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Research Article

Localized Automorphisms and Endomorphisms

Abdelgabar Adam Hassan[1,2]

[1] *Department of Mathematics, College of Science, Jouf University, Kingdom of Saudi Arabia*
[2] *Department of Mathematics, University of Nyala, Nyala, Sudan*
jabra69b@gmail.com

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Abstract. We give a practical criterion of invertibility of endomorphisms of O_n corresponding to unitaries in the normalizer of the diagonal inside the uniformly hyperfinite subalgebra. We also analyze the action of such localized automorphisms on the spectrum of the diagonal thus obtaining criteria of outerness. Unital endomorphisms of the Cuntz algebra O_n which preserve the canonical uniformly hyperfinite-subalgebra $F_n \subseteq O_n$ are investigated. We give examples of such endomorphisms $\lambda = \lambda_u$ for which the associated unitary element u in O_n.

Keywords. Algebra, Endomorphism, Automorphism, Uniformly hyperfinite, Cuntz algebra

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1. Introduction

The C^*-algebra generated by all words of the form $S_\alpha S_\beta^*$, $\alpha, \beta \in W_n^k$, and it is isomorphic to the matrix algebra $M_{n^k}(C)$. The F_n, the norm closure of $\bigcup_{k=0}^{\infty} F_n^k$, is the uniformly hyperfinite-algebra of type n^∞, called the core uniformly hyperfinite-subalgebra of O_n. It is the fixed point algebra for the periodic gauge action of the reals: $\alpha : R \to Aut(O_n)$ defined on generators as $\alpha_t(S_i) = e^{it} S_i$, $t \in R$ (see Cuntz [9]).

We denote by S_n the group of those unitaries in O_n which can be written as finite sums of words, i.e., in the form $u = \sum_{j=1}^{m} S_{\alpha_j} S_{\beta_j}^*$ for some $\alpha_j, \beta_j \in W_n$. It turns out that S_n is isomorphic to the Higman-Thompson group $G_{n,1}$. We also denote $P_n = S_n \cap U(F_n)$. Then $P_n = \bigcup_k P_n^k$, where P_n^k

are permutation unitaries in $U(F_n^k)$. That is, for each $u \in P_n^k$ there is a unique permutation σ of multi-indices W_n^k such that $u = \sum_{\alpha \in W_n^k} S_{\alpha(\alpha)} S_\alpha^*$ (Nekrashevych [23]).

For u a unitary in O_n we denote by λ_u the unital endomorphism of O_n determined by $\lambda_u(S_i) = uS_i$, $i = 1, \ldots, n$. We denote by φ the canonical shift: $\varphi(x) = \sum_i S_i x S_i^*$, $x \in O_n$. Note that φ commutes with the action α. If $u \in U(O_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u)\ldots\phi^{k-1}(u).$$

We agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i a = \varphi(a) S_i$, valid for all $i = 1, \ldots, n$ and $a \in O_n$ (Cuntz [9], and Nekrashevych [23]).

2. The Cuntz Algebra Which Preserve the Diagonal Subalgebra

If n is an integer greater than 1, then the Cuntz algebra O_n is unital, simple C^*-algebra generated by n isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^{n} S_i S_i^* = 1$, we denote by W_n^k the set of k-tuples $\alpha = (\alpha^1, \ldots, \alpha^k)$ with $\alpha^m \in \{1, \ldots, n\}$, and we denote by W_n the union $\bigcup_{k=0}^{\infty} W_n^k$ where $W_n^0 = \{0\}$. Elements of W_n are called multi-indices and if $\alpha \in W_n^k$ then $l(\alpha) = k$, the length of α. If $\alpha = (\alpha^1, \ldots, \alpha^k) \in W_n^k$, then $S_\alpha = S_{\alpha^1}, \ldots, S_{\alpha^k}$, with $S_0 = 1$ by convention [7, 14, 15]. Every S_α is an isometry and its range projection are $S_\alpha S_\alpha^*$. Every word in $\{S_i, S_i^*, i = 1, \ldots, n\}$ can be uniquely expressed as $S_\alpha S_\beta^*$ for some $\alpha, \beta \in W_n$ (Cuntz [7, 9], and Hong [14]).

