Communications in Mathematics and Applications
Vol. 15, No. 1, pp. 179–184, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i1.2067

http://www.rgnpublications.com

Research Article

Fixed Point Theorems for (ξ,α)-Expansive Mappings in Metric Like Spaces

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Received: September 24, 2022 **Accepted:** December 21, 2023

Abstract. In this paper, we shall introduce the (ξ,α)-expansive mappings and prove some fixed point theorems for these mappings in complete metric like spaces. We shall also provide examples to illustrate the main results.

Keywords. Fixed point, (ξ,α)-expansive mappings, Metric like spaces

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

In 2000, Hitzler and Seda [6] introduced the notion of metric like (dislocated metric) space and generalized Banach [5] contraction principle in such spaces. In 2012, Amini-Harandi [4] discovered dislocated metric spaces and proved some fixed point theorems in these spaces. Many authors proved fixed point theorems in these spaces (see [1–3, 7, 9, 10, 14, 15]).

In 1984, Wang *et al.* [13] established some fixed point theorems for expansive mappings in complete metric spaces. Recently in 2012, Shahi *et al.* [12] proved fixed point theorems for (ξ,α)-expansive mappings in complete metric space.

In this paper, we shall prove fixed point theorems for (ξ,α)-expansive mappings in metric like spaces.

Definition 1.1 ([4])**.** Let X be a non-empty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions:

(i) $d(x,y)=0 \Rightarrow x=y$; (ii) $d(x,y)=d(y,x)$; (iii) $d(x,z) \le d(x,y)+d(y,z)$,

for all $x,y,z \in X$.

Then d is called metric like (dislocated metric) and (X,d) is called metric like (dislocated) metric space.

Definition 1.2 ([4])**.** Let (X,d) be a metric like space:

(i) A sequence $\{x_n\}$ in X is a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n,y_m)$ exists and is finite.

(ii) (X,d) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x\in X$, that is,
$$\lim_{n\to\infty} d(x,x_n)=d(x,x)=\lim_{n,m\to\infty} d(x_n,x_m).$$

(iii) A mapping $T:(X,d)\to(X,d)$ is continuous if for any sequence x_n in X such that
$$d(x_n,x)\to d(x,x) \text{ as } n\to\infty,$$
we have
$$d(Tx_n,Tx)\to d(Tx,Tx) \text{ as } n\to\infty.$$

Lemma 1.3 ([8])**.** *[8]Let (X,d) be a metric like space. Let $\{X_n\}$ be a sequence in X such that $X_n\to x$ where $x\in X$ and $d(x,x)=0$. Then for all $y\in X$, we have*
$$\lim_{n\to\infty} d(x_n,y)=d(x,y).$$

Definition 1.4 ([11])**.** Let $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$. We say that T is an α-admissible mapping if
$$\alpha(x,y)=1 \implies \alpha(Tx,Ty)\ge 1, \quad \text{for all } x,y\in X.$$

In 2012, Shahi *et al.* [12] gave the following family of functions:

Let χ denote all functions $\xi:[0,\infty)\to[0,\infty)$ which satisfy the following properties:

(i) ξ is non decreasing;

(ii) $\sum_{n=1}^{+\infty}\xi^n(a)<+\infty$ for each $a>0$ where ξ^n is the nth iterate of ξ;

(iii) $\xi(a+b)=\xi(a)+\xi(b)$,

for all $a,b\in[0,\infty)$.

Lemma 1.5 ([11])**.** *If $\xi:[0,\infty)\to[0,\infty)$ is a non decreasing function, then for each $a>0$, $\lim_{n\to\infty}\xi^n(a)=0$ implies $\xi(a)<a$.*

2. Main Results

In this section, we shall prove the fixed point theorems for expansive mappings.

Definition 2.1. Let (X,d) be a metric like space and $T:X\to X$ be a given mapping. We say that T is a (ξ,α)-expansive mapping if there exists two functions $\xi\in\chi$ and $\alpha:X\times X\to[0,\infty)$ such that
$$\xi(d(Tx,Ty))\ge\alpha(x,y)d(x,y), \tag{2.1}$$
for all $x,y\in X$.

