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Research Article

Vertex *k*-Prime Labeling of Cyclic Snakes

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Abstract. For each positive integer k, a simple graph G of order p is said to be k-prime labeling if there exists an injective function f whose labels are from k to $k+p-1$ that induces a function $f^+: E(G) \to N$ of the edges of G defined by $f^+(uv) = \gcd(f(u), f(v))$, $\forall\ e = uv \in E(G)$ such that every pair of neighbouring vertices are relatively prime. This type of graph is known as a k-prime graph. In this paper, we redefine the labeling as vertex k-prime labeling for some k positive integers and study some cyclic snake graphs and corona graphs of the form $mC_n \odot K_1$ which admit vertex k-prime labeling.

Keywords. Vertex k-prime labeling, Triangular snakes, Pentagonal snakes, Cyclic snakes, Corona graphs

Mathematics Subject Classification (2020). 05C12, 05C78, 05C90

1. Introduction

A graph labeling consists of assigning integers to its vertices, edges, or both depending on certain constraints. To know more about the vast study of different graph labeling, one can refer to Gallian [4]. A prime labeling is when the integers from 1 to n are assigned to the vertices, edges, or both, with the condition that each labeled pair of adjacent vertices is comparatively prime. The k-prime labeling concept was proposed by Vaidya and Prajapathi [5] where they proved that all path graph P_m is k-prime for every k positive integers. We investigated the results on tree related graphs such as Y-tree, X-tree and extend to one point union of path graphs and proved that they admit k-prime labeling [1]. The term cyclic snakes was introduced by Barrientos [2] and showed that the kC_4-snakes, kC_8-snakes and kC_{12}-snakes are graceful.

This paper exhibits that mC_n-snake for $m > 1$ and $n \geq 3$ is vertex k-prime. We also define a generalised mC_n-snake and prove that for even positive integer n, mC_n is vertex k-prime. Further we prove that Corona graph $mC_n \odot K_1$ is also vertex k-prime.

2. Preliminaries

To begin with, we revise the k-prime labeling concept proposed by Vaidya and Prajapati [5], and redefine the labeling as vertex k-prime labeling.

Definition 2.1. A vertex k-prime labeling of a graph G is a bijective function $f : V \to \{k, k+1, k+2, \dots, k+|V|-1\}$ for some positive integer k such that $\gcd(f(u), f(v)) = 1$ $\forall\ e = uv \in E(G)$. A graph G that admits vertex k-prime labeling is called a vertex k-prime graph.

Definition 2.2 ([2])**.** A mC_n-snake is a m-block connected graph, each of the blocks is isomorphic to the cycle C_n, such that the path is created by the block cut point graph. We call mC_n-snake as a cyclic snake.

Note. For $n = 5$, mC_5-snake is called as Pentagonal snake and for $n = 9$, mC_9-snake is called as nanogonal snake.

Definition 2.3 ([3])**.** The *corona* $G_1 \odot G_2$ *of two graphs* G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 and p_1-copies of G_2 and then joining by a line the ith vertex of G_1 to every vertex in the ith copy of G_2.

3. Main Results

Theorem 3.1. *Triangular Snake T_n is vertex k-prime for $n > 1$ and odd k.*

Proof. We represent the point set and line set of T_n as

$$V(T_n) = \{y_z : 1 \leq z \leq n\} \cup \{x_z : 1 \leq z \leq n-1\};$$
$$E(T_n) = \{y_z y_{z+1}, x_z y_z, x_z y_{z+1} : 1 \leq z \leq n-1\}.$$

Hence the number of points in T_n is $2n-1$ and the number of lines in T_n is $3n-3$.
Now define a function f from points of $T(n)$ to $k, k+1, \dots, k+2n-2$ as given below:

$$f(y_1) = k,$$
$$f(x_1) = k+1,$$
$$f(y_z) = f(y_{z-1}) + 2, \quad 2 \leq z \leq n,$$
$$f(x_z) = f(x_{z-1}) + 2, \quad 2 \leq z \leq n-1.$$

For any $y_z y_{z+1} \in E(T_n)$, $\gcd(f(y_z), f(y_{z+1}) = \gcd(f(y_{z-1}) + 2,\ f(y_z) + 2) = 1$ since $f(y_{z-1}) + 2$ and $f(y_z) + 2$ are consecutive odd positive integers. For any $x_z y_z \in E(T_n)) = \gcd(f(x_z), f(y_z)) = \gcd(f(x_{z-1}) + 2, f(y_{z-1}) + 2) = 1$ since $f(x_{z-1}) + 2$ and $f(y_{z-1}) + 2$ are consecutive positive integers. For any $x_z y_{z+1} \in E(T_n) = \gcd(f(x_z), f(y_{z+1})) = \gcd(f(x_{z-1}) + 2, f(y_z) + 2) = 1$ since $f(x_{z-1}) + 2$ and $f(y_z) + 2$ are consecutive positive integers.
Hence T_n admits vertex k-prime labeling. □

Theorem 3.2. *Pentagonal Snake mC_5 is vertex k-prime for $m \geq 2$ and odd k.*

Proof. Let m blocks of C_5 form a pentagonal snake mC_5. We represent the point set and line set of mC_5 as

$$V(mC_5) = \{x_{a,b} : 1 \leq b \leq 5, 1 \leq a \leq m\},$$

$$E(mC_5) = \begin{cases} \{x_{a,b}x_{a,b+1} : 1 \leq b \leq 4,\ 1 \leq a \leq m\}, \\ \{x_{a,5}x_{a,1} : 1 \leq a \leq m\}. \end{cases}$$