Lemma 2.1 ([6]). *If $\lambda_w \in Aut(O_n, D_n)$ and $w \in F_n$ then $\lambda_w(F_n) \subseteq F_n$.*

Proof. Let γ be the standard gauge action of the circle group on O_n, for which F_n is the fixed-point algebra. Then for each $z \in U(1)$ we have $\lambda_w \gamma_z = \gamma_z \lambda_w$. Thus, also $\lambda_w^{-1} \gamma_z = \gamma_z \lambda_w^{-1}$ and consequently λ_w^{-1} preserves the fixed-point algebra of γ. That is $\lambda_w^{-1}(F_n) \subseteq F_n$, as required.

Since $N_{D_n}(O_n) = U(D_n) \rtimes S_n$ by Cuntz [9], it easily follows that $N_{D_n}(F_n) = U(D_n) \rtimes P_n$, where $P_n = S_n \cap F_n$. We see that P_n is contained in the algebraic part $\bigcup_{k=0}^{\infty} F_n^k$ of F_n, and write $P_n^k = P_n \cap F_n^k$. It is not difficult to see that unitaries in P_n are related to permutations of multi-indices, as follows. Let P_n^k denote the set of permutations of W_n^k, and let $P_n = \bigcup_{k=0}^{\infty} P_n^k$. Then, for each unitary $w \in P_n^k$ there exists a permutation $\sigma \in P_n^k$ such that

$$w = \sum_{\alpha \in W_n^k} S_{\sigma(\alpha)} S_\alpha^*. \tag{2.1}$$

In that case we write $w \sim \sigma$ and $\lambda_w = \lambda_\sigma$. We denote

$$\lambda(P_n)^{-1} = \{\lambda_w \in Aut(O_n) : w \in P_n\}. \tag{2.2}$$

Theorem 2.2. *$Aut(O_n, D_n) \cap Aut(O_n, F_n) \cong U(D_n) \cong \lambda(P_n)^{-1}$. In particular, $\lambda(P_n)^{-1}$ is a subgroup of $Aut(O_n, D_n) \cap Aut(O_n, F_n)$.*

If $u \in U(O_n)$, then $Ad(u) = \lambda_{\Phi(u)u^*}$ is the inner automorphism of O_n determined by u. We denote by $Inn(O_n)$ the group of inner automorphisms of O_n (Conti *et al*. [6]).

Theorem 2.3. *If $u \in P_n$ and λ_u is invertible then the following conditions are equivalent:*

(i) *automorphism λ_u has infinite order,*

(ii) *the Z action on O_n generated by λ_u is outer,*

(iii) *the Z action on X_n generated by λ_u is topologically free.*

Proof. (i)⇒(ii): This follows from the fact that if $\lambda_w \in Inn(O_n)$ then λ_u has finite order.

(ii)⇒(iii): If the action is not topologically free then for some m the set of h_u^m has a non-empty interior. Thus, there exists $(x_1,\dots,x_r)$ such that h_u^m fixes each sequence (y_i) whose initial segment coincides with $(x_1,\dots,x_r)$. But then h_u^m is inner.