Theorem 2.2. *Let (X,d) be a complete metric-like space and $T:X\to X$ be a bijective, (ξ,α)-expansive mapping satisfying the following conditions:*

(i) *T^{-1} is a-admissible;*

(ii) *There exists $x_0\in X$ such that $a(x_0,T^{-1}x_0)\ge 1$;*

(iii) *T is continuous.*

Then there exists a $w \in X$ such that $d(w,w)=0$. Assume in addition that

$$\text{if } d(x,x)=0, \quad \text{for some } x \in X,$$

then $\alpha(T^{-1}x,T^{-1}x)\geq 1$. Then such w is a fixed point of T, that is, $Tw=w$.

Proof. Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1}, \quad \text{for all } n \in \mathbb{N},$$

where $x_0 \in X$ such that $\alpha(x_0,T^{-1}x_0)\geq 1$. Now if $x_m = x_{m+1}$ for some $m \in \mathbb{N}$, then $Tx_{m+1}=x_m$. So T has a fixed point and we are done.
So, let us assume that

$$x_n \neq x_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

It is given that $\alpha(x_0,x_1)=\alpha(x_0,T^{-1}x_0)\geq 1$. Recalling that T^{-1} is α-admissible, therefore, we have

$$\alpha(T^{-1}x_0,T^{-1}x_1)=\alpha(x_1,x_2)\geq 1.$$

Continuing this process, we get

$$\alpha(x_n,x_{n+1})\geq 1, \tag{2.2}$$

for all $n \in \mathbb{N}$.
Using equation (2.2) and applying equation (2.1) with $x=x_n$, $y=x_{n+1}$, we obtain

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \alpha(x_n,x_{n+1})d(x_n,x_{n+1}) \\ &\leq \xi(d(Tx_n,Tx_{n+1})) \\ &= \xi(d(x_{n-1},x_n)). \end{aligned}$$

Therefore, by repetition of inequality, we have

$$d(x_n,x_{n+1})\leq \xi^n(d(x_0,x_1)),$$

for all $n \in \mathbb{N}$.
For any $n>m\geq 0$, we have

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m+1})+d(x_{m+1},x_{m+2})+d(x_{m+2},x_{m+3})+\ldots+d(x_{n-1},x_n) \\ &\leq \xi^m(d(x_0,x_1))+\ldots+\xi^{n-1}(d(x_0,x_1)). \end{aligned}$$

From $\sum \xi^n(a)<\infty$ for all $a>0$, it follows that $\{x_n\}$ is Cauchy sequence in the complete metric like space (X,d). So, there exists $w \in X$ such that

$$\lim_{n\to\infty} d(x_n,w)=d(w,w)=\lim_{n,m\to\infty} d(x_n,x_m)=0. \tag{2.3}$$

Since $T^{-1}$is continuous, from equation (2.3), we obtain that

$$\lim_{n\to\infty} d(x_{n+1},T^{-1}w)=\lim_{n\to\infty} d(T^{-1}x_n,T^{-1}w)=d(T^{-1}w,T^{-1}w). \tag{2.4}$$

On the other hand, by equation (2.3) and Lemma 1.3, we have

$$\lim_{n\to\infty} d(x_{n+1},T^{-1}w)=d(w,T^{-1}w). \tag{2.5}$$

Comparing equations (2.4) and (2.5), we get

$$d(T^{-1}w,T^{-1}w)=d(w,T^{-1}w).$$

Now from equation (2.1) using hypothesis, we have

$$\begin{aligned} d(w,T^{-1}w) &= d(T^{-1}w,T^{-1}w) \\ &\leq \alpha(T^{-1}w,T^{-1}w)d(T^{-1}w,T^{-1}w) \\ &\leq \xi(d(w,w)) \\ &< d(w,w) \end{aligned}$$

which implies that

$$d(w,T^{-1}w)=0,$$

that is $w=T^{-1}w$ implies that

$$Tw=w.$$

In the next theorem we omit continuity by the following hypothesis:

(H). *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $\to\infty$, then $\alpha(T^{-1}x_n,T^{-1}x)\geq 1$.*

Theorem 2.3. *If in Theorem 2.2, we replace continuity by the hypothesis (H), then still T has a fixed point.*

Proof. Following the proof of Theorem 2.2, we know that sequence $\{x_n\}$ is Cauchy sequence in X.
Since (X,d) is a complete metric like space, $x_n\to w$ as $n\to\infty$.
So, $\alpha(T^{-1}x_n,T^{-1}w)\geq 1$.
From equation (2.1), we get

$$\begin{aligned}
d(T^{-1}w,w) &\leq d(T^{-1}w,x_{n+1})+d(x_{n+1},w)\\
&= d(T^{-1}x_n,T^{-1}w)+d(x_{n+1},w)\\
&\leq \alpha(T^{-1}x_n,T^{-1}w)d(T^{-1}x_n,T^{-1}w)+d(x_{n+1},w)\\
&\leq \xi(d(x_n,w))+d(x_{n+1},w).
\end{aligned}$$

Taking limit as $n\to\infty$ and using continuity of ξ at $t=0$ and equation (2.3), we obtain that $d(T^{-1}w,w)=0$, that is

$$T^{-1}w=w$$

implies that

$$Tw=w. \qquad \square$$

Now we shall a condition to get a uniqueness of fixed point.

Theorem 2.4. *If in Theorems 2.2 and 2.3, we add the following condition:*

If $w\in X$ such that $Tw=w$, then for all $v\in X$, $\alpha(w,v)\geq 1$.

Then T has a unique fixed point.

Proof. Following the proof of Theorems 2.2 and 2.3, we obtain that w is the fixed point of T.
So let us suppose, if possible T have w and v two distinct fixed point.

Then equation (2.1) using above condition and definition of ξ, implies that

$$\begin{aligned}
d(w,v) &\leq \alpha(w,v)d(w,v)\\
&\leq \xi(d(Tw,Tv))\\
&< d(Tw,Tv)\\
&= d(w,v)
\end{aligned}$$

a contradiction.
So, T has a unique fixed point. $\square$

Example 2.5. Let $X=[0,\infty)$ and $d:[0,\infty)\times[0,\infty)\to[0,\infty)$ as $d(x,y)=\max\{x,y\}$.
Clearly, (X,d) is a complete metric like space.

Define $T:[0,\infty)\to[0,\infty)$ as $Tx=4x$ and

$$\xi(a)=\frac{a}{2},\quad \text{for all } a\geq 0,$$

$$\alpha(x,y)=1,\quad \text{for all } x,y\in X.$$

Now, without loss of generality assume that $x\geq y$, then left hand side of equation (2.1) is

$$\xi(d(Tx,Ty))=\frac{1}{2}4x=2x. \tag{2.6}$$

Similarly, right hand side of equation (2.1) becomes

$$\alpha(x,y)d(x,y)=x. \tag{2.7}$$

From equations (2.6) and (2.7), we obtain that
T is bijective (ξ,α)-expansive mapping.
Clearly, T^{-1} is α-admissible.
If we take $x_0=1$, then clearly $\alpha(1,T^{-1}1)\geq 1$.
Also $d(0,0)=0$, so it is clear that $\alpha(0,T^{-1}0)\geq 1$ and T^{-1} is continuous.
So, all the conditions of Theorems 2.2 and 2.3 are satisfied.
Hence T has an unique fixed point.
Clearly, 0 is the unique fixed point of$T$.

