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Research Article

Detour Pebbling Number on Some Commutative Ring Graphs

A. Lourdusamy[1], S. Kither Iammal*[2] and I. Dhivviyanandam[2]

[1] Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai 627002, Tamil Nadu, India
[2] PG and Research Department of Mathematics, St. Xavier's College (Autonomous), Manonmaniam Sundaranar University, Abisekapatti 627012, Tamilnadu, India
***Corresponding author:** cathsat86@gmail.com

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Abstract. The detour pebbling number of a graph G is the least positive integer $f^*(G)$ such that these pebbles are placed on the vertices of G, we can move a pebble to a target vertex by a sequence of pebbling moves each move taking two pebbles off a vertex and placing one of the pebbles on an adjacent vertex using detour path. In this paper, we compute the detour pebbling number for the commutative ring of zero-divisor graphs, sum and the product of zero divisor graphs.

Keywords. Pebbling number, Detour pebbling number, Zero-divisor, Sum and the product of zero-divisor graph

Mathematics Subject Classification (2020). 05E15, 05C12, 05C25, 05C38, 05C76

1. Introduction

R. A. Beeler *et al*. [2] stated that Lagarias and Saks suggested the concept of graph pebbling to solve a number theoretic conjecture. Then, Chung [5] introduced graph pebbling into the literature. The researchers can get details of graph pebbling by reading the paper "Survey on graph pebbling" by Hurlbert [6].

The detour pebbling was introduced by Lourdusamy *et al*. [7] using detour path in any connected graph and they determined the detour pebbling number for complete graphs, path graphs, wheel graphs, star graphs, middle graph of path and square of some graphs [3, 8].

Detour pebbling number guarantees the reachable of a pebble even though if there are any blocks in the movement of supply.

Throughout the paper, G stands for a simple connected graph. Let us now explain the detour pebbling number of a vertex v in a graph G. It is the least positive integer $f^*(G,v)$ with the following property: With every possible configuration of $f^*(G,v)$ pebbles there is a possibility to move a pebble to v by a sequence of pebbling moves using detour path where pebbling move is defined as removal of two pebbles from a vertex throwing one pebble away and placing another pebble on the adjacent vertex.

In this paper, we discuss the detour pebbling concept for some zero-divisor graphs, sum and product of zero divisor graphs. In Section 2, we have given preliminaries which are used for the subsequent sections. In Section 3, we find the detour pebbling number for some zero-divisor graphs. In Section 4, we find the detour pebbling number for sum of zero-divisor graphs. In Section 5, we find the detour pebbling number for the product of two zero-divisor graphs.

2. Preliminaries

For graph theoretic terminologies, the reader can refer to [7].

Definition 2.1. In [1], the definition of the zero-divisor graph of a ring R is given as follows: The zero-divisor graph of a ring R is a simple graph whose set of vertices consists of all (non-zero) zero-divisors, with an edge defined between x and y if and only if $xy = 0$. It will be denoted by $\Gamma(Z)$.

Note that $2,3,4$ in Z_6 are zero-divisors. For the element 2 in Z_6 we use y_2, for the element 3 in Z_6 we use y_3 and for the element 4 in Z_6 we use y_4. In general, for the element i in Z_n we use y_i.

Definition 2.2. In [3], we find the definition of sum of two graphs as follows: Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be the simple connected graphs. Then $G_1 \cup G_2$ is the graph $G(V,E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ and $G_1 + G_2$ is $G_1 \cup G_2$ together with the edges joining elements of V_1 to elements of V_2.

Definition 2.3. In [8], the detour pebbling number of a vertex is defined as follows: The detour pebbling number of a vertex v in a graph G is the smallest number $f^*(G,v)$ such that for any placement of $f^*(G,v)$ pebbles on the vertices of G it is possible to move a pebble to v using a detour path by a sequence of pebbling moves. The detour pebbling number of a graph is denoted by $f^*(G)$, is the maximum $f^*(G,v)$ over all the vertices of G.