(iii)⇒(i): This is obvious. □

We now give a practical criterion of invertibility of endomorphisms corresponding to permutations (Szymański [25]). First recall that $End(O_n)$ contains a distinguished endomorphism Φ, called shift, such that

$$\Phi(a) = \sum_{i=1}^{n} S_i a S_i^* . \tag{2.3}$$

Let $u \in p_n^k$. If $k \geq 2$ then we define

$$B_w = \{w, \Phi(w), \dots, \Phi^{k-2}(w)\}' \cap f_n^{k-1} . \tag{2.4}$$

Here prime denotes the commutant. $k \leq 1$ then we set $B_w = C1$. One checks that $b \in f_n^{k-1}$ belongs to B_w if and only if for each pair $\alpha, \beta \in W_n^l$, $l \in \{0,1,\dots,k-2\}$, $S_\alpha^* b S_\beta$ commutes with w. We define a vector space V_w as the quotient

$$V_w = \frac{f_n^{k-1}}{B_w} . \tag{2.5}$$

Now for each pair $i,j \in \{1,\dots,n\}$ we define a linear $a_{ij}^w : f_n^{k-1} \to f_n^{k-1}$ such that

$$a_{ij}^w(b) = S_i^* w b w^* S_j . \tag{2.6}$$

One checks that $a_{ij}^w(B_w) \subseteq B_w$ for each i,j. Thus, a_{ij}^w induces a linear map

$$\tilde{a}_{ij}^w : V_w \to V_w . \tag{2.7}$$

With this preparation we make the following definition:

$$A_w = \text{the subring of } End(V_w) \text{ generated by } \{\tilde{a}_{ij}^w : i,j = 1,\dots,n\}. \tag{2.8}$$

Now we are ready to showed the following.

Theorem 2.4. *If $w \in P_n$ then endomorphism λ_w is invertible if and only if the corresponding ring A_w is nilpotent.*

Proof. Let $w \in p_n^k$ and suppose that λ_w is invertible. There exists $u \in p_n$ such that $\lambda_w^{-1} = \lambda_u$. Thus, there exists positive integer l such that $\lambda_w^{-1}(f_n^{k-1}) \subseteq f_n^l$. For each $a \in f_n^l$ the sequence $Ad(w^*\Phi(w^*)\dots\Phi^m(w^*))(a)$ stabilizes from $m = l-1$ at the value $\lambda_w(a)$. Consequently, for each $b \in f_n^{k-1}$ the sequence $Ad(\Phi^m(w)\dots\Phi(w)w)(b)$ stabilizes from $m = l-1$ at the value $\lambda_w^{-1}(b)$. There exist elements $c_{\mu\nu}(b) \in f_n^{k-1}$, $\mu, \nu \in W_n^l$, such that for each $r \geq 1$ we have

$$\sum_{\mu,\nu\in W_n^l} S_\mu c_{\mu\nu}(b)S_\nu^* = Ad(\Phi^{l-1}(w)\dots\Phi(w)w)(b)$$
$$= Ad(\Phi^{l-1+r}(w)\dots\Phi(w)w)(b)$$
$$= \sum_{\mu,\nu\in W_n^l} S_\mu Ad(\Phi^{r-1}(w))(c_{\mu\nu}(b))S_\nu^*.$$

Hence $c_{\mu\nu}(b) = Ad(\Phi^{r-1}(w))(c_{\mu\nu}(b))$. Thus $\{c_{\mu\nu}(b) : b \in f_n^{k-1}, \mu,\nu \in W_n^l\} \subseteq B_w$. If $\alpha = (i_1,\cdots,i_l)$ and $\beta = (j_1,\cdots,j_l)$, then, let $T_{\alpha,\beta} = a^w_{i_j j_j}\dots a^w_{i_1 j_1}$. For each $b \in f_n^{k-1}$, we have $T_{\alpha,\beta}(b) = c_{\alpha,\beta}(b)$. Consequently, $A_w^l = \{0\}$ and A_w is nilpotent.