Example 2.6. Let $X=[0,\infty)$ and $d:[0,\infty)\times[0,\infty)\to[0,\infty)$ as $d(x,y)=\max\{x,y\}$.
Clearly, (X,d) is a complete metric like space.
Define $T:[0,\infty)\to[0,\infty)$ as $Tx=5x$ and

$$\xi(a)=\frac{a}{2},\quad \text{for all } a\geq 0,$$

$$\alpha(x,y)=\frac{3}{2},\quad \text{for all } x,y\in X.$$

Now, without loss of generality assume that $x\geq y$, then left hand side of equation (2.1) is

$$\xi(d(Tx,Ty))=\frac{1}{2}5x=\frac{5}{2}x. \tag{2.8}$$

Similarly, right hand side of equation (2.1) becomes

$$\alpha(x,y)d(x,y)=\frac{3}{2}x. \tag{2.9}$$

From equations (2.8) and (2.9), we obtain that
T is bijective (ξ,α)-expansive mapping.
Clearly, T^{-1} is α-admissible.
If we take $x_0=1$, then clearly $\alpha(1,T^{-1}1)\geq 1$.
Also, $d(0,0)=0$, so it is clear that $\alpha(0,T^{-1}0)\geq 1$ and hypothesis (H) holds.
So, all the conditions of Theorem 2.3 are satisfied.
Hence T has an unique fixed point.
Clearly, 0 is the unique fixed point of T.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. T. Aage and J. N. Salunke, Some results of fixed point theorem in dislocated quasi metric space, *Bulletin of Marathwada Mathematical Society* **9** (2008), 1 – 5.

[2] C. T. Aage and J. N. Salunke, The results of fixed points in dislocated and dislocated quasi-metric space, *Applied Mathematical Sciences* **2**(59) (2008), 2941 – 2948.

[3] M. U. Ali and T. Kamran, On (α^*,ψ)-contractive multi-valued mappings, *Fixed Point Theory and Applications* **2013** (2013), Article number: 137, DOI: 10.1186/1687-1812-2013-137.

[4] A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, *Fixed Point Theory and Applications* **2012** (2012), Article number: 204, DOI: 10.1186/1687-1812-2012-204.

[5] B. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3**(1) (1992), 133 – 181, URL: https://eudml.org/doc/213289.

[6] A. Hitzler and A. K. Seda, Dislocated topologies, *Journal of Electrical Engineering* **51**(12/s) (2000), 3 – 7, URL: http://iris.elf.stuba.sk/JEEEC/data/pdf/12s_100-02.pdf.

[7] A. Isufati, Fixed point theorems in dislocated quasi-metric spaces, *Applied Mathematical Sciences* **4**(5) (2010), 217 – 223.

[8] E. Karapınar and P. Salimi, Dislocated metric space to metric spaces with some fixed point theorems, *Fixed Point Theory and Applications* **2013** (2013), Article number: 222, DOI: 10.1186/1687-1812-2013-222.

[9] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, *Fixed Point Theory* **4**(1) (2003), 79 – 89, URL: https://www.math.ubbcluj.ro/~nodeacj/vol__4(2003)_no_1.htm.

[10] M. Kohli, R. Shrivastava and M. Sharma, Some results on fixed point theorems in dislocated quasi metric space, *International Journal of Theoretical & Applied Sciences* **2**(1) (2010), 27 – 28.

[11] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive types mappings, *Nonlinear Analysis: Theory, Methods & Applications* **75**(4) (2012), 2154 – 2165, DOI: 10.1016/J.NA.2011.10.014.

[12] P. Shahi, J. Kaur and S. S. Bhatia, Fixed point theorems for (ξ,α)-expansive mappings in complete metric spaces, *Fixed Point Theory and Applications* **2012** (2012), Article number: 157, DOI: 10.1186/1687-1812-2012-157.

[13] S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iseki, Some fixed point theorems on expansion mappings, *Mathematica Japonica* **29** (1984), 631 – 636.

[14] Z. M. Zeyada, G. H. Hassab and M. A. Ahmad, A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi metric space, *Arabian Journal for Science and Engineering* **31**(2005), 111 – 114, DOI: ref.

[15] K. Zoto, E. Houxha and A. Isufati, Some new results in dislocated and dislocated quasi-metric space, *Applied Mathematical Sciences* **6**(71) (2012), 3519 – 3526.