Definition 2.4. In [4], we find the definition of product of two graphs as follows: If $G = (V_G,E_G)$ and $H = (V_H,E_H)$ are two graphs, the direct product of G and H is the graph, $G \times H$, whose vertex set is the Cartesian product $V(G \times H) = V_G \times V_H = \{(x,y) : x \in V_G,\, y \in V_H\}$ and whose edge set is given by $E_{G\times H} = \{\{(x,y),(x',y')\} : x = x' \text{ and } (y,y') \in E_H \text{ or } (x,x') \in E_G \text{ and } y = y'\}$.

Theorem 2.1 ([7]). *For any path P_n with n vertices, the detour pebbling number is $f^*(P_n) = 2^{n-1}$.*

Theorem 2.2 ([7]). *Let $K_{1,n}$ be an n-star where $n > 1$. The detour pebbling number for the n-star graph is $f^*(K_{1,n}) = n+2$.*

Note 2.1. Let $p(v)$ denotes the number of pebbles at the vertex v and $v \in V(\Gamma(Z_n))$.

3. Detour Pebbling Number for Zero-Divisor Graphs

In this section, we compute the detour pebbling number of zero-divisor graphs

Theorem 3.1. *For $\Gamma(Z_6)$, $f^*(\Gamma(Z_6)) = 4$.*

Proof. Let $V(\Gamma(Z_6))$ be $\{y_2, y_3, y_4\}$ and $E(\Gamma(Z_6))$ be $\{(y_2, y_3), (y_3, y_4)\}$. Since $\Gamma(Z_6) \cong K_{1,2}$, by Theorem 2.2 the result follows. □

Theorem 3.2. *For $\Gamma(Z_8)$, $f^*(\Gamma(Z_8)) = 4$.*

Proof. Let $V(\Gamma(Z_8)) = \{y_2, y_4, y_6\}$ and $E(\Gamma(Z_8)) = \{(y_2, y_4), (y_4, y_6)\}$. Since $\Gamma(Z_8) \cong K_{1,2}$, then by Theorem 2.2 the result follows. □

Theorem 3.3. *For $\Gamma(Z_9)$, $f^*(\Gamma(Z_9)) = 2$.*

Proof. Let $V(\Gamma(Z_9))$ be $\{y_3, y_6\}$ and $E(\Gamma(Z_9))$ be $\{(y_3, y_6)\}$. This is isomorphic to P_2. Hence, by Theorem 2.1 we are done. □

Theorem 3.4. *For $\Gamma(Z_{10})$, $f^*(\Gamma(Z_{10})) = 6$.*

Proof. Let $V(\Gamma(Z_{10}))$ be $\{y_2, y_4, y_5, y_6, y_8\}$ and $E(\Gamma(Z_{10}))$ be $\{(y_2, y_5), (y_4, y_5), (y_6, y_5), (y_8, y_5)\}$. Since $\Gamma(Z_{10}) \cong K_{1,4}$, by Theorem 2.2 $f^*(\Gamma(Z_{10})) = 6$. □

Theorem 3.5. *For $\Gamma(Z_{12})$, $f^*(\Gamma(Z_{12})) = 33$.*

Proof. Let $V(\Gamma(Z_{12})) = \{y_2, y_3, y_4, y_6, y_8, y_9, y_{10}\}$, $E(\Gamma(Z_{12})) = \{(y_2, y_6), (y_6, y_8), (y_6, y_4), (y_6, y_{10}), (y_8, y_9), (y_4, y_9), (y_4, y_3), (y_8, y_3)\}$. Place one pebble on y_2 and 31 pebbles on y_9. Then, we cannot move a pebble to y_{10} using the detour path. Hence, $f^*(\Gamma(Z_{12})) \geq 33$.

Let us consider the distribution of 33 pebbles on $\Gamma(Z_{12})$.