We refer the vertex $x_{a+1,1}$ as $x_{a,5}$ for $1 \leq a \leq m-1$ to facilitate defining of lines. The number of points and lines for mC_5 is $|V(mC_5)| = 4m+1$ and $|E(mC_5)| = 5m$.
Now define a function f from points of mC_5 to $k, k+1, \ldots, k+4m$ as given below:

$$f(x_{a,b}) = k + 4a + b - 5, \qquad 1 \leq b \leq 5,\ 1 \leq a \leq m,$$
$$f(x_{a,5}) = f(x_{a+1,1}) = k + 4a, \quad 1 \leq a \leq m.$$

For any $x_{a,b}x_{a,b+1} \in E(mC_5)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+4a+b-5, k+4a+b-4) = 1$ since $k+4a+b-5$ and $k+4a+b-4$ are consecutive positive integers. For any $x_{a,5}x_{a,1} \in E(mC_5)$, $\gcd(f(x_{a,5}), f(x_{a,1})) = \gcd(k+4a, k+4a-4) = 1$ since k is odd.
Hence mC_5 admits vertex k-prime labeling. □

Theorem 3.3. *Nanogonal snake mC_9 is vertex k-prime for $m \geq 2$ and odd k.*

Proof. Let m blocks of C_9 form a nanogonal snake mC_9. We represent the point set and line set of mC_9 as

$$V(mC_9) = \{x_{a,b} : 1 \leq b \leq 9, 1 \leq a \leq m\}$$

$$E(mC_9) = \begin{cases} \{x_{a,b}x_{a,b+1} : 1 \leq b \leq 8,\ 1 \leq a \leq m\}, \\ \{x_{a,9}x_{a,1} : 1 \leq a \leq m\}. \end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,9}$ for $1 \leq a \leq m-1$ to facilitate defining of lines. The number of points and lines for mC_9 is $|V(mC_9)| = 8m+1$ and $|E(mC_9)| = 9m$.
Now define a function f from points of mC_9 to $k, k+1, \ldots, k+8m$ as given below:

$$f(x_{a,b}) = k + 8a + b - 9, \qquad 1 \leq b \leq 9,\ 1 \leq a \leq m,$$
$$f(x_{a,9}) = f(x_{a+1,1}) = k + 8a, \quad 1 \leq a \leq m.$$

For any $x_{a,b}x_{a,b+1} \in E(mC_9)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+8a+b-9, k+8a+b-8) = 1$ since $k+8a+b-9$ and $k+8a+b-8$ are consecutive positive integers. For any $x_{a,9}x_{a,1} \in E(mC_9)$, $\gcd(f(x_{a,9}), f(x_{a,1})) = \gcd(k+8a, k+8a-8) = 1$ since k is odd.
Hence mC_9 admits vertex k-prime labeling. □

Theorem 3.4. *mC_n is vertex k-prime for even positive integer $n \geq 4$ and $k \not\equiv 0(\bmod(n-1))$, where $n-1$ is prime.*

Proof. Let m blocks of C_n form a cyclic snake mC_n. We represent the point set and line set of mC_n as

$$V(mC_n) = \{x_{a,b} : 1 \leq b \leq n,\ 1 \leq a \leq m\},$$

$$E(mC_n)=\begin{cases}\{x_{a,b}x_{a,b+1}:1\le b\le n-1,1\le a\le m\},\\ \{x_{a,n}x_{a,1}:1\le a\le m\}.\end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,n}$ for $1\le a\le m-1$ to facilitate defining of lines. The number of points and lines for mC_n is $|V(mC_n)|=(n-1)m+1$ and $|E(mC_n)|=nm$ (see Figure 1).

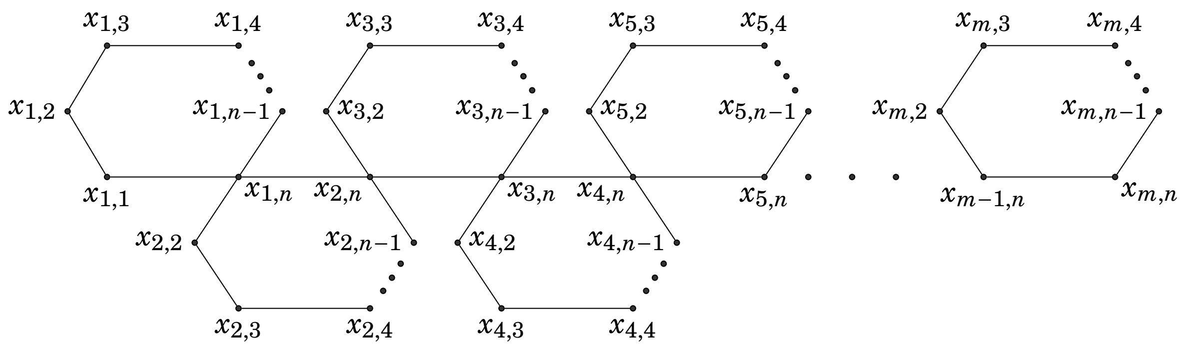

Figure 1. mC_n-snake

Now define a function f from points of mC_n to $k,k+1,\dots,k+(n-1)m$ as given below:

$$f(x_{a,b})=k+(n-1)a+b-n,\qquad 1\le b\le n,\ 1\le a\le m,$$
$$f(x_{a+1,1})=f(x_{a,n})=k+(n-1)a,\quad 1\le a\le m.$$

