Communications in Mathematics and Applications
Vol. 14, No. 2, pp. 745–758, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i2.2043

http://www.rgnpublications.com

Research Article

On the Radio Antipodal Mean Number of Some Grid Related Graphs

T. Arputha Jose[1], R. Prabakaran*[2] and R. Gomathy[3]

[1] *Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Anna University, Chennai, India*
[2] *Department of Mathematics, St. Joseph's Institute of Technology, Anna University, Chennai, India*
[3] *Department of Education, Sri Muthukumaran College of Education, Tamiladu Teacher's Education University, Chennai, India*
***Corresponding author:** prabaperception@gmail.com

Received: September 5, 2022 **Accepted:** January 10, 2023

Abstract. The radio antipodal mean labeling of a graph G is a function f that assigns to each vertex u, a non-negative integer $f(u)$ such that $f(u) \neq f(v)$ if $d(u,v) < \operatorname{diam}(G)$ and $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq \operatorname{diam}(G)$, where $d(u,v)$ represents the shortest distance between any pair of vertices u and v of G and $\operatorname{diam}(G)$ denotes the diameter of G. The radio antipodal mean number of f, denoted by $r_{amn}(f)$ is the maximum number assigned to any vertex of G. The radio antipodal mean number of G, denoted by $r_{amn}(G)$ is the minimum value of $r_{amn}(f)$ taken over all antipodal mean labeling f of G. In this paper, the exact values of radio antipodal mean number of some grid related graphs have been obtained.

Keywords. Communication networks, Channel assignment problem, Radio labeling, Radio antipodal mean number, Triangular grid, Torus grid

Mathematics Subject Classification (2020). 05C12, 05C15, 05C78

1. Introduction

The graphs considered here are simple, finite, connected and undirected. For terms not defined here, one can refer to West [18]. The process of assigning labels (non-negative integers) to either vertices or edges or to both subject to certain condition is known as graph labeling (Gallian [5]). Many researchers around the world are working on different kind of graph labeling till date even though the first paper on graph labeling was presented by Rosa in 1966 [15]. The main reason behind is that it has a wide range of applications in various fields. To list a few, graph labeling is useful in coding theory, astronomy, circuit design,communication network addressing, secret sharing (Giridaran *et al*. [6]).

One of the major application of graph labeling is in telecommunication networks. An important problem in telecommunication network is *channel* (*frequency*) *assignment problem*, which can be addressed using graph labeling technique. The process of assigning channels (frequencies) efficiently to all radio transmitters is popularly known as the *channel* (*frequency*) *assignment problem* (Kang *et al*. [13]). The elementary challenge in designing mobile networks is also similar to channel (frequency) assignment problem, where set of radio frequency band has to be assigned to radio transmitters such that the interference is avoided (Havet [10]). The channel assignment problem has become an important problem since there is a speedy growth in the wireless communication services. Even though the radio spectrum bandwidth is costly, the insufficiency of the same is also high (Saha and Panigrahi [16]). Mathematically, the channel assignment problem can be viewed as an optimization problem where the radio spectrum bandwidth has to be minimized and the usage of it should be maximum. This problem can be viewed as a graph theoretical problem where the transmitters are considered as vertices and adjacent transmitters are connected by an edge (Jose and Giridaran [12]). Hale [9] formulated the channel assignment problem as a graph coloring problem in the year 1980. This was further developed by Griggs and Yeh [8]. This development led them in defining a new graph labeling technique called $L(2,1)$ labeling or *distance* 2 *labeling*. The $L(2,1)$ labeling (Griggs and Yeh [8]) was defined as follows. Given a real number $d>0$, an $L_d(2,1)$-labeling of G is a non-negative real-valued function $f:V(G)\to[0,\infty)$ such that, whenever x and y are two adjacent vertices in V, then $|f(x)-f(y)|\geq 2d$, and whenever the distance between x and y is 2, then $|f(x)-f(y)|\geq d$.

The generalized form of the $L(2,1)$ labeling is known as *radio labeling* which was introduced by Chartrand *et al*. [3] in the year 2001. The *radio labeling* of a graph G is an injection from the set of vertices of G to the set of natural numbers such that, $d(u,v)+|f(u)-f(v)|\geq \operatorname{diam}(G)+1$, where $d(u,v)$ represents the shortest distance between every distinct pair of vertices u and v of G. The *span* of a radio labeling f is $\max\{|f(u)-f(v)|: u,v\in V(G)\}$. The *radio number* of G is the minimum span of all radio labeling of G and it is denoted by $rn(G)$. It has been proved that the problem of finding the radio number of an arbitrary graph is NP complete (Kchikech *et al*. [14]).

Since there is a scarcity for the radio spectrum bandwidth and also because of its high cost, it is very important that the network operators have to maximize the usage of available radio spectrum efficiently. In order to achieve the efficient spectrum usage, the concept of frequency reuse is used (Janssen [11]). That is, in different locations of the communication network(at sufficiently large distance) the same frequency can be reused. If we reuse frequencies to the nearby radio transmitters, it may result in interference. Therefore, to avoid the interference, the difference between the channels assigned to the nearby radio stations must be sufficiently large (Vaidya and Vihol [17]).

