, respectively and let W_{wv}, W_{rb} and W_{wb} represent the average waiting times of customers in the system throughout working vacation, typical busy and repair periods, respectively. Then

(a) Differentiating $(\mathscr{V}_1^*(z,0)+\mathscr{V}_2^*(z,0))$ with regard to z and calculating at $z=1$ gives the expected number of customers in the system (L_{wv}).

$$L_{wv}=\frac{\lambda\mathscr{V}_0\begin{bmatrix}\eta(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)\{(1-z_0)(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)\\ \times S^*_{wv_2}(\eta)-1)+(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)-\lambda(1-z_0)(qS^{*'}_{wv_1}(\eta)\\ +pS^{*'}_{wv_1}(\eta)S^*_{wv_2}(\eta)+pS^*_{wv_1}(\eta)S^{*'}_{wv_2}(\eta))\}-(1-z_0)(qS^*_{wv_1}(\eta)\\ +pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)\{-\lambda(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)\\ +\eta(-q\lambda S^{*'}_{wv_1}(\eta)-p\lambda S^{*'}_{wv_1}(\eta)S^*_{wv_2}(\eta)-p\lambda S^*_{wv_1}(\eta)S^{*'}_{wv_2}(\eta)-1)\}\end{bmatrix}}{[\eta(qS^*_{wv_1}(\eta)+pS^*_{wv_1}(\eta)S^*_{wv_2}(\eta)-1)]^2}.$$

We used Little's formula to calculate $W_{wv}=\frac{L_{wv}}{\lambda}$ and obtained the expected number of waiting customers in the system during the working vacation.

(b) Now, differentiating $(\mathscr{B}_1^*(z,0)+\mathscr{B}_2^*(z,0))$ with regard to z and calculating at $z=1$ provides the expected number of customers in the system during typical busy (L_{rb}).

$$L_{rb}=\frac{N_{rb}}{D_{rb}},$$

$$N_{rb}=\delta(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)\{(\lambda\mathscr{B}_0(1-z_1)+\eta(\mathscr{V}_1^*(1,0)-\mathscr{V}_1^*(z_1,0))$$