From the labeling pattern defined, mC_n satisfies the conditions of vertex k-prime labeling as $f(x_{a,b}x_{a,b+1})=\gcd(f(x_{a,b}),f(x_{a,b+1}))=\gcd(k+(n-1)a+b-n,k+(n-1)a+b-n+1)=1$ since $k+(n-1)a+b-n$ and $k+(n-1)a+b-n+1$ are consecutive positive integers; $f(x_{a,n}x_{a,1})=\gcd(f(x_{a,n}),f(x_{a,1}))=\gcd(k+(n-1)a,k+(n-1)a+1-n)=1$ since $k\not\equiv 0(\mathrm{mod}(n-1))$, where $n-1$ is prime.

Hence mC_n admits vertex k-prime labeling. □

Theorem 3.5. *mC_n is vertex k-prime for even positive integer $n\ge 10$ and k not multiple factor of $(n-1)$, where $n-1$ is not prime.*

Proof. Let m blocks of C_n form a cyclic snake mC_n. We represent the point set and line set of mC_n as

$$V(mC_n)=\{x_{a,b}:1\le b\le n,1\le a\le m\},$$
$$E(mC_n)=\begin{cases}\{x_{a,b}x_{a,b+1}:1\le b\le n-1,1\le a\le m\},\\ \{x_{a,n}x_{a,1}:1\le a\le m\}.\end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,n}$ for $1\le a\le m-1$ to facilitate defining of lines. The number of points and lines for mC_n is $|V(mC_n)|=(n-1)m+1$ and $|E(mC_n)|=nm$.

Now define a function f from points of mC_n to $k,k+1,\dots,k+(n-1)m$ as given below:

$$f(x_{a,b})=k+(n-1)a+b-n,\qquad 1\le b\le n,1\le a\le m,$$
$$f(x_{a,n})=f(x_{a+1,1}),\qquad 1\le a\le m.$$

From the labeling pattern defined, mC_n satisfies the conditions of vertex k-prime labeling as $f(x_{a,b}x_{a,b+1})=\gcd(f(x_{a,b}),f(x_{a,b+1}))=\gcd(k+(n-1)a+b-n,k+(n-1)a+b-n+1)=1$ since $k+(n-1)a+b-n$ and $k+(n-1)a+b-n+1$ are consecutive positive integers; $f(x_{a,n}x_{a,1})=$

$\gcd(f(x_{a,n}), f(x_{a,1})) = \gcd(k+(n-1)a, k+(n-1)a+1-n) = 1$ since k not multiple factor of $(n-1)$, where $n-1$ is not prime.
Hence mC_n admits vertex k-prime labeling when $n-1$ is not prime.
An illustration is given in Figure 2.

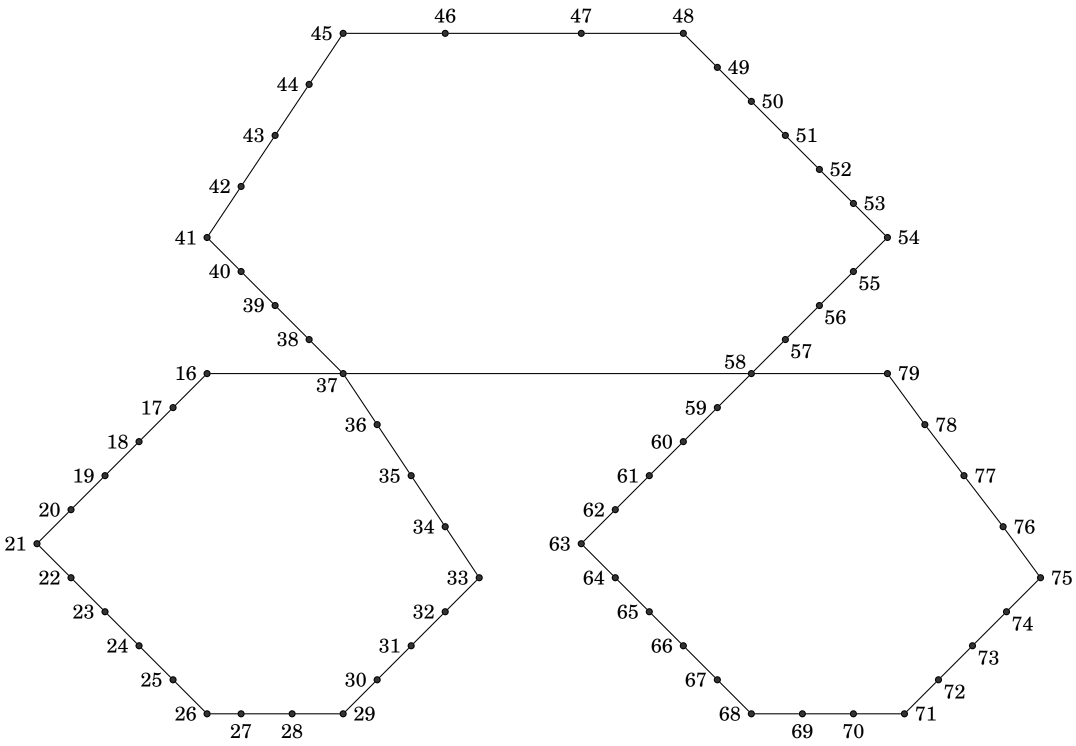

Figure 2. Vertex k-prime labeling of $3C_{22}$ for $k = 16$ □

Theorem 3.6. *The corona $T_n \odot K_1$ is vertex k-prime for $n > 1$.*

Proof. We represent the point set and line set of $T_n \odot K_1$ as

$$V(T_n \odot K_1) = \{u_a : 1 \le a \le n-1\} \cup \{v_a : 1 \le a \le n\} \cup \{x_a : 1 \le a \le n-1\} \cup \{y_a : 1 \le a \le n\},$$