Based on the concept of frequency reuse, Chartrand *et al*. [2] introduced a new graph labeling technique called *radio antipodal labeling* in the year 2002. The radio antipodal labeling of a graph G is a function $f:V(G)\to N$ such that $d(u,v)+|f(u)-f(v)|\geq \operatorname{diam}(G)$. The span obtained by radio antipodal labeling of a graph is less compared to radio labeling of a graph as the vertices at diametric distances are assigned the same label in antipodal labeling.

Based on the concept of radio antipodal labeling, Xavier and Thivyarathi [19] in the year 2018, introduced a new graph labeling technique called *radio antipodal mean labeling*. The radio antipodal mean labeling of graph G is a function f that assigns to each vertex u, a non-negative integer $f(u)$ such that $f(u) \neq f(v)$ if $d(u,v) < \text{diam}(G)$ and $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq \text{diam}(G)$. If $d(u,v) = \text{diam}(G)$, then $f(u) = f(v)$. The antipodal mean number of f, denoted by $r_{amn}(f)$ is the maximum number assigned to any vertex of G. The antipodal mean number of G, denoted by $r_{amn}(G)$ is the minimum value of $r_{amn}(f)$ taken over all antipodal mean labeling f of G. For our convenience, throughout this paper $r_{amn}(G)$ will be denoted as $ramn(G)$. Xavier and Thivyarathi [19] has obtained the upper bounds of mesh related networks.

In this paper, the radio antipodal mean number of triangular grid and torus grid graphs have been obtained.

2. Radio Antipodal Mean Number of Triangular Grid

We begin this section by defining T^∞ graph from which triangular grid graph is formed and have obtained the radio antipodal mean number of triangular grid graph.

The triangular grid graph is obtained from an infinite graph which denotes the arrangements of transmitters in a network. The transmitters in the triangular lattice is considered as vertices and adjacent transmitters are connected by an edge. It is assumed that such arrangement of transmitters in the network gives a good coverage (Havet [10]). Based on this pattern of arrangements of transmitters, a triangular grid graph has been defined as follows:

Definition 2.1 ([7])**.** The *infinite graph* T^∞ associated with the two dimensional triangular grid graph or triangular tiling graph is a graph drawn in the plane with straight line edges and vertices defined as follows.

A linear combination $xp + yq$ of two vectors $p = (1,0)$ and $q = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ with integers x and y represents the vertices of T^∞. Thus the vertices of T^∞ are points with Cartesian coordinates $\left(x + \frac{y}{2}, y\frac{\sqrt{3}}{2}\right)$. Two vertices of T^∞ are adjacent if and only if the Euclidean distance between them is equal to 1.

Definition 2.2. A *triangular grid graph* is a finite induced sub-graph of T^∞. The nth dimension of triangular grid graph is denoted by $TG(n)$. The construction of $TG(n)$ from T^∞ is shown in Figure 1.

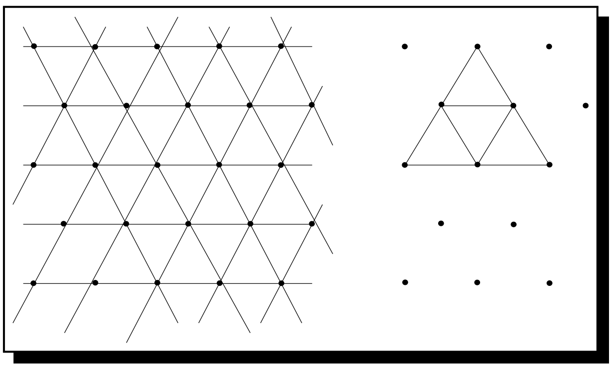

Figure 1. Construction of $TG(n)$ from T^∞

Remark 1. The vertices of triangular grid graph is arranged in such a way that the first row contains a single vertex, second row with 2 vertices and so on. In general, ith row of $TG(n)$, denoted by r_i will have i vertices.

Lemma 2.1. *Radio antipodal mean number of $TG(2)$ is 4.*

Proof. $TG(2)$ has 6 vertices, 9 edges and it's diameter is 2. Out of this 6 vertices, two pair of vertices are at diametric distance and hence they can receive same label.
Therefore, $f(v_1) = f(v_4) = f(v_6) = 1$.
Since the distance between any two vertices will be at least 1, the remaining 3 vertices can be labeled with the labels 2, 3 and 4. Hence, $ramn(TG(2)) \geq 4$.
From Figure 2, it is observed that $ramn(TG(2)) \leq 4$.
Therefore, $ramn(TG(2)) = 4$. □

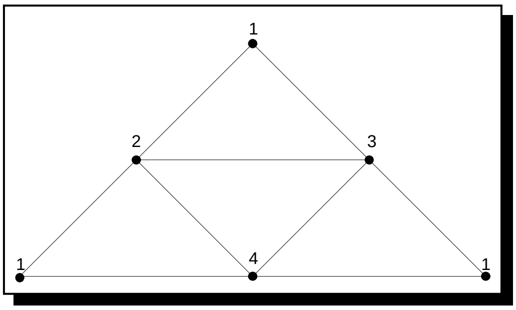

Figure 2. $TG(2)$

Lemma 2.2. *The radio antipodal mean number of $TG(3)$ is 8.*

Proof. It is easy to observe from Figure 3 that $ramn(TG(3)) \leq 8$.
The graph $TG(3)$ has 10 vertices out of which 2 pairs of vertices are at diametric distance. Therefore these vertices can be assigned same label. To label the remaining 7 vertices, we need at least 7 labels. Hence, all together we need at least 8 labels to label all the vertices of $TG(3)$. Therefore, $ramn(TG(3)) \geq 8$.
As a consequence, we get $ramn(TG(3)) = 8$. □