$$
\begin{aligned}
&+\gamma(\mathscr{R}_1^*(1,0)-\mathscr{R}_1^*(z_1,0))+\eta(S^*_{rb_2}(\delta)\mathscr{V}_2^*(1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{V}_2^*(z_1,0))\\
&+\gamma(S^*_{rb_2}(\delta)\mathscr{R}_2^*(1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{R}_2^*(z_1,0)))(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)\\
&+(\lambda\mathscr{B}_0+\eta\mathscr{V}_1^{*'}(1,0)+\gamma\mathscr{R}_1^{*'}(1,0)-\eta S^*_{rb_2}(\delta)\mathscr{V}_2^*(1,0)-\eta\lambda S^{*'}_{rb_2}(\delta)\mathscr{V}_2^*(1,0)\\
&+\eta S^*_{rb_2}(\delta)\mathscr{V}_2^{*'}(1,0)-\gamma S^*_{rb_2}(\delta)\mathscr{R}_2^*(1,0)-\gamma\lambda S^{*'}_{rb_2}(\delta)\mathscr{R}_2^*(1,0)+\gamma S^*_{rb_2}(\delta)\mathscr{R}_2^{*'}(1,0))\\
&\times(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)+(\lambda\mathscr{B}_0(1-z_1)+\eta(\mathscr{V}_1^*(1,0)-\mathscr{V}_1^*(z_1,0))\\
&+\gamma(\mathscr{R}_1^*(1,0)-\mathscr{R}_1^*(z_1,0))+\eta(S^*_{rb_2}(\delta)\mathscr{V}_2^*(1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{V}_2^*(z_1,0))\\
&+\gamma(S^*_{rb_2}(\delta)\mathscr{R}_2^*(1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{R}_2^*(z_1,0)))(-q\lambda S^{*'}_{rb_1}(\delta)-p\lambda S^{*'}_{rb_1}(\delta)S^*_{rb_2}(\delta)\\
&-p\lambda S^*_{rb_1}(\delta)S^{*'}_{rb_2}(\delta))-\{(\eta\mathscr{V}_2^*(1,0)+\gamma\mathscr{R}_2^*(1,0))(-q\lambda S^{*'}_{rb_1}(\delta)-p\lambda S^{*'}_{rb_1}(\delta)S^*_{rb_2}(\delta)\\
&-p\lambda S^*_{rb_1}(\delta)S^{*'}_{rb_2}(\delta)-1)(S^*_{rb_2}(\delta)-1)+(\eta\mathscr{V}_2^{*'}(1,0)+\gamma\mathscr{R}_2^{*'}(1,0))\\
&\times(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)(S^*_{rb_2}(\delta)-1)-\lambda(\eta\mathscr{V}_2^*(1,0)+\gamma\mathscr{R}_2^*(1,0))\\
&\times(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)S^{*'}_{rb_2}(\delta)\}\\
&-\{(\lambda\mathscr{B}_0(1-z_1)+\eta(\mathscr{V}_1^*(1,0)-\mathscr{V}_1^*(z_1,0))+\gamma(\mathscr{R}_1^*(1,0)-\mathscr{R}_1^*(z_1,0))\\
&+\eta(S^*_{rb_2}(\delta)\mathscr{V}_2^*(1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{V}_2^*(z_1,0))\\
&+\gamma(S^*_{rb_2}(\delta)\mathscr{R}_2^*(1,0)-z_1^{-1}S^*_{rb_2}(\delta+\lambda-\lambda z_1)\mathscr{R}_2^*(z_1,0)))(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)\\
&-1)-(\eta\mathscr{V}_2^*(1,0)+\gamma\mathscr{R}_2^*(1,0))(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)(S^*_{rb_2}(\delta)-1)\}\\
&\{-\lambda(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)+\delta(-q\lambda S^{*'}_{rb_1}(\delta)-p\lambda S^{*'}_{rb_1}(\delta)S^*_{rb_2}(\delta)-p\lambda S^*_{rb_1}(\delta)\\
&\times S^{*'}_{rb_2}(\delta)-1)\},
\end{aligned}
$$

$$D_{rb}=[\delta(qS^*_{rb_1}(\delta)+pS^*_{rb_1}(\delta)S^*_{rb_2}(\delta)-1)]^2.$$

We used Little's formula to calculate $W_{rb}=\frac{L_{rb}}{\lambda}$ and obtained the expected number of waiting customers in the system during typical busy.

(c) Then differentiating $(\mathscr{R}_1^*(z,0)+\mathscr{R}_2^*(z,0))$ with regard to z and calculating at $z=1$ gives the expected number of customers in the system during repair periods (L_{wb}).

$$
L_{wb}=\frac{\lambda\mathscr{R}_0\left[\begin{aligned}&\gamma(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\{(1-z_2)(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)\\
&\times S^*_{wb_2}(\gamma)-1)+(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)-\lambda(1-z_2)(qS^{*'}_{wb_1}(\gamma)\\
&+pS^{*'}_{wb_1}(\gamma)S^*_{wb_2}(\gamma)+pS^*_{wb_1}(\gamma)S^{*'}_{wb_2}(\gamma))\}-(1-z_2)(qS^*_{wb_1}(\gamma)\\
&+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\{-\lambda(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)\\
&+\gamma(-q\lambda S^{*'}_{wb_1}(\gamma)-p\lambda S^{*'}_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-p\lambda S^*_{wb_1}(\gamma)S^{*'}_{wb_2}(\gamma)-1)\}\end{aligned}\right]}{[\gamma(qS^*_{wb_1}(\gamma)+pS^*_{wb_1}(\gamma)S^*_{wb_2}(\gamma)-1)]^2}.
$$

We used Little's formula to calculate $W_{wb}=\frac{L_{wb}}{\lambda}$ and obtained the expected number of waiting customers in the system during repair periods.