$$E(T_n \odot K_1) = \begin{cases} \{v_a v_{a+1}, v_a u_a, u_a v_{a+1} : 1 \le a \le n-1\}, \\ \{u_a x_a : 1 \le a \le n-1\}, \\ \{v_a y_a : 1 \le a \le n\}. \end{cases}$$

The number of points and lines of $T_n \odot K_1$ is $|V(T_n \odot K_1)| = 4n-2$ and $|E(T_n \odot K_1)| = 5n-4$.

Case 1. k is odd

Now define a function f from points of $T_n \odot K_1$ to $k, k+1, \dots, k+4n-3$ as given below:

$$f(v_1) = k$$
$$f(v_a) = k+4a-4, \quad 2 \le a \le n,$$
$$f(u_a) = k+4a-2, \quad 1 \le a \le n-1,$$
$$f(x_a) = k+4a-1, \quad 1 \le a \le n-1,$$
$$f(y_a) = k+4a-3, \quad 1 \le a \le n.$$

For any $v_av_{a+1} \in E(T_n \odot K_1)$, $\gcd(f(v_a), f(v_{a+1})) = \gcd(k+4a-4, k+4a) = 1$ since k is odd. For any $v_au_a \in E(T_n \odot K_1)$, $\gcd(f(v_a), f(u_a)) = \gcd(k+4a-4, k+4a-2) = 1$ since $k+4a-4$ and $k+4a-2$ are consecutive odd positive integers. For any $u_av_{a+1} \in E(T_n \odot K_1)$, $\gcd(f(u_a), f(v_{a+1})) = \gcd(k+4a-2, k+4a) = 1$ since $k+4a-2$ and $k+4a$ are consecutive odd positive integers. For any $u_ax_a \in E(T_n \odot K_1)$, $\gcd(f(u_a), f(x_a)) = \gcd(k+4a-2, k+4a-1) = 1$ since $k+4a-2$ and $k+4a$ are consecutive positive integers. For any $v_ay_a \in E(T_n \odot K_1)$, $\gcd(f(v_a), f(y_a)) = \gcd(k+4a-4, k+4a-3) = 1$ since $k+4a-4$ and $k+4a-3$ are consecutive positive integers.

Case 2. k is even

Now define a function f from points of $T_n \odot K_1$ to $k, k+1, \dots, k+4n-3$ as given below:

$$\begin{aligned} &f(y_1) = k, \\ &f(y_a) = k+4a-4, \quad 2 \le a \le n, \\ &f(v_a) = k+4a-3, \quad 1 \le a \le n, \\ &f(u_a) = k+4a-1, \quad 1 \le a \le n-1, \\ &f(x_a) = k+4a-2, \quad 1 \le a \le n-1. \end{aligned}$$

For any $v_av_{a+1} \in E(T_n \odot K_1)$, $\gcd(f(v_a), f(v_{a+1})) = \gcd(k+4a-3, k+4a+1) = 1$ since $k+1$ is odd. For any $v_au_a \in E(T_n \odot K_1)$, $\gcd(f(v_a), f(u_a)) = \gcd(k+4a-3, k+4a-1) = 1$ since $k+4a-3$ and $k+4a-1$ are consecutive odd positive integers. For any $u_av_{a+1} \in E(T_n \odot K_1)$, $\gcd(f(u_a), f(v_{a+1})) = \gcd(k+4a-1, k+4a+1) = 1$ since $k+4a-1$ and $k+4a+1$ are consecutive odd positive integers. For any $u_ax_a \in E(T_n \odot K_1)$, $\gcd(f(u_a), f(x_a)) = \gcd(k+4a-1, k+4a-2) = 1$ since $k+4a-1$ and $k+4a-2$ are consecutive positive integers. For any $v_ay_a \in E(T_n \odot K_1)$, $\gcd(f(v_a), f(y_a)) = \gcd(k+4a-3, k+4a-4) = 1$ since $k+4a-3$ and $k+4a-4$ are consecutive positive integers.

Hence $T_n \odot K_1$ admits vertex k-prime labeling. □

Theorem 3.7. *The corona $mC_5 \odot K_1$ is vertex k-prime for $m \ge 1$.*

Proof. Let m blocks of C_5 form a pentagonal snake mC_5. We represent the point set and line set of $mC_5 \odot K_1$ as

$$V(mC_5 \odot K_1) = \{x_{a,b} : 1 \le b \le 5, 1 \le a \le m\} \cup \{y_r^s : 1 \le r \le 5, 1 \le s \le m\},$$

$$E(mC_5 \odot K_1) = \begin{cases} \{x_{a,b}x_{a,b+1} : 1 \le b \le 4, 1 \le a \le m\}, \\ \{x_{a,5}x_{a,1} : 1 \le a \le m\}, \\ \{x_{a,b}y_r^s : 1 \le b \le 5, 1 \le a \le m, 1 \le r \le 5, 1 \le s \le m\} \end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,5}$ for $1 \le a \le m-1$ and y_1^{s+1} as y_5^s for $1 \le s \le m-1$ to facilitate defining of lines. The number of points and lines of $mC_5 \odot K_1)$ is $|V(mC_5 \odot K_1)| = 8m+2$ and $|E(mC_5 \odot K_1)| = 9m+1$.