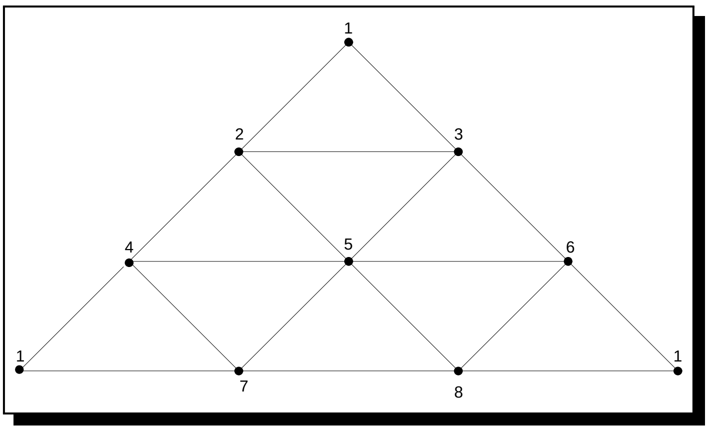

Figure 3. $TG(3)$

Corollary 2.1. *It is easy to verify that,*

(i) $\mathrm{ramn}(TG(4)) = 13$.

(ii) $\mathrm{ramn}(TG(5)) = 20$.

Lemma 2.3. *The radio antipodal mean number of triangular grid graph,* $\mathrm{ramn}(TG(n)) \geq n\left(\frac{n+5}{2}\right) - 5$, $n \geq 6$.

Proof. The graph $TG(n)$ has $\frac{1}{2}(n^2+3n+2)$ vertices and $\frac{3}{2}n(n+1)$ edges. In the vertex set of $TG(n)$, the vertices v_1, $v_{\left\lfloor \frac{n^2+n+3}{2} \right\rfloor}$, $v_{\frac{n^2+3n+2}{2}}$ are at diametric distance and hence they can be given the same label. The diameter of $TG(n)$ is n. Here, the vertex v_1 is labeled as $n-3$. If not, suppose if the vertex v_1 receives the label less than $n-3$ for instance let $f(v_1) = n-5$. Then by the definition of radio antipodal mean labeling, the label $n-4$ can be assigned to a vertex v_i which should be at a distance $n-2$ from v_1. The label $n-3$ can be then given to a vertex which is at a distance $n-3$ from the vertex v_i which is not possible as $d(v_i,v_j) < n-3$, $j = \left\lfloor \frac{n^2+n+3}{2} \right\rfloor$, $\frac{n^2+3n+2}{2}$. Hence, the vertex label starts from $n - \left\lfloor \frac{n}{2} \right\rfloor$ and all the labels from 1 to $n - \left\lfloor \frac{n}{2} \right\rfloor - 1$ are left out. Since, there are $\frac{1}{2}(n^2+3n+2)$ vertices to be labeled, we need at least $n\left(\frac{n+5}{2}\right) - 5$ labels to label all the vertices of $TG(n)$.

Therefore, $ramn(TG(n)) \geq n\left(\frac{n+5}{2}\right) - 5$, $n \geq 6$. □

Theorem 2.1. *The radio antipodal mean number of triangular grid graph,* $\mathrm{ramn}(TG(n)) \leq n\left(\frac{n+5}{2}\right) - 5$, $n \geq 6$.

Proof. Let the vertex set of $TG(n)$ be denoted as $v_1, v_2, \dots, v_{\frac{1}{2}(n^2+3n+2)}$. In this vertex set, there are 2 pairs of vertices which are at diametric distance and hence these vertices can be given same labels. Therefore, $f(v_1) = f\left(v_{\frac{n^2+3n+2}{2}}\right) = f\left(v_{\left\lfloor \frac{n^2+n+3}{2} \right\rfloor}\right) = n-3$.

Next, we label the vertices v_2, v_3, v_4 and v_5 as follows. The vertex v_2 is labeled as n, v_3 as $n+1$. Similarly, $f(v_4) = n-2$ and $f(v_5) = n-1$.

The remaining vertices of $TG(n)$ are labeled by the mapping:

$$f(v_i) = \begin{cases} n+i-4, & 6 \leq i \leq \frac{n(n+1)}{2}, \\ n+i-5, & \left\lceil \frac{n^2+n+3}{2} \right\rceil \leq i \leq \frac{1}{2}n(n+3). \end{cases} \tag{2.1}$$

Now we claim that the above mapping (2.1) is an valid radio antipodal mean labeling.

Let u,v be any two vertices of $TG(n)$. In order to prove the above claim, the following cases are considered.

Case 1. If the vertices u and v lies in the same row r_i, where $3 < i < n+1$.

Case 1.1. If the vertices u and v are adjacent in r_i.

By the mapping (2.1), $f(u_j) = n+j-4$ and $f(v_k) = n+k-4$.

Also, $d(u_j, v_k) = 1$.

Thus, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq n$.

Case 1.2. If the vertices are not adjacent in r_i.

As the vertices are not adjacent, $d(u,v) > 1$.

Using function (2.1), $f(u_j) = n+j-4$ and $f(v_k) = n+k-4$.

This implies, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq n$.