5. Statistical Outcomes

Setting $q=0.8$, $p=0.2$, $\mu_{rb_2}=2.6$, $\mu_{wb_1}=2.7$, $\mu_{wb_2}=2.8$, $\mu_{wv_1}=2.5$, $\mu_{wv_2}=2.6$, $\eta=2.1$, $\delta=2.5$, $\gamma=4.8$, $z_0=0.5$, $z_1=0.5$, $z_2=0.5$ and changing η from 2.1 to 2.3 insteps of 0.1, δ from 2.5 to 3.5 insteps of 0.5 and γ from 4.8 to 6.8 insteps of 1.0. We calculated the measured value of L_{wv}

and tabulated them in Table 1, Table 2, and Table 3, respectively. The correlating graphs for λ versus L_{wv} are seen in Figure 1, Figure 2 and Figure 3, respectively. As λ rises, L_{wv} falls for various values of η, whereas L_{wv} rises for various values of δ and γ, as seen in the graphs.

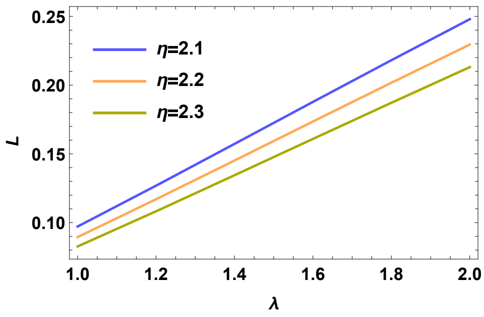

Figure 1. λ vs. L_{wv}

λ	$\eta = 2.1$	$\eta = 2.2$	$\eta = 2.3$
1.0	0.0971	0.0895	0.0827
1.2	0.1269	0.1161	0.1082
1.4	0.1572	0.1452	0.1344
1.6	0.1878	0.1735	0.1608
1.8	0.2181	0.2017	0.1872
2.0	0.2471	0.2295	0.2131

Table 1. λ vs. L_{wv}

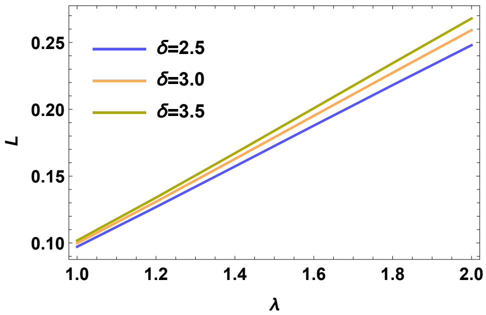

Figure 2. λ vs. L_{wv}

λ	$\delta = 2.5$	$\delta = 3.0$	$\delta = 3.5$
1.0	0.0971	0.0998	0.1018
1.2	0.1269	0.1309	0.1339
1.4	0.1572	0.1628	0.1671
1.6	0.1878	0.1951	0.2008
1.8	0.2181	0.2274	0.2345
2.0	0.2471	0.2593	0.2671

Table 2. λ vs. L_{wv}

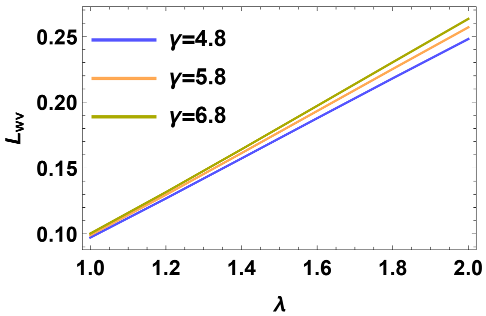

Figure 3. λ vs. L_{wv}

λ	$\gamma = 4.8$	$\gamma = 5.8$	$\gamma = 6.8$
1.0	0.0971	0.0989	0.1002
1.2	0.1269	0.1297	0.1317
1.4	0.1572	0.1613	0.1642
1.6	0.1878	0.1932	0.1972
1.8	0.2181	0.2252	0.2303
2.0	0.2471	0.2568	0.2633