Case 1: Let the target be y_2.
The detour distance from the vertex y_2 to any other vertex is $d^*(y_2, y_i) \leq 5$ where $i = \{3, 4, 6, 8, 9, 10\}$. Without loss of generality, let us consider the detour path $P_1 : y_2, y_6, y_4, y_3, y_8, y_9$. Path P_1 covers all the vertices except y_{10}. By Theorem 2.1, if we distribute 32 pebbles on path P_1, we are able to pebble the target. If $p(y_10) = 0$, then by placing 32 pebbles on P_1 we are done. If $p(y_{10}) = 1$ then by placing 32 pebbles on P_1 we are done. if $31 \leq p(y_{10}) \leq 2$, then by placing $33 - p(y_{10})$ pebbles on P_1 we are done. If $p(y_{10}) \geq 32$ then by Theorem 2.1, we pebble the target.

Similarly, we can prove for the vertices y_2, y_3, y_9 and y_{10}.

Case 2: Let the target vertex be y_6.
The detour distance from y_6 to any other vertex is $d^*(y_6, y_j) \leq 4$ where $j = \{2,3,4,8,9,10\}$. Without loss of generality, let us consider the detour path $P_2 : y_6, y_4, y_3, y_8, y_9$. By Theorem 2.1, using 16 pebbles on P_2 we can reach the target. If $y_2 = 1$ and $y_{10} = 1$ then we need to place 16 pebbles on P_2 to reach the target. If $y_2 \geq 2$ and $y_{10} \geq 2$ then directly we are done.

Case 3: Let y_4 to be the target.
The detour distance from y_4 to any other vertex is $d^*(y_4, y_l) \leq 4$ where $l = \{2,3,6,8,9,10\}$. Let us consider the detour path $P_3 : y_4, y_3, y_8, y_6, y_2$ or $P_4 : y_4, y_3, y_8, y_6, y_{10}$. Without loss of generality, let us consider the detour path P_3. This path does not contain 2 vertices of $\Gamma(Z_{12})$. By using *Case* 2 we are done. Similarly, we can prove for the vertex y_8.
Hence, the detour pebbling number of $\Gamma(Z_{12})$ is $f^*(\Gamma(Z_{12})) = 33$. □

Theorem 3.6. *For* $\Gamma(Z_{14})$*,* $f^*(\Gamma(Z_{14})) = 8$.

Proof. The $V(\Gamma(Z_{14}))$ be $\{y_2, y_4, y_6, y_7, y_8, y_{10}, y_{12}\}$ and $E(\Gamma(Z_{14}))$ be $\{(y_2, y_7), (y_4, y_7), (y_6, y_7), (y_8, y_7), (y_{10}, y_7), (y_{12}, y_7)\}$. Since $\Gamma(Z_{14}) \cong K_{1,6}$, then by Theorem 2.2, $f^*(\Gamma(Z_{14})) = 8$. □

Theorem 3.7. *For* $\Gamma(Z_{15})$*,* $f^*(\Gamma(Z_{15})) = 16$.

Proof. Let $V(\Gamma(Z_{15}))$ be $\{y_3, y_5, y_6, y_9, y_{10}, y_{12}\}$ and $V(\Gamma(Z_{15}))$ be $\{(y_3, y_5), (y_6, y_5), (y_6, y_{10}), (y_9, y_5), (y_{12}, y_5), (y_{10}, y_3), (y_{10}, y_9), (y_{10}, y_{12})\}$. Let y_3 be the target vertex. The detour path of $\Gamma(Z_{15})$ is $P : y_3, y_5, y_9, y_{10}, y_{12}$. If we place 15 pebbles on y_{12}, we cannot reach the target. Hence, $f^*(\Gamma(Z_{15})) \geq 16$.

Case 1: Let us assume the target is y_9.
The detour distance from y_9 to any other vertex is $d^*(y_9, y_i) \leq 4$ where $i = \{10,12,5,3,6\}$. Let the detour path be $P = y_3, y_5, y_6, y_{10}, y_9$. Let $p(y_{12}) = 0$ then by Theorem 2.1 we are done by using 16 pebbles. If $p(y_{12}) = 1$, then by placing 14 or 15 pebbles on P we are done. If $p(y_{12}) \geq 2$, then by placing $16 - p(y_12)$ on P and we are done. By symmetry, we can prove for y_3, y_{12}, y_6.