Case 2. If the vertices $u \in r_i$ and $v \in r_j$, $3 < i,\ j < n+1$, $i \neq j$.

Case 2.1. If r_i and r_j are neighbours.

In this case, $d(u_i,v_j) \geq 1$.

By (2.1), $f(u_i) = n+i-4$ and $f(v_j) = n+j-4$.

Hence, $d(u_i,v_j)+\left\lceil\frac{f(u_i)+f(v_j)}{2}\right\rceil \geq 1+\left\lceil\frac{2n+i+j-8}{2}\right\rceil > n$.

Case 2.2. If r_i and r_j are non-neighbours.

This case will be similar to the previous case except the fact that $d(u_i,v_j) > 1$.

Therefore, $d(u_i,v_j)+\left\lceil\frac{f(u_i)+f(v_j)}{2}\right\rceil \geq n$.

Case 3. If the vertices lies in the $(n+1)$th row.

Case 3.1. Let $u_i, v_j \in V_i$, $\left\lceil\frac{n^2+n+3}{2}\right\rceil \leq i \leq \frac{1}{2}n(n+3)$.
Here, $d(u,v) \geq 1$.
Also by the function (2.1), $f(u_i) = n+i-5$ and $f(v_j) = n+j-5$.
This guarantees that $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq n$.

Case 3.2. $u_i \in V_i$, $\left\lceil\frac{n^2+n+3}{2}\right\rceil \leq i \leq \frac{1}{2}n(n+3)$ and $v = v_{\left\lfloor\frac{n^2+n+3}{2}\right\rfloor}$.
In this case, the distance between the vertices u_i and v will be at least 1.
By function (2.1), $f(u_i) = n+i-5$ and $f(v) = n-1$.
Hence, in this case $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq n$.

Case 3.3. $u_i \in V_i$, $\left\lceil\frac{n^2+n+3}{2}\right\rceil \leq i \leq \frac{1}{2}n(n+3)$ and $v = v_{\frac{n^2+3n+2}{2}}$.

This case will be similar to the previous case and hence $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq n$ is assured.

Case 4. If the vertex $u \in r_1$ and $v \in r_{n+1}$.

Case 4.1. Suppose $u = u_1$ and $v \in V_i$, $\left\lceil\frac{n^2+n+3}{2}\right\rceil \leq i \leq \frac{1}{2}n(n+3)$.
In the considered case, the distance between the vertices u and v will be n which is the diameter of G.
From (2.1), $f(u_1) = n-3$ and $f(v_i) = n+i-5$.
Therefore, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq n$.

Case 4.2. If $u = u_1$ and $v = v_i$, $i = \left\lfloor\frac{n^2+n+3}{2}\right\rfloor, \frac{n^2+3n+2}{2}$.
The distance between the vertices u and v will be equal to diameter of G.
Also, $\left\lceil\frac{f(u)+f(v)}{2}\right\rceil = 1$.
Therefore, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq n$.

Case 5. Suppose $u = v_1$ and $v \in r_2$.
Here, $d(u,v) = 1$.

Also by the function (2.1), $f(u) = n-3$ and $f(v) \geq n$.
This assures that, $\lceil \frac{f(u)+f(v)}{2} \rceil \geq \lceil \frac{2n-3}{2} \rceil \geq n$.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq n$.

Case 6. Suppose $u,v \in r_2$.
In this case, $u = v_2$ and $v = v_3$. Since u and v are neighbours, $d(u,v) = 1$.
Also by the function (2.1), we have $f(u) = n$ and $f(v) = n+1$.
Therefore, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{2n+1}{2} \rceil \geq n$.

Case 7. Let $u,v \in r_3$.
In this case, the distance between the vertices u and v will be at least 1.
Also, $f(u) \geq n-2$ and $f(v) = n+2$.
This guarantees that, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq n$.

Case 8. If $u = v_1$ and $v \in r_3$.
Here, the distance between the vertices u and v will be at least 2.
By mapping (2.1), $f(v_1) = n-3$ and the label of $f(v_i)$, $i = 2,3$ will be at least n.
Therefore, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 2 + \lceil \frac{2n-5}{2} \rceil \geq n$.

Case 9. Let $u \in r_2$ and $v \in r_3$.
From the mapping (2.1), $f(u) \geq n$ and $f(v) = n-2$.
Thus, $\lceil \frac{n+n-2}{2} \rceil = n-1$.
Also, here the distance between u and v will be at least 1.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq n$.

Case 10. Let $u \in r_2$ and $v \in r_{n+1}$.
Here, $d(u,v) = n-1$.
By mapping (2.1), $f(u) \geq n$ and $f(v) = n-3$.
Therefore, $\lceil \frac{f(u)+f(v)}{2} \rceil \geq \lceil \frac{2n-3}{2} \rceil \geq n$.
This guarantees that, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq n$.

Case 11. Suppose $u \in r_3$ and $v \in r_{n+1}$.
Using mapping (2.1), $f(u) \geq n-2$ and $f(v) = n-3$.
Hence, $\lceil \frac{f(u)+f(v)}{2} \rceil \geq \lceil \frac{2n-5}{2} \rceil \geq n$.
In this case, $d(u,v) = n-2$.
Consequently, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq n$.