Case 1. k is odd

Now define a function f from points of $mC_5 \odot K_1$ to $k, k+1, \dots, k+8m+1$ as given below:

$$\begin{aligned} &f(x_{1,1}) = k, \\ &f(x_{a,b}) = k+8a+2b-10, \quad 2 \le b \le 5, 1 \le a \le m, \\ &f(y_r^s) = k+2r+8s-9, \quad 1 \le r \le 5, 1 \le s \le m, \\ &f(x_{a,5}) = f(x_{a+1,1}), \quad 1 \le a \le m. \end{aligned}$$

For any $x_{a,b}x_{a,b+1} \in E(mC_5 \odot K_1), \gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+8a+2b-10, k+8a+2b-8) = 1$ since $k+8a+2b-10$ and $k+8a+2b-8$ are consecutive odd positive integers. For any $x_{a,5}x_{a,1} \in E(mC_5 \odot K_1) = \gcd(f(x_{a,5}), f(x_{a,1})) = \gcd(k+8a, k+8a-8)$ since k is odd. For any $x_{a,b}y_r^s \in E(mC_5 \odot K_1) = \gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+8a+2b-10, k+2r+8s-9) = 1$ since $k+8a+2b-10$ and $k+2r+8s-9$ are consecutive positive integers.

Case 2. k is even

Now define a function f from points of $mC_5 \odot K_1$ to $k, k+1, \dots, k+8m+1$ as given below:

$$\begin{aligned}
&f(y_1^1) = k,\\
&f(y_r^s) = k+2r+8s-10, && 2 \le r \le 5, 1 \le s \le m,\\
&f(x_{a,b}) = k+8a+2b-9, && 1 \le b \le 5, 1 \le a \le m,\\
&f(x_{a,5}) = f(x_{a+1,1}), && 1 \le a \le m.
\end{aligned}$$

For any $x_{a,b}x_{a,b+1} \in E(mC_5 \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+8a+2b-9, k+8a+2b-7) = 1$ since $k+8a+2b-9$ and $k+8a+2b-7$ are consecutive odd positive integers. For any $x_{a,5}x_{a,1} \in E(mC_5 \odot K_1) = \gcd(f(x_{a,5}), f(x_{a,1})) = \gcd(k+8a+1, k+8a-7)$ since k is odd. For any $x_{a,b}y_r^s \in E(mC_5 \odot K_1) = \gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+8a+2b-9, k+2r+8s-10) = 1$ since $k+8a+2b-9$ and $k+2r+8s-10$ are consecutive positive integers.

Hence $mC_5 \odot K_1$ admits vertex k-prime labeling. □

Theorem 3.8. *The corona graphs $mC_9 \odot K_1$ is vertex k-prime for $m \ge 1$.*

Proof. Let m blocks of C_9 form a nanogonal snake mC_9. We represent the point set and line set of $mC_9 \odot K_1$ as

$$V(mC_9 \odot K_1) = \{x_{a,b} : 1 \le b \le 9, 1 \le a \le m\} \cup \{y_r^s : 1 \le r \le 9, 1 \le s \le m\},$$

$$E(mC_9 \odot K_1) = \begin{cases} \{x_{a,b}x_{a,b+1} : 1 \le b \le 8, 1 \le a \le m\},\\ \{x_{a,9}x_{a,1} : 1 \le a \le m\},\\ \{x_{a,b}y_r^s : 1 \le b \le 9, 1 \le a \le m, 1 \le r \le 9, 1 \le s \le m\}. \end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,9}$ for $1 \le a \le m-1$ and y_1^{s+1} as y_9^s for $1 \le s \le m-1$ to facilitate defining of lines. The number of points and lines of $mC_9 \odot K_1$ is $|V(mC_9 \odot K_1)| = 16m+2$ and $|E(mC_9 \odot K_1)| = 17m+1$.

Case 1. k is odd

Now define a function f from points of $mC_9 \odot K_1$ to $k, k+1, \dots, k+16m+1$ as given below:

$$\begin{aligned}
&f(x_{1,1}) = k,\\
&f(x_{a,b}) = k+16a+2b-18, && 2 \le b \le 9, 1 \le a \le m,\\
&f(y_r^s) = k+2r+16s-17, && 1 \le r \le 9, 1 \le s \le m,\\
&f(x_{a,9}) = f(x_{a+1,1}), && 1 \le a \le m.
\end{aligned}$$

For any $x_{a,b}x_{a,b+1} \in E(mC_9 \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+16a+2b-18, k+16a+2b-16) = 1$ since $k+16a+2b-18$ and $k+16a+2b-16$ are consecutive odd positive integers. For any $x_{a,9}x_{a,1} \in E(mC_9 \odot K_1) = \gcd(f(x_{a,9}), f(x_{a,1})) = \gcd(k+16a, k+16a-16)$ since k is odd. For any $x_{a,b}y_r^s \in E(mC_9 \odot K_1) = \gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+16a+2b-18, k+2r+16s-17) = 1$ since $k+16a+2b-18$ and $k+2r+16s-17$ are consecutive positive integers.