Case 2: Let the target vertex be y_5.
The detour distance from y_5 to any other vertex is $d^*(y_5, y_j) \leq 3$ where $j = \{3,9,6,10,12\}$. Without loss of generality, let us consider the path $P_1 : y_5, y_3, y_{10}, y_{12}$. Let $X = \{y_9, y_6\}$ be the vertex set which is not on P_1. If $p(X) = 0$, then to pebble the target $p(P_1) = 8$ is sufficient. If $1 \leq p(X) \leq 2$, then by using $4 \leq p(P_1) \leq 6$ and we are done. If $p(X) \geq 3$, then with $2 \leq p(P_1) \leq 3$ and we are done. By symmetry, we can prove for y_{10}.
Hence, the detour pebbling number of $f^*(\Gamma(Z_{15})) = 16$. □

Theorem 3.8. *For* $\Gamma(Z_{16})$*,* $f^*(\Gamma(Z_{16})) = 11$.

Proof. Let $V(\Gamma(Z_{16})) = \{y_2, y_4, y_6, y_8, y_{10}, y_{12}, y_{14}\}$ and $E(\Gamma(Z_{16})) = \{(y_8, y_2), (y_8, y_4), (y_8, y_6), (y_8, y_{10}), (y_8, y_{12}), (y_8, y_{14}), (y_4, y_{12})\}$. Let us distribute 10 pebbles on the graph $\Gamma(Z_{16})$. If we place 7 pebbles on y_2 and 1 pebble each on the vertices y_6, y_{10} and y_{14}, then we cannot reach the vertex y_{12} by using the detour path. Hence, $f^*(\Gamma(Z_{16})) \geq 11$. Now we prove the sufficient part.

Case 1: Let y_4 to be the target.
The detour distance from y_4 to any other vertex is $d^*(y_4, y_j) \leq 3$, where $j = \{2, 6, 10, 12, 14\}$. Without loss of generality, let us consider the path $P : y_4, y_{12}, y_8, y_2$. The detour path P does not contain 3 vertices of $V(\Gamma(Z_{16}))$. Therefore, by placing one pebble each on those vertices and distributing 8 pebbles on the detour path P, we are done. By symmetry, we can prove for y_{12}.

Case 2: Let y_8 to be the target.
The detour distance from y_8 to any other vertex is $d^*(y_8, y_k) \leq 2$, where $k = \{2, 4, 6, 10, 12, 14\}$. Let us consider the path $P_1 : y_8, y_4, y_{12}$. This particular path does not contain the rest of the vertices of $\Gamma(Z_{16})$. Let us consider $p(< y_2, y_6, y_{10}, y_{14} >) \leq 1$. Distributing 4 pebbles on the detour path P_1 and placing 0 pebbles on the remaining vertices, we can reach the target. If we place one pebble each on the uncovered vertices of the detour path P_1 and 4 pebbles on the path P_1, we are done.

Case 3: Let y_2 to be the target vertex.
The detour distance from y_2 to any other vertex is $d^*(y_2, y_k) \leq 3$, where $k = \{8, 4, 6, 10, 12, 14\}$. Let us consider the detour path $P_2 : y_2, y_8, y_4, y_{12}$ which does not contain the vertices $\{x_6, x_{10}, x_{14}\}$ of $\Gamma(Z_{16})$. By Theorem 2.1, Distributing 8 pebbles on P_2 we are done. If $p(y_6, y_{10}, y_{14}) \leq 3$, then placing 8 pebbles on P_2 we are done. Similarly, we can prove for the vertices y_6, y_{10} and y_{14}.
Therefore, the detour pebbling number of $f^*(\Gamma(Z_{16})) = 11$. □

Theorem 3.9. *For* $\Gamma(Z_{18})$*,* $f^*(\Gamma(Z_{18})) = 37$.