Case 12. If $u_i \in V_i$, $6 \leq i \leq \frac{n(n+1)}{2}$ and $v_j \in V_j$, $\lceil \frac{n^2+n+3}{2} \rceil \leq j \leq \frac{n(n+3)}{2}$.
In the case considered, $d(u,v) \geq 1$.
From function (2.1), $f(u_i) = n+i-4$ and $f(v_j) = n+j-5$.
Hence, $\lceil \frac{f(u_i)+f(v_j)}{2} \rceil = \lceil \frac{2n+i+j-9}{2} \rceil$.
This assures that $d(u,v) + \lceil \frac{f(u_i)+f(v_j)}{2} \rceil \geq n$.

Case 13. Let $u_i \in r_2$, $v_i \in r_j$, $4 \le j \le n$.
In this case, by the function (2.1) $f(u_i) \ge n$. Also, $f(v_j) = n + j - 4$.
Also, the distance between the vertices u_i and v_j will be at least 2.
Therefore, $d(u_i, v_j) + \lceil \frac{f(u_i)+f(v_j)}{2} \rceil \ge n$.

Hence, in all the above considered cases, the mapping (2.1) satisfies the radio antipodal mean labeling condition.
Therefore, the mapping (2.1) is an valid radio antipodal mean labeling.
By this mapping, the vertex $v_{\lfloor \frac{1}{2}(n^2+3n+1) \rfloor}$ receives the maximum label, which is given by, $n\left(\frac{n+5}{2}\right) - 5$.
Therefore, $ramn(TG(n)) \le n\left(\frac{n+5}{2}\right) - 5$, $n \ge 6$. □

Theorem 2.2. *For $n \ge 6$,* $\mathrm{ramn}(TG(n)) = n\left(\frac{n+5}{2}\right) - 5$.

Proof. The proof follows from Lemma 2.3 and Theorem 2.1. □

3. Radio Antipodal Mean Number of Torus Grid

We commence this section by providing the application of torus grid graph followed by the definition and then have obtained the radio antipodal mean number of the same.

In super computing, a torus network is now found every where due to it's efficient parallel processing by which it can maximize the computing performance. For the high-performance systems, the torus topology has been proven to be the popular interconnect (Bossard [1]). This motivated us to interpret channel assignment problem on torus grid network.

Definition 3.1 ([4])**.** Torus grid graph are obtained from the Cartesian product of two cycles $C_m \times C_n$. Here we restrict ourself to consider only the case $m = n$, that is $C_n \times C_n$. The nth dimension of torus grid graph is denoted by $T(n \times n)$. $T(n \times n)$ has n^2 vertices and $2n^2$ edges. $T(3 \times 3)$ has been shown in Figure 4.

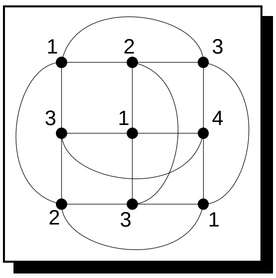

Figure 4. Labeled $T(3 \times 3)$

Lemma 3.1. *The radio antipodal mean number of Torus grid graph of order* 3 *is* 4.

Proof. $TG(3 \times 3)$ has 9 vertices and 18 edges. The diameter of $T(3 \times 3)$ is 2. The vertices v_1, v_5 and v_9 are at diametric distance and hence they can receive same label. Let $f(v_i) = 1$, $i = 1, 5, 9$. Similarly, the vertices v_2 and v_7 are at diametric distance and hence they can receive same label

which is given by, $f(v_i) = 2$, $i = 2,7$. The vertices v_3, v_4 and v_8 are also at diametric distance and hence they are labeled as 3. The remaining vertex v_6 is labeled as 4. Hence, $ramn(T(3\times 3)) \geq 4$. From Figure 4, it can be observed that $ramn(T(3\times 3)) \leq 4$.
Therefore, $ramn(T(3\times 3)) = 4$. □

Corollary 3.1. *It is easy to check that* $\mathrm{ramn}(T(4\times 4)) = 8$.

Lemma 3.2. *The radio antipodal mean number of torus grid graph,* $\mathrm{ramn}(T(n\times n)) \geq n^2+n-6$, $n>4$.

Proof. The graph $T(n\times n)$ has n^2 vertices and $2n^2$ edges. In the vertex set of $T(n\times n)$, the vertices v_1 and $v_{\frac{n(n+1)}{2}+1}$ are at diametric distance and hence they can be given the same label. The diameter of $T(n\times n)$ is $2\lfloor \frac{n}{2}\rfloor$. If the vertex v_1 is assigned the label $n-5$, then by the definition of radio antipodal mean labeling, the label $n-4$ can be assigned to a vertex which is at distance $n-2$ from v_1. This is not possible since the vertex which is at distance $n-2$ from v_1 has distance less than $n-2$ from $v_{\frac{n(n+1)}{2}+1}$ and hence it is not possible to assign the label $n-4$ to any vertex. Therefore, the first $n-5$ labels are excluded and from $n-4$ the vertices of $T(n\times n)$ are labeled. Since there are n^2 vertices, we need at least n^2+n-6 labels to label all the vertices of $T(n\times n)$. Hence, $ramn(T(n\times n)) \geq n^2+n-6$, $n>4$. □

Theorem 3.1. *The radio antipodal mean number of torus grid graph,* $\mathrm{ramn}(T(n\times n)) \leq n^2+n-6$, $n\equiv 0 \pmod 2$, $n\geq 6$.