Table 3. λ vs. L_{wv}

Setting $q = 0.8$, $p = 0.2$, $\mu_{wb_1} = 2.3$, $\mu_{wb_2} = 2.4$, $\mu_{rb_1} = 2.2$, $\mu_{rb_2} = 2.1$, $\mu_{wv_1} = 2.5$, $\mu_{wv_2} = 2.3$, $\eta = 2.1$, $\delta = 2.5$, $\gamma = 2.8$, $z_0 = 0.5$, $z_1 = 0.5$, $z_2 = 0.5$ and changing η from 2.1 to 4.1 insteps of 1.0, δ from 2.5 to 4.5 insteps of 1.0 and γ from 2.8 to 3.6 insteps of 0.4. We calculated the measured value of L_{rb} and tabulated them in Table 4, Table 5 and Table 6, respectively. The correlating graphs for λ versus L_{rb} are seen in Figure 4, Figure 5 and Figure 6, respectively. As λ rises, L_{rb} rises for various values of η and γ, L_{rb} falls for various values of δ, as seen in the graphs.

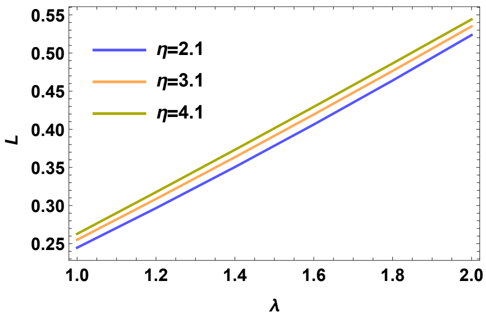

Figure 4. λ vs. L_{rb}

λ	$\eta = 2.1$	$\eta = 3.1$	$\eta = 4.1$
1.0	0.2447	0.2573	0.2628
1.2	0.2966	0.3112	0.3176
1.4	0.3505	0.3663	0.3731
1.6	0.4063	0.4225	0.4294
1.8	0.4631	0.4798	0.4864
2.0	0.5236	0.5382	0.5441

Table 4. λ vs. L_{rb}

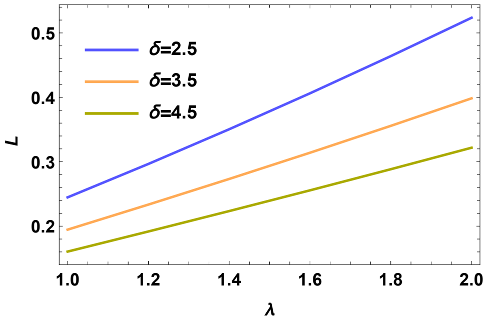

Figure 5. λ vs. L_{rb}

λ	$\delta = 2.5$	$\delta = 3.5$	$\delta = 4.5$
1.0	0.2447	0.1946	0.1605
1.2	0.2966	0.2335	0.1917
1.4	0.3505	0.2733	0.2234
1.6	0.4063	0.3140	0.2556
1.8	0.4631	0.3557	0.2884
2.0	0.5236	0.3985	0.3218

Table 5. λ vs. L_{rb}

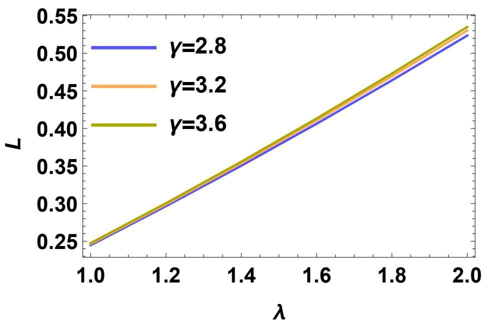

Figure 6. λ vs. L_{rb}

λ	$\gamma = 2.8$	$\gamma = 3.2$	$\gamma = 3.6$
1.0	0.2447	0.2464	0.2474
1.2	0.2966	0.2981	0.3005
1.4	0.3505	0.3537	0.3558
1.6	0.4063	0.4105	0.4133
1.8	0.4631	0.4694	0.4730
2.0	0.5236	0.5303	0.5350