Proof. Let $V(\Gamma(Z_{18}))$ be $\{y_2, y_3, y_4, y_6, y_8, y_{10}, y_{12}, y_{14}, y_{16}, y_9, y_{15}\}$ and $E(\Gamma(Z_{18}))$ be $\{(y_9, y_i), (y_6, y_j), (y_{12}, y_{15}), (y_{12}, y_3)\}$ where $i = 2, 4, 6, 8, 10, 12, 14, 16$ and $j = 3, 12, 15$. To prove the necessary part, let us consider the target vertex to be y_3. Without loss of generality, consider the detour path $P : y_2, y_9, y_{12}, y_{15}, y_6, y_3$. If we place 31 pebbles on y_2 and one pebble each on y_4, y_8, y_{10}, y_{14} and y_{16}, then we cannot reach the target. Hence, $f^*(\Gamma(Z_{18})) \geq 37$.

For the sufficient part, let us consider the following cases.

Case 1: Let y_3 to be the target.
The detour distance from y_3 to any other vertex is ≤ 5. Consider the same detour path P as defined in necessary part. By Theorem 2.1, if we distribute 32 pebbles on P, then we can reach the target. If we place one pebble each on $y_i : i = 4, 8, 10, 14, 16$ and 32 pebbles on P we can reach the target. Let $A = \{4, 8, 10.14.16\}$. If $\sum_{i \in A} \lfloor \frac{p(y_i)}{2} \rfloor + \lfloor \frac{p(y_2)}{2} \rfloor + p(y_9) \geq 16$, then we can reach the target. Otherwise, if $\sum_{i \in A} \lfloor \frac{p(y_i)}{2} \rfloor + \lfloor \frac{p(y_2)}{2} \rfloor + p(y_9) \leq 15$, then there will be

$37-\left[\sum_{i\in A}p(y_i)+p(y_2)+p(y_9)\right]$ pebbles on P excluding y_2 and y_9. In this configuration, we can easily reach target. Similarly, we can prove for all the vertices of the graph except for y_6, y_9 and y_{12}.

Case 2: Let y_9 to be the target.
The length of the detour path from y_9 to any other vertex is ≤ 4. Consider the detour path $P_1 : y_9, y_{12}, y_{15}, y_6, y_3$. By Theorem 2.1, if we distribute 16 pebbles on P_1, then we can reach the target. If we place one pebble on each vertex y_k where $k = \{2,4,8,10,14,16\}$ and distributing 16 pebbles on P_1, then we are done. If $p(y_k)\ge 2$, then we are done. Similarly we can prove for y_6 and y_{12}.
Therefore, the detour pebbling number of $\Gamma(Z_{18})$ is $f^*(\Gamma(Z_{18})) = 37$. □

4. Detour Pebbling Number for the Union of Two Zero-Divisor Graphs

In this section, we are going to find the detour pebbling number for the union of any two zero-divisor graphs.

Theorem 4.1. *For* $\Gamma(Z_6)+\Gamma(Z_4)$*,* $f^*(\Gamma(Z_6)+\Gamma(Z_4)) = 8$.

Proof. Let $V(\Gamma(Z_6))$ be $\{y_2, y_3, y_4\}$. The graph $\Gamma(Z_6)$ is isomorphic to Z_6. Let the vertex set of $\Gamma(Z_4)$ ia a singleton set and denoted as x_1. Let the edge set of $(\Gamma(Z_6)+\Gamma(Z_4))$ be $\{(y_2,y_3),(y_3,y_4),(y_i,x_1)\}$ where $i = 2,3,4$. Let the target vertex be x_1. The detour distance from x_1 to any other vertex is $d^*(x_1,y_i)\le 3$. Let us consider the path $P : x_1, y_4, y_3, y_2$. Since it contains all the vertices of the graph $(\Gamma(Z_6)+\Gamma(Z_4))$, then by Theorem 2.1, the detour pebbling number of $f^*(\Gamma(Z_6)+\Gamma(Z_4)) = 8$.