Case 2. k is even

Now define a function f from points of $mC_9 \odot K_1$ to $k, k+1, \dots, k+16m+1$ as given below:

$$
\begin{aligned}
&f(y_1^1) = k, \\
&f(y_r^s) = k + 2r + 16s - 118, && 2 \le r \le 9, 1 \le s \le m, \\
&f(x_{a,b}) = k + 16a + 2b - 17, && 1 \le b \le 9, 1 \le a \le m, \\
&f(x_{a,9}) = f(x_{a+1,1}), && 1 \le a \le m.
\end{aligned}
$$

For any $x_{a,b}x_{a,b+1} \in E(mC_9 \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+16a+2b-17, k+16a+2b-15) = 1$ since $k+16a+2b-17$ and $k+16a+2b-15$ are consecutive odd positive integers. For any $x_{a,9}x_{a,1} \in E(mC_9 \odot K_1) = \gcd(f(x_{a,9}), f(x_{a,1})) = \gcd(k+16a+1, k+16a-15)$ since k is odd. For any $x_{a,b}y_r^s \in E(mC_9 \odot K_1) = \gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+16a+2b-17, k+2r+16s-18) = 1$ since $k+16a+2b-17$ and $k+2r+16s-18$ are consecutive positive integers.

Hence $mC_9 \odot K_1$ admits vertex k-prime labeling. □

Theorem 3.9. *The corona graph $mC_n \odot K_1$ is vertex k-prime for even positive integer $n \ge 4$ and $k \not\equiv 0 (\mathrm{mod}(n-1))$, where $n-1$ is prime.*

Proof. Let m blocks of C_n form a cyclic snake mC_n. We represent the point set and line set of $mC_n \odot K_1$ as

$$V(mC_n \odot K_1) = \{x_{a,b} : 1 \le b \le n, 1 \le a \le m\} \cup \{y_r^s : 1 \le r \le n, 1 \le s \le m\},$$

$$E(mC_n \odot K_1) = \begin{cases} \{x_{a,b}x_{a,b+1} : 1 \le b \le n-1, 1 \le a \le m\}, \\ \{x_{a,n}x_{a,1} : 1 \le a \le m\}, \\ \{x_{a,b}y_r^s : 1 \le b \le n, 1 \le a \le m, 1 \le r \le n, 1 \le s \le m\}. \end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,n}$ for $1 \le a \le m-1$ and y_1^{s+1} as y_n^s for $1 \le s \le m-1$ to facilitate defining of lines. The number of points and lines of $mC_n \odot K_1$ is $|V(mC_n \odot K_1)| = 2(n-1)m+2$ and $|E(mC_n \odot K_1)| = (2n-1)m+1$ (see Figure 3).

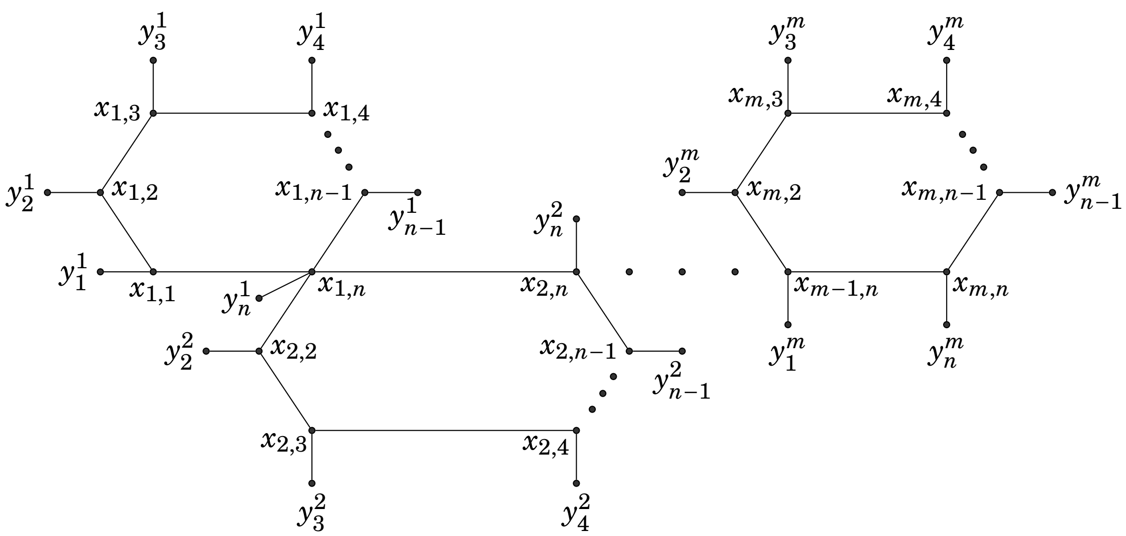

Figure 3. $mC_n \odot K_1$

Case 1. k is odd

Now define a function f from points of $mC_n \odot K_1$ to $k, k+1, \dots, k+2(n-1)m+1$ as given below:

$$\begin{aligned} & f(x_{1,1}) = k, & \\ & f(x_{a,b}) = k+(2n-2)a+2b-2n, & 2 \le b \le n, 1 \le a \le m, \\ & f(y_r^s) = k+2r+(2n-2)s-(2n-1), & 1 \le r \le n, 1 \le s \le m, \\ & f(x_{a,n}) = f(x_{a+1,1}), & 1 \le a \le m. \end{aligned}$$