Proof. Let the vertex set of $T(n\times n)$ be $\{v_1, v_2, \dots, v_{n^2-1}, v_{n^2}\}$. In this vertex set, the vertices v_1 and $v_{\frac{n(n+1)}{2}+1}$ are at diametric distance and hence these two vertices can be given same label. The remaining vertices of $T(n\times n)$ are labeled by the mapping,

$$f(v_i) = \begin{cases} n+i-3, & 1\leq i\leq 2n-1, \\ n-3, & i=2n, \\ n+i-4, & 2n+1\leq i\leq \frac{n(n+1)}{2}, \\ n+i-5, & \frac{n(n+1)+4}{2}\leq i\leq n^2-2, \\ n-4, & i=n^2-1, \\ n+i-6, & i=n^2. \end{cases} \tag{3.1}$$

Claim. The mapping (3.1) is an valid radio antipodal mean labeling.

Let u,v be any two vertices of $T(n\times n)$. The following cases are considered in-order to prove the above claim.

Case 1. Let $u,v\in v_i, 1\leq i\leq 2n-1$

Case 1.1. If u and v are adjacent.
Here, $f(u_i) = n+i-3$ and $f(v_j) = n+j-3$.
Also, $d(u_i, v_j) = 1$.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2}\rceil \geq 1 + \lceil \frac{2n+i+j-6}{2}\rceil \geq n$.
This assures, $d(u,v) + \lceil \frac{f(u)+f(v)}{2}\rceil \geq \mathrm{diam}(G)$.

Case 1.2. If u and v are non adjacent.
In this case, the distance between the vertices u_i and v_j will be at least 2.
By mapping (3.1), $f(u_i)=n+i-3$ and $f(v_j)=n+j-3$.
Thus, $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq 2+\lceil\frac{2n+i+j-6}{2}\rceil \geq \text{diam}(G)$.

Case 2. Suppose, $u,v\in v_i$, $2n+1\leq i\leq \frac{n(n+1)}{2}$.

Case 2.1. If the vertices u and v are adjacent.
In this case, $d(u,v)=1$.
From (3.1), $f(u_i)=n+i-4$ and $f(v_j)=n+j-4$.
This guarantees that $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq \text{diam}(G)$.

Case 2.2. Suppose the vertices u and v are not adjacent.
In the case considered, by the function (3.1), we have $f(u_i)=n+i-4$ and $f(v_j)=n+j-4$.
Also, $d(u,v)>1$.
Thus, $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq \text{diam}(G)$.

Case 3. If $u,v\in v_i$, $\frac{n(n+1)+4}{2}\leq i\leq n^2-2$.

Case 3.1. Let u and v be neighbours.
By mapping (3.1), $f(u_i)=n+i-5$ and $f(v_j)=n+j-5$.
It is obvious that, $d(u,v)=1$.
Hence, $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq 1+\lceil\frac{2n+i+j-10}{2}\rceil \geq \text{diam}(G)$.

Case 3.2. If u and v are not neighbours.
Then, $d(u,v)>1$.
Also, by (3.1), $f(u_i)=n+i-5$ and $f(v_j)=n+j-5$.

Therefore, $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq 2+\lceil\frac{2n+i+j-10}{2}\rceil > \text{diam}(G)$.

Case 4. Let $u\in v_i$, $1\leq i\leq 2n-1$ and $v=v_{2n}$.

Case 4.1. If $d(u,v)=1$.
Here, $u=u_i$, $i=n$, $n+1$, $2n-1$ and $v=v_{2n}$.
By (3.1), $f(u_i)=n+i-3$, $f(v_{2n})=n-3$.
Hence, $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq \text{diam}(G)$.

Case 4.2. Suppose $d(u,v)>1$.
In the case considered,$u=u_i$, for $i\neq n,n+1,2n-1$.
By the mapping (3.1), $f(u_i)=n+i-3$, $f(v_{2n})=n-3$.
This guarantees that $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq \text{diam}(G)$.

Case 5. $u\in v_i$, $2n+1\leq i\leq \frac{n(n+1)}{2}, v=v_{2n}$.
The distance between the vertices u and v will be at least 1.
By (3.1), $f(u_i)=n+i-4$ and $f(v_{2n})=n-3$.
Therefore, $d(u,v)+\lceil\frac{f(u)+f(v)}{2}\rceil \geq \text{diam}(G)$.

Case 6. $u \in v_i$, $\frac{n(n+1)+4}{2} \leq i \leq n^2-2$ and $v = v_{2n}$.
By (3.1), $f(u_i) = n+i-5$, $f(v_{2n}) = n-3$.
Also, $d(u,v) \geq 1$.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{2n+i-8}{2} \rceil \geq n$.

Case 7. $u \in v_i$, $2n+1 \leq i \leq \frac{n(n+1)}{2}$, $v \in v_i$, $\frac{n(n+1)+4}{2} \leq i \leq n^2-2$.
In this case, $d(u,v) \geq 1$.
By the function (3.1), $f(u_i) = n+i-4$, $f(v_j) = n+j-5$.
Thus, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{2n+i+j-9}{2} \rceil > \text{diam}(G)$.

Case 8. $u = u_{n^2-1}$ and $v = v_{2n}$.
The distance between the vertices u and v in this case will be more than 1, i.e., $d(u,v) > 1$.
From the function (3.1), $f(u_{n^2-1}) = n-4$, $f(v_{2n}) = n-3$.
Therefore, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{2n-7}{2} \rceil \geq \text{diam}(G)$.