Theorem 4.2. *For* $\Gamma(Z_{10})+\Gamma(Z_4)$*,* $f^*(\Gamma(Z_{10})+\Gamma(Z_4)) = 16$.

Proof. Let $V(\Gamma(Z_{10}))$ be $\{y_2, y_4, y_6, y_8, y_5\}$. The graph $\Gamma(Z_{10})$ is isomorphic to Z_{10}. The vertex set of $\Gamma(Z_4)$ is a singleton set $\{x_1\}$. The edge set of $(\Gamma(Z_{10})+\Gamma(Z_4))$ is $\{(y_5,y_j),(y_i,x_1)\}$ where $i = 2,4,5,6,8$ and $j = 2,4,6,8$. Let the target vertex be y_2. The detour distance from y_2 to any other vertex of $\Gamma(Z_{10})+\Gamma(Z_4)$ is ≤ 4. Let us choose the detour path $P = y_8, x_1, y_6, y_5, y_2$. If we place 15 pebbles on y_8, then we fail to reach the target. Thus, $f^*(\Gamma(Z_{10})+\Gamma(Z_4))\ge 16$. Now, let us prove $f^*(\Gamma(Z_{10})+\Gamma(Z_4))\le 16$.

Case 1: Let y_2 to be the target.
If $p(y_4) = 0$, then by Theorem 2.1 we can pebble the target by using 16 pebbles. If $p(y_4)\ge 1$, then using the remaining pebbles on P we can reach the target by using the detour path through y_4. By symmetry we can prove for y_4, y_6, y_8.

Case 2: Let x_1 to be the target.
The detour distance from x_1 to any other vertex is $d^*(x_1,y_i)\le 3$. Let us consider the path $P_1 : y_2, y_5, y_4, x_1$. Let Q be the set of vertices which are not on P_1. By Theorem 2.1, if we

distribute 8 pebbles on P_1, then we are done. If $1 \le p(Q) \le 7$, then using $8 - p(Q)$ pebbles on P_1 we can reach the target by using the detour path through Q. If $p(Q) \ge 8$ by Theorem 2.1 we are done. By symmetry we can prove for y_5.
Hence, the detour pebbling number of $f^*(\Gamma(Z_{10}) + \Gamma(Z_4)) = 16$. □

Theorem 4.3. *Let s be any prime number. Then for $\Gamma(Z_{2s}) + \Gamma(Z_4)$, $f^*(\Gamma(Z_{2s}) + \Gamma(Z_4)) = s + 9$ where $s \ge 7$.*

Proof. Let $V(\Gamma(Z_{2s})+\Gamma(Z_4))$ be $\{y_2, y_4, \dots, y_{2s-2}, y_s, x_1\}$ and the edge set be $\{y_i y_s, x_1 y_i, x_1 y_s\}$ where $i = 2,4,\dots,2s-2$. The graph $\Gamma(Z_{2s})+\Gamma(Z_4)$ is isomorphic to $K_{1,n} \times \{x_1\}$. Let the target vertex be y_2. The detour distance from y_2 to any other vertex is ≤ 4. Without loss of generality, let us consider the path $P : y_2, y_s, y_4, x_1, y_6$. This particular path P contains 5 vertices of $\Gamma(Z_{2s}) + \Gamma(Z_4)$. Let W be the set of vertices which are not on the detour path P. Note that $|W| = s - 4$. Suppose, we distribute 11 pebbles on y_6 and one pebble each on the vertices of $\Gamma(Z_{2s})+\Gamma(Z_4)$ except x_1, y_s and the target. In this configuration, we cannot reach the target. Hence, $f^*(\Gamma(Z_{2s}) + \Gamma(Z_4)) \ge s + 9$. To prove $f^*(\Gamma(Z_{2s}) + \Gamma(Z_4)) \le s + 9$, let us consider the distribution of $s + 9$ pebbles on the graph.