Based on the labeling pattern defined, the lines of $mC_n \odot K_1$ satisfy the conditions of vertex k-prime labeling as, for any $x_{a,b}x_{a,b+1} \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+(2n-2)a+2b-2n, k+(2n-2)a+2b-2n+2) = 1$ since $k+(2n-2)a+2b-2n$ and $k+(2n-2)a+2b-2n+2$ are consecutive odd positive integers. For any $x_{a,n}x_{a,1} \in E(mC_n \odot K_1) = \gcd(f(x_{a,n}), f(x_{a,1})) = \gcd(k+(2n-2)a, k+(2n-2)a-2n+2) = 1$ since $k \not\equiv 0(\text{mod}(n-1))$, where $n-1$ is prime. For any $x_{a,b}y_r^s \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+(2n-2)a+2b-2n, k+2r+(2n-2)s-(2n-1)) = 1$ since $k+(2n-2)a+2b-2n$ and $k+2r+(2n-2)s-(2n-1)$ are consecutive positive integers.

Case 2. k is even

Now define a function f from points of $mC_n \odot K_1$ to $k, k+1, \dots, k+2(n-1)m+1$ as given below:

$$\begin{aligned} & f(y_1^1) = k, & \\ & f(y_r^s) = k+2r+(2n-2)s-2n, & 2 \le r \le n, 1 \le s \le m, \\ & f(x_{a,b}) = k+(2n-2)a+2b-(2n-1), & 1 \le b \le n, 1 \le a \le m, \\ & f(x_{a,n}) = f(x_{a+1,1}), & 1 \le a \le m. \end{aligned}$$

Based on the labeling pattern defined, the lines of $mC_n \odot K_1$ satisfy the conditions of vertex k-prime labeling as, for any $x_{a,b}x_{a,b+1} \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+(2n-2)a+2b-(2n-1), k+(2n-2)a+2b-2n+3) = 1$ since $k+(2n-2)a+2b-(2n-1)$ and $k+(2n-2)a+2b-2n+3$ are consecutive odd positive integers. For any $x_{a,n}x_{a,1} \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,n}), f(x_{a,1})) = \gcd(k+(2n-2)a+1, k+(2n-2)a-2n+3) = 1$ since $k \not\equiv 0(\text{mod}(n-1))$, where $n-1$ is prime. For any $x_{a,b}y_r^s \in E(mC_n \odot K_1) = \gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+(2n-2)a+2b-(2n-1), k+2r+(2n-2)s-2n) = 1$ since $k+(2n-2)a+2b-(2n-1)$ and $k+2r+(2n-2)s-2n$ are consecutive positive integers. Hence $mC_n \odot K_1$ admits vertex k-prime labeling. □

Theorem 3.10. *The corona graph $mC_n \odot K_1$ is vertex k-prime for even positive integer $n \ge 10$ and k not multiple factor of $(n-1)$, where $n-1$ is not prime.*

Proof. Let m blocks of C_n form a cyclic snake mC_n. We represent the point set and line set of $mC_n \odot K_1$ as

$$V(mC_n \odot K_1) = \{x_{a,b} : 1 \le b \le n, 1 \le a \le m\} \cup \{y_r^s : 1 \le r \le n, 1 \le s \le m\},$$

$$E(mC_n \odot K_1) = \begin{cases} \{x_{a,b}x_{a,b+1} : 1 \le b \le n-1, 1 \le a \le m\}, \\ \{x_{a,n}x_{a,1} : 1 \le a \le m\}, \\ \{x_{a,b}y_r^s : 1 \le b \le n, 1 \le a \le m, 1 \le r \le n, 1 \le s \le m\}. \end{cases}$$

We refer the point $x_{a+1,1}$ as $x_{a,n}$ for $1 \le a \le m-1$ and y_1^{s+1} as y_n^s for $1 \le s \le m-1$ to facilitate defining of lines. The number of points and lines of $mC_n \odot K_1$ is $|V(mC_n \odot K_1)| = 2(n-1)m+2$ and $|E(mC_n \odot K_1)| = (2n-1)m+1$.

Case 1. k is odd

Now define a function f from points of $mC_n \odot K_1$ to $k, k+1, \dots, k+2(n-1)m+1$ as given below:

$$f(x_{1,1}) = k,$$
$$f(x_{a,b}) = k + (2n-2)a + 2b - 2n, \qquad 2 \le b \le n, 1 \le a \le m,$$
$$f(y_r^s) = k + 2r + (2n-2)s - (2n-1), \quad 1 \le r \le n, 1 \le s \le m,$$
$$f(x_{a,n}) = f(x_{a+1,1}), \qquad 1 \le a \le m.$$

Based on the labeling pattern defined, the lines of $mC_n \odot K_1$ satisfy the conditions of vertex k-prime labeling as, for any $x_{a,b}x_{a,b+1} \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+(2n-2)a+2b-2n, k+(2n-2)a+2b-2n+2) = 1$ since $k+(2n-2)a+2b-2n$ and $k+(2n-2)a+2b-2n+2$ are consecutive odd positive integers. For any $x_{a,n}x_{a,1} \in E(mC_n \odot K_1) = \gcd(f(x_{a,n}), f(x_{a,1})) = \gcd(k+(2n-2)a, k+(2n-2)a-2n+2) = 1$ since k not multiple factor of $(n-1)$, where $n-1$ is not prime. For any $x_{a,b}y_r^s \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+(2n-2)a+2b-2n, k+2r+(2n-2)s-(2n-1)) = 1$ since $k+(2n-2)a+2b-2n$ and $k+2r+(2n-2)s-(2n-1)$ are consecutive positive integers.