Table 6. λ vs. L_{rb}

Setting $q = 0.8$, $p = 0.2$, $\mu_{wb_1} = 2.7$, $\mu_{wb_2} = 2.8$, $\mu_{wv_1} = 2.5$, $\mu_{rb_2} = 2.6$, $\eta = 2.2$, $\delta = 2.7$, $\gamma = 2.9$, $z_0 = 0.5$, $z_1 = 0.5$, $z_2 = 0.5$ and changing η from 2.2 to 2.6 insteps of 0.2, δ from 2.7 to 4.7 insteps of 1.0 and γ from 2.9 to 4.9 insteps of 1.0. We calculated the measured value of L_{wb} and tabulated them in Table 7, Table 8 and Table 9, respectively. The correlating graphs for λ versus L_{wb} are seen in Figure 7, Figure 8 and Figure 9, respectively. As λ rises, L_{wb} falls for various values of η and γ whereas L_{wb} rises for various values of δ, as seen in the graphs.

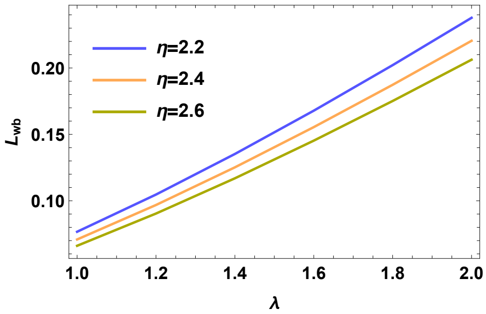

Figure 7. λ vs. L_{wb}

λ	$\eta = 2.2$	$\eta = 2.4$	$\eta = 2.6$
1.0	0.0767	0.0709	0.0661
1.2	0.1047	0.0968	0.0904
1.4	0.1352	0.1252	0.1169
1.6	0.1678	0.1555	0.1453
1.8	0.2021	0.1873	0.1751
2.0	0.2378	0.2205	0.2062

Table 7. λ vs. L_{wb}

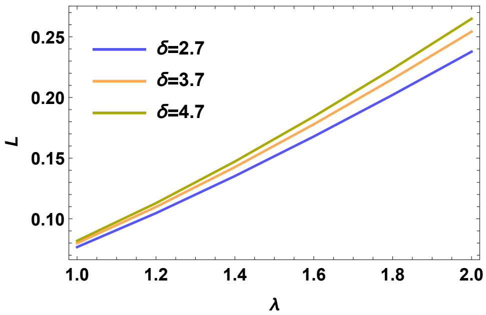

Figure 8. λ vs. L_{wb}

λ	$\delta = 2.7$	$\delta = 3.7$	$\delta = 4.7$
1.0	0.0767	0.0799	0.0819
1.2	0.1047	0.1098	0.1129
1.4	0.1352	0.1426	0.1472
1.6	0.1678	0.1779	0.1843
1.8	0.2021	0.2153	0.2236
2.0	0.2378	0.2543	0.2649

Table 8. λ vs. L_{wb}

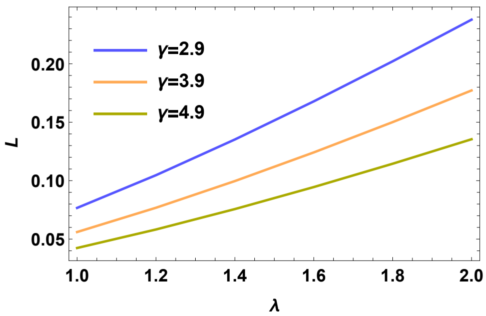

Figure 9. λ vs. L_{wb}

λ	$\gamma = 2.9$	$\gamma = 3.9$	$\gamma = 4.9$
1.0	0.0767	0.0560	0.0424
1.2	0.1047	0.0768	0.0582
1.4	0.1352	0.0997	0.0757
1.6	0.1678	0.1242	0.0945
1.8	0.2021	0.1501	0.1145
2.0	0.2378	0.1772	0.1354

Table 9. λ vs. L_{wb}

6. Conclusion

We looked at a M/G/1 queue with second optional service, disasters, working breakdowns and working vacation is evaluated. We established steady state queue length distributions for idle server, typical busy service and delayed service using the PGF and supplementary variable technique. Also, we perform some special cases. Eventually, the statistical outcomes and metrics of effectiveness are computed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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