Case 9. If $u = v_{n^2}$ and $v = v_{2n}$.
By the function (3.1), $f(u_{n^2}) = n^2+n-6$, $f(v_{2n}) = n-3$.
The distance between the vertices u and v will be 2.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 2 + \lceil \frac{n^2+2n-9}{2} \rceil > \text{diam}(G)$.

Case 10. $u \in v_i$, $1 \leq i \leq 2n-1$ and $v = v_{n^2-1}$.
From the mapping (3.1), $f(u_i) = n+i-3$ and $f(v_{n^2-1}) = n-4$.
Also, $d(u,v) > 1$ in this case.
Thus, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 2 + \lceil \frac{2n+i-7}{2} \rceil \geq \text{diam}(G)$.

Case 11. $u \in v_i$, $\frac{n(n+1)+4}{2} \leq i \leq n^2-2$ and $v = v_{n^2-1}$.
In this case, the distance between the vertices u and v will be at least 1.
Also by mapping (3.1), $f(u_i) = n+i-5$ and $f(v_{n^2-1}) = n-4$.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{2n+i-9}{2} \rceil \geq \text{diam}(G)$.

Case 12. $u \in v_i$, $\frac{n(n+1)+4}{2} \leq i \leq n^2-2$ and $v = n^2$.
By the function (3.1), $f(u_i) = n+i-5$ and $f(v_{n^2}) = n^2+n-6$.
Also, $d(u,v) \geq 2$.
Hence, $d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{n^2+2n+i-11}{2} \rceil > \text{diam}(G)$.

Case 13. $u \in v_i$, $2n+1 \leq i \leq \frac{n(n+1)}{2}$, $v = v_{n^2-1}$.
The distance between the vertices u and v will be at least 1.
By (3.1), $f(u_i) = n+i-4$ and $f(v_{n^2-1}) = n-4$,
$d(u,v) + \lceil \frac{f(u)+f(v)}{2} \rceil \geq 1 + \lceil \frac{2n+i-8}{2} \rceil \geq \text{diam}(G)$.

Case 14. $u \in v_i$, $2n+1 \leq i \leq \frac{n(n+1)}{2}$, $v = v_{n^2}$.

Here, $d(u,v) > 1$.

By the function (3.1), $f(u_i) = n+i-4$ and $f(v_{n^2}) = n^2+n-6$.

$d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq 2+\left\lceil\frac{n^2+2n+i-10}{2}\right\rceil > \operatorname{diam}(G)$.

Case 15. $u=n^2-1$ and $v=n^2$.

The distance between the vertices u and v will be 1.

From (3.1), $f(u_{n^2-1})=n-4$ and $f(v_{n^2})=n^2+n-6$.

Thus, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq 1+\left\lceil\frac{n^2+2n-10}{2}\right\rceil > \operatorname{diam}(G)$.

Case 16. $u\in v_i$, $1\leq i\leq 2n-1$ and $v\in v_j$, $2n+1\leq j\leq \frac{n(n+1)}{2}$.

Here by the function (3.1), $f(u_i)=n+i-3$ and $f(v_j)=n+j-4$.

Also in this case, the distance between the vertices u and v will be at least 1.

Therefore, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq 1+\left\lceil\frac{2n+i+j-7}{2}\right\rceil \geq \operatorname{diam}(G)$.

Case 17. $u\in v_i$, $1\leq i\leq 2n-1$ and $v\in v_j$, $\frac{n(n+1)+4}{2}\leq j\leq n^2-2$.

From (3.1), $f(u_i)=n+i-3$ and $f(v_j)=n+j-5$.

In this case, the distance between the vertices u and v will be at least 1.

Therefore, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq 1+\left\lceil\frac{2n+i+j-8}{2}\right\rceil \geq \operatorname{diam}(G)$.

Thus from all the cases considered above, it is evident that $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq \operatorname{diam}(G)$ for any pair of vertices $u,v\in T(n\times n)$.

Therefore, the mapping (3.1) is an valid radio antipodal mean labeling.

By the mapping (3.1), the vertex v_{n^2} receives the maximum label which is given by, n^2+n-6.

Therefore, $ramn(T(n\times n))\leq n^2+n-6$, $n\equiv 0(\operatorname{mod} 2)$, $n\geq 6$. □

Theorem 3.2. *The radio antipodal mean number of torus grid graph,* $\operatorname{ramn}(T(n\times n))\leq n^2+n-6$, $n\equiv 1(\operatorname{mod} 2)$*,* $n\geq 5$.

Proof. Let $\{v_1,v_2,\dots,v_{n^2-1},v_{n^2}\}$ be the vertex set of $T(n\times n)$, $n\equiv 1(\operatorname{mod} 2)$. Here the vertices v_1 and $v_{\frac{n(n+1)}{2}-2}$ are at diametric distance and hence they can be given same label. Therefore, $f(v_1)=f\left(v_{\frac{n(n+1)}{2}-2}\right)$. The vertices of $T(n\times n)$ are labeled by the function,

$$f(v_i)=\begin{cases} n+i-4, & 1\leq i\leq 2n-1,\\ n-4, & i=2n,\\ n+i-5, & 2n+1\leq i\leq \frac{n(n+1)}{2}-3,\\ n+i-6, & \frac{n(n+1)}{2}-1\leq i\leq n^2. \end{cases} \tag{3.2}$$

Claim. The mapping (3.2) is an valid radio antipodal mean labeling.