Case 1: Let x_1 to be the target.
The detour distance from x_1 to any other vertex is ≤ 3. Choose the detour path $P_1 = \{x_1, y_2, y_s, y_4\}$. P_1 covers 4 vertices of the graph. Let Q be the set vertices which are not on P_1. Note that $|Q| = s - 3$. With $1 \le p(Q) \le s - 1$ and $(s + 9) - p(Q)$ pebbles on the detour path P_1 we can reach the target by using an alternate detour path different from P_1. If $p(Q) \ge s$, then we can find an alternate detour path different from P_1 to reach the target. Similarly, we can prove for the vertex y_s.

Case 2: Let the target vertex be y_4.
The length of the detour path from y_4 to any other vertex is ≤ 4. Consider a detour path P_2 be $\{y_4, x_1, y_2, y_s, y_6\}$. By Theorem 2.1 we can reach the target using 16 pebbles on P_2. Let W be the set of vertices which are not on P_2. Clearly $W = s - 4$. If we place one pebble each on the vertices of W and $p(s + 9) - p(W)$ pebbles on P_2, then we can transfer a pebble to y_4 with an alternating the detour path through one of the vertices of W. If $1 \le p(W) \le s - 2$ and $(s + 9) - p(Q)$ pebbles on the detour path P_2 we can reach the target by using an alternate detour path different from P_2. If $p(Q) \ge s$, then we can find an alternate detour path different from P_2 to reach the target. Similarly, we can prove for y_i where $i = \{2, 6, 8, \dots, 2s - 2\}$. Hence, the detour pebbling number of $\Gamma(Z_{2s}) + \Gamma(Z_4)$ is $f^*(\Gamma(Z_{2s}) + \Gamma(Z_4)) = s + 9$. □

Corollary 4.1. *Let s be any prime number. Then for $\Gamma(Z_{2s}) + \Gamma(Z_s)$, $f^*(\Gamma(Z_{2s}) + \Gamma(Z_s)) \cong f^*(\Gamma(Z_{2s}) + \Gamma(Z_4)) = s + 9$, where $s \ge 7$.*

Theorem 4.4. *For $\Gamma(Z_{2s}) + \Gamma(Z_{2s})$, $f^*(\Gamma(Z_{2s}) + \Gamma(Z_{2s})) = 2^{2s-1}$, where s is any prime number.*

Proof. Let $\Gamma(Z_{2s})$ and $\Gamma(Z_{2s})$ are the two copies of zero-divisor graph $\Gamma(Z_{2s})$. Let $V(\Gamma(Z_{2s})+\Gamma(Z_{2s})) = \{v_2,v_4,\dots,v_{2s-2},v_s,u_2,u_4,\dots,u_{2s-2},u_s\}$ and $E(\Gamma(Z_{2s})+\Gamma(Z_{2s})) = \{v_iv_s,u_ju_s,v_iu_j\}$ where $i,j=2,4,\dots,2s-2$. The length of the detour path of the graph $\Gamma(Z_{2s})+\Gamma(Z_{2s})$ is $2s-1$ for any vertex. The detour path covers all the vertices of the graph. Thus, by Theorem 2.1, the detour pebbling number of $\Gamma(Z_{2s})+\Gamma(Z_{2s})$ is $f^*(\Gamma(Z_{2s})+\Gamma(Z_{2s})) = 2^{2s-1}$.

Result 4.1. *For* $\Gamma(Z_s)+\Gamma(Z_s)$, $f^*(\Gamma(Z_s)+\Gamma(Z_s)) = 2$.

5. Detour Pebbling Number for the Product of Two Zero-Divisor Graphs

In this section, we find the detour pebbling number for the product of two zero-divisor graphs.

We now give the following trivial results on the detour pebbling number for the product of two zero-divisor graphs:

Result 5.1. (i) *For* $\Gamma(Z_{2s})\times\Gamma(Z_s)$, $f^*(\Gamma(Z_{2s})\times\Gamma(Z_s)) = s+1$.