Now, Let $w \in p_n^k$ and suppose that $A_w^l = \{0\}$. Let $b \in f_n^{k-1}$ and define $T_{\alpha,\beta}$ as above. $T_{\alpha,\beta}(b)$ commutes with $Ad(\Phi^m(w))$ for any m. Hence if $r \geq 1$, then we have

$$Ad(\Phi^{l-1+r}(w)\dots\Phi(w)w)(b) = \sum_{\mu,\nu\in W_n^l} S_\mu Ad(\Phi^{r-1}(w))(T_{\mu\nu}(b))S_\nu^*$$
$$= \sum_{\mu,\nu\in W_n^l} S_\mu T_{\mu\nu}(b)S_\nu^*.$$

Thus, for each $b \in f_n^{k-1}$ the sequence $Ad(\Phi^m(w)\dots\Phi(w)w)(b)$ stabilizes $m = l-1$. We have $w^* = \sum_{i,j=1}^{n} S_i b_{ij} S_j^*$ for some $b_{ij} \in f_n^{k-1}$. It follows from the above argument that the sequence

$$Ad(\Phi^{l-1+r}(w)\dots\Phi(w)w)(w^*) = \sum_{ij} Ad(\Phi(\Phi^{m-1}(w)\dots\Phi(w)w))(S_i B_{ij} S_\nu^*)$$
$$= \sum_{ij} S_i Ad((\Phi^{m-1}(w)\dots\Phi(w)w))(b_{ij})S_j^*$$

stabilizes from $m = l$ at the value $\lambda_w^{-1}(w^*)$. Consequently, λ_w is invertible (Izumi [16], Kawamura [20], and Pask and Rennie [24]). □

3. Canonical Uniformly Hyperfinite-subalgebra

We expand the initial observations on endomorphisms preserving the canonical uniformly hyperfinite-subalgebra in a more systematic manner (Cuntz [8]). We study a particularly interesting class of such endomorphisms related to certain elements in the normalizer of the canonical (Cuntz [9]).

Proposition 3.1 ([6]). *Let u be a unitary in O_n and let v be a unitary in the relative commutant $\lambda_u(F_n)' \cap O_n$. Define $w := u\varphi(v)$. Then the restrictions of endomorphisms λ_u and λ_w coincide on F_n. Likewise, if $\tilde{w} = vu$ then the restrictions of endomorphisms λ_u and $\lambda_{\tilde{w}}$ coincide on F_n.*

Proof. It is enough to compute the action of λ_w on all elements of the form $S_{\alpha_1}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_1}$ for every integer $k \geq 1$ and all α_i and β_j in $\{1,\dots,n\}$, for all $1 \leq i,\ j \leq k$. To this end, we verify by induction on k that

$$\lambda_w(S_{\alpha_1}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_1}) = \lambda_u(S_{\alpha_1}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_1}).$$

Indeed, for $k = 1$, we have

$$\lambda_w(S_{\alpha_1}S^*_{\beta_1}) = wS_{\alpha_1}S^*_{\beta_1}w^* = u\varphi(v)S_{\alpha_1}S^*_{\beta_1}\varphi(v)^*u^* = uS_{\alpha_1}S^*_{\beta_1}u_* = \lambda_u(S_{\alpha_1}S^*_{\beta_1}).$$

Since $\varphi(v)$ and $S_{\alpha_1}S^*_{\beta_1}$ commute. Now assuming the identity holds for $k-1$, we have

$$\begin{aligned}\lambda_w(S_{\alpha_1}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_1}) &= \lambda_w(S_{\alpha_1})\lambda_w(S_{\alpha_2}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_2})\lambda_w(S^*_{\beta_1})^* \\ &= u\varphi(v)S_{\alpha_1}\lambda_u(S_{\alpha_2}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_2})S^*_{\beta_1}\varphi(v)^*u^* \\ &= uS_{\alpha_1}v\lambda_u(S_{\alpha_2}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_2})v^*S^*_{\beta_1}u^* \\ &= uS_{\alpha_1}\lambda_u(S_{\alpha_2}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_2})S^*_{\beta_1}u^* \\ &= \lambda_u(S_{\alpha_1}\dots S_{\alpha_k}S^*_{\beta_k}\dots S^*_{\beta_1}),\end{aligned}$$

since v is in the commutant of $\lambda_u(F_n)$. The proof of the remaining claim is similar. □