In order to prove this claim, we have to show that for any two vertices of $T(n\times n)$, the radio antipodal mean labeling condition, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq \operatorname{diam}(G)$ is preserved.

The cases will be similar to the previous proof and hence it is left to the reader.

Hence, in all the cases considered it can be seen that, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq \operatorname{diam}(G)$. Thus the mapping (3.2) is an valid radio antipodal mean labeling.

By the function (3.2), the vertex v_{n^2} receives the maximum label given by n^2+n-6.

Therefore, $ramn(T(n\times n))\leq n^2+n-6$, $n\equiv 1(\operatorname{mod} 2)$, $n\geq 5$ □

Theorem 3.3. $\text{ramn}(T(n \times n)) = n^2 + n - 6$, $n \equiv 1 (\text{mod}\, 2)$, $n \geq 5$.

Proof. The proof can be obtained directly from Lemma 3.2 and Theorems 3.1 and 3.2. □

4. Conclusion

In this paper, the radio antipodal mean number of triangular grid and torus grid graph have been obtained. This work can be further extended to other communication networks like butterfly and benes which are under investigation.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Bossard, On solving the decycling problem in a torus network, *Wireless Communications and Mobile Computing* **2021** (2021), Article ID 5598173, 6 pages, DOI: 10.1155/2021/5598173.

[2] G. Chartrand, D. Erwin and P. Zhang, Radio antipodal colorings of graphs, *Mathematica Bohemica* **127** (2002), 57 – 69, DOI: 10.21136/MB.2002.133978.

[3] G. Chartrand, D. Erwin, P. Zhang and F. Harary, Radio labelings of graphs, *Bulletin of the Institute of Combinatorics and its Applications* **33** (2001), 77 – 85.

[4] S. N. Daoud, Edge odd graceful labeling of cylinder and torus grid graphs, *IEEE Access* **7** (2019), 10568 – 10592, DOI: 10.1109/ACCESS.2018.2889293.

[5] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics* **DS6**(Version 25) (2022), 623 pages, URL: https://www.combinatorics.org/files/Surveys/ds6/ds6v25-2022.pdf.

[6] M. Giridaran, T. A. Jose and E. A. Jeony, On the radio antipodal geometric mean number of ladder related graphs, *Malaya Journal of Matematik* **10**(1) (2022), 98 – 109, DOI: 10.26637/mjm1001/009.

[7] V. S. Gordon, Y. L. Orlovich and F. Werner, Hamiltonian properties of triangular grid graphs, *Discrete Mathematics* **308**(24) (2008), 6166 – 6188, DOI: 10.1016/j.disc.2007.11.040.

[8] J. R. Griggs and R. K. Yeh, Labelling graphs with a condition at distance 2, *SIAM Journal on Discrete Mathematics* **5**(4) (1992), 586 – 595, DOI: 10.1137/0405048.

[9] W. K. Hale, Frequency assignment: Theory and applications, *Proceedings of the IEEE* **68**(12) (1980), 1497 – 1514, DOI: 10.1109/PROC.1980.11899.

[10] F. Havet, Channel assignment and multicolouring of the induced subgraphs of the triangular lattice, *Discrete Mathematics* **233**(1-3) (2001), 219 – 231, DOI: 10.1016/S0012-365X(00)00241-7.

[11] J. C. M. Janssen, Channel assignment and graph labeling, Chapter 5, in: *Handbook of Wireless Networks and Mobile Computing*, I. Stojmenović (editor), John Wiley & Sons, Inc., pp. 95 – 117 (2002), DOI: 10.1002/0471224561.

[12] T. A. Jose and M. Giridaran, Radial radio mean labeling of Mongolian tent and diamond graphs, *International Journal for Research in Applied Science & Engineering Technology* **8**(7) (2020), 2078 –– 2083, DOI: 10.22214/ijraset.2020.30759.

[13] S. M. Kang, S. Nazeer, W. Nazeer, I. Kousar and C. Y. Jung, Radio labeling and radio number of caterpillar related graphs, *Mitteilungen Klosterneuburg* **65**(5) (2015), 149 – 159.

[14] M. Kchikech, R. Khennoufa and O. Togni, Linear and cyclic radio k-labelings of trees, *Discussiones Mathematicae Graph Theory* **27**(1) (2007), 105 – 123, DOI: 10.7151/dmgt.1348.

[15] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs* (International Symposium, Rome, July 1966), Dunod Gordon & Breach Science Publishers, Inc., New York and Dunod, Paris, pp. 349 – 355 (1967).

[16] L. Saha and P. Panigrahi, A new graph radio k-coloring algorithm, *Discrete Mathematics, Algorithms and Applications* **11**(1) (2019), 1950005, DOI: 10.1142/S1793830919500058.

[17] S. K. Vaidya and P. L. Vihol, Radio labeling for some cycle related graphs, *International Journal of Mathematics and Soft Computing* **2**(2) (2012), 11 – 24.

[18] D. B. West, *Introduction to Graph Theory*, 2nd edition, Pearson Education, Inc., xix + 512 pages (2001).

[19] D. A. Xavier and R. C. Thivyarathi, Radio antipodal mean number of certain graphs, *International Journal of Mathematics Trends and Technology* **54**(6) (2018), 467 – 470.