(ii) *For* $\Gamma(Z_s)\times\Gamma(Z_s)$, $f^*(\Gamma(Z_s)\times\Gamma(Z_s)) = 1$.

Theorem 5.1. *For* $\Gamma(Z_{2s})\times\Gamma(Z_{2s})$, $f^*(\Gamma(Z_{2s})\times\Gamma(Z_{2s})) = 2^{4s-4}$, *where s be any prime number.*

Proof. Let us consider two copies of zero-divisor graphs $\Gamma(Z_{2s})$. Let the vertex set of the first copy be $\{v_2,v_4,\dots,v_{2s-2},v_s\}$ and that of the second copy be $\{u_2,u_4,\dots,u_{2s-2},u_s\}$. The total number of vertices in $\Gamma(Z_{2s})\times\Gamma(Z_{2s})$ is s^2. The detour distance from (v_s,u_s) to any other vertex of the given graph is $\le 4s-4$. Let us choose the path $P = \{(v_s,u_s),(v_s,u_{2s-j}),(v_s,u_{s-i}),(v_{s-i},u_s),(v_{2s-j},u_s),(v_l,u_l),(v_l,u_{l+2}),(v_{(2s-2)},u_{(2s-4)})\}$ where $i=1,3,\dots,s-2$, $j=2,4,\dots,s-1$ and $l=2,4,\dots,2s-2$. Note that the detour path P has $(4s-3))$ vertices. The number of vertices on $\Gamma(Z_{2s})\times\Gamma(Z_{2s})$ which are not on P is $(s^2-(4s-3))$ and let Q be the set of those $(s^2-(4s-3))$ vertices. If we place $2^{4s-4}-1$ pebbles on the vertex (v_{2s-2},u_{2s-2}), we cannot reach the target (v_s,u_s). Hence, $f^*(\Gamma(Z_{2s})\times\Gamma(Z_{2s})) \ge 2^{4s-4}$.
Now let us prove the sufficient condition.

Case 1: Let the target be (v_s,u_s).
The detour distance from (v_s,u_s) to any other vertex is $\le 4s-4$. Consider the same detour path of P as in necessary part. By Theorem 2.1, distributing 2^{4s-4} pebbles on P we can reach the target. If $p(Q)\ge 1$ and there are $2^{4s-4}-p(Q)$ pebble on P, then we can reach the target by having an alternate detour path passing through at least any one of the vertices of Q which has a pebble on it. By symmetry, we can prove for (v_i,u_j) where $i,j=2,4,\dots,2s-2$.

Case 2: Let the target be (v_s,u_j), $j=2,4,\dots,2s-2$.
Without loss of generality, let it be (v_s,u_4). The length of the detour path from (v_s,u_4) to any other vertex is $\le 4s-5$. Choose a detour path $P_1 : \{(v_s,u_s),(v_s,u_{2s-j}),(v_s,u_{s-i}),(v_{s-i},u_s),(v_{2s-j},u_s),(v_l,u_l),(v_l,u_{l+2})\}$ where $i=1,3,\dots,s-2$, $j=2,4,\dots,s-1$ and $l=2,4,\dots,2s-2$. Note that W be set of vertices which are not on P_1. Let $|W| = (s^2-(4s-4))$. By Theorem 2.1, distributing 2^{4s-5}

pebbles on P_1 we can reach the target. If $1 \leq p(W) \leq 2^{4s-5}$ and there are $2^{4s-5} - p(W)$ pebbles on the vertices of P_1, then we can reach the target by travelling through another detour path having at least a vertex of W which has a pebble on it. By symmetry, we can prove for (v_i, u_s) where $i = 2, 4, \ldots, 2s-2$.

Thus, the detour pebbling number of $\Gamma(Z_{2s}) \times \Gamma(Z_{2s})$ is $f^*(\Gamma(Z_{2s}) \times \Gamma(Z_{2s})) = 2^{4s-4}$. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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