Case 2. k is even

Now define a function f from points of $mC_n \odot K_1$ to $k, k+1, \dots, k+2(n-1)m+1$ as given below:

$$f(y_1^1) = k,$$
$$f(y_r^s) = k + 2r + (2n-2)s - 2n, \qquad 2 \le r \le n, 1 \le s \le m,$$
$$f(x_{a,b}) = k + (2n-2)a + 2b - (2n-1), \quad 1 \le b \le n, 1 \le a \le m,$$
$$f(x_{a,n}) = f(x_{a+1,1}), \qquad 1 \le a \le m.$$

Based on the labeling pattern defined, the lines of $mC_n \odot K_1$ satisfy the conditions of vertex k-prime labeling as, for any $x_{a,b}x_{a,b+1} \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,b}), f(x_{a,b+1})) = \gcd(k+(2n-2)a+2b-(2n-1), k+(2n-2)a+2b-2n+3) = 1$ since $k+(2n-2)a+2b-(2n-1)$ and $k+(2n-2)a+2b-2n+3$ are consecutive odd positive integers. For any $x_{a,n}x_{a,1} \in E(mC_n \odot K_1)$, $\gcd(f(x_{a,n}), f(x_{a,1})) = \gcd(k+(2n-2)a+1, k+(2n-2)a-2n+3) = 1$ since k not multiple factor of $(n-1)$, where $n-1$ is not prime. For any $x_{a,b}y_r^s \in E(mC_n \odot K_1) = \gcd(f(x_{a,b}), f(y_r^s)) = \gcd(k+(2n-2)a+2b-(2n-1), k+2r+(2n-2)s-2n) = 1$ since $k+(2n-2)a+2b-(2n-1)$ and $k+2r+(2n-2)s-2n$ are consecutive positive integers.

Hence $mC_n \odot K_1$ admits vertex k-prime labeling when $n-1$ is not prime.

An illustration is given in Figure 4.

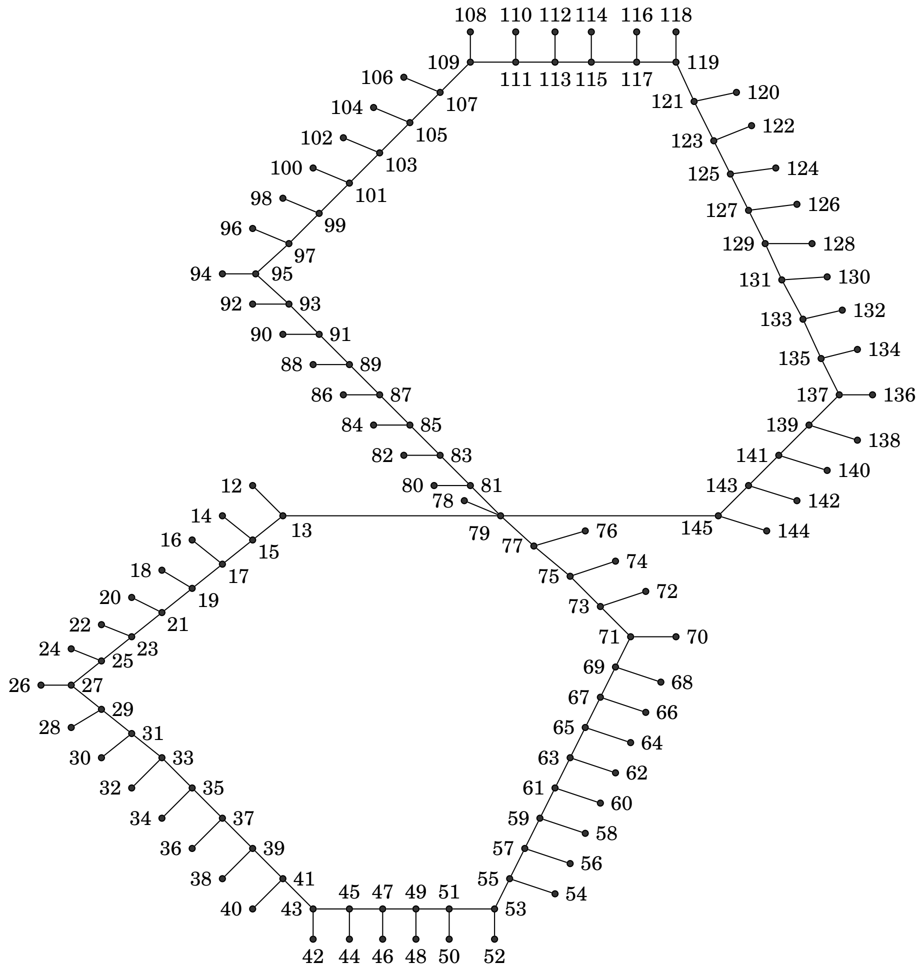

Figure 4. Vertex k-prime labeling of $2C_{34} \odot K_1$ for $k = 12$ □

4. Conclusion

The results presented in this paper is on cyclic snake graphs and corona graphs of the form $mC_n \odot K_1$ that satisfy the conditions of vertex k-prime labeling.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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