Proposition 3.2. *Let u be a unitary in O_n, then*

$$\lambda_u(F_n)'\cap O_n = \bigcap_{k\ge 1}(Adu\circ\phi)^k(O_n).$$

Proof. Clearly, an element $x\in O_n$ lies in $\lambda_u(F_n)'\cap O_n$ if and only if, for all $k\ge 1$ and all $y\in F_n^k$, x commutes with $\lambda_u(y)=u_k y u_k^*$, i.e.,

$$u_k^* x u_k \in (F_n^k)'\cap O_n = \varphi^k(O_n).$$

This means precisely that, for each $k\ge 1$, x lies in the range of $Ad(u_k)\varphi^k = (Adu\circ\varphi)^k$.

It is also useful to observe that $Adu\circ\varphi$ restricts to an automorphism of $\lambda_u(F_n)'\cap O_n$. This follows from the following simple lemma. □

Lemma 3.3. *Let A be a unital C^*-algebra and ρ an injective unital $*$-endomorphism of A, then ρ restricts to a $*$-automorphism of*

$$A_\rho := \bigcap_{k\in N}\rho^k(a).$$

Proof. One has a descending tower of unital C^*-subalgebras of A,

$$A\supset\rho(a)\supset\rho^2(a)\supset\dots,$$

thus A_ρ is a unital C^*-subalgebra of A. A_n element $x\in A_\rho$ satisfies

$$x=\rho(x_1)=\rho^2(x_2)=\dots=\rho^k(x_k)=\dots$$

for elements $x_1,\dots,x_k,\dots$ in A. It is then clear that ρ maps A_ρ into itself, and moreover $x_1,\dots,x_k,\dots\in A_\rho$ so that in particular $\rho(A_\rho)=A_\rho$.

Endomorphisms ρ for which $A_\rho = C1$ are often called shifts.

To this end, it suffices to find a unitary $u\in F_n$ such that the relative commutant $\lambda_u(F_n)'\cap O_n$ is not contained in F_n. This is possible. In fact, one can even find unitaries in a matrix algebra F_n^k such that $\lambda_u(F_n)'\cap O_n$ is not contained in F_n. The existence of such unitaries was demonstrated in [8], [18], [22]. The relative commutant $\lambda_u(O_n)'\cap O_n$ coincides with the space (λ_u,λ_u) of self-intertwiners of the endomorphism λ_u, which can be computed as

$$(\lambda_u,\lambda_u)=\{x\in O_n : x=(Adu\circ\varphi)(x)\}.$$ □

Proposition 3.4. *There are sequences $\{v_n\}$ and $\{w_n\}$ of unitaries in F_2 such that*

(i) *$\{\lambda_{v_n}\}$ is asymptotically central in O_2,*

(ii) *$\|w_n\lambda_{v_{n+1}}(S_j)w_n^* - \lambda_{v_n}(S_j)\| < 2^{-n}$, for all $n\in N$ and for $j=1,2$,*

(iii) $\|w_n S_j w_n^* - S_j\| < 2^{-n}$, *for all* $n \in N$ *and for* $j = 1,2$.

Proof. Let $\{v_n\}$ be as in Proposition 3.2. Then $\{v_n\}$ and any subsequence there of will satisfy (i). Upon passing to a subsequence we can assume that

$$\|\lambda_{v_m}(S_i)\lambda_{v_n}(S_j) - \lambda_{v_n}(S_j)\lambda_{v_m}(S_i)\| < \frac{1}{n}, \tag{3.1}$$

for all $m > n \geq 1$ and for all $i, j = 1,2$. We claim that one can find a sequence $\{w_n\}$ of unitaries in F_2 satisfying (ii) and (iii) above — provided that we again pass to a subsequence of $\{v_n\}$. It suffices to show that for each $\delta > 0$ there exists a natural number n such that for each natural number $m > n$ there is a unitary $w \in F_2$ for which

$$\|w\lambda_{v_m}(S_j)w^* - \lambda_{v_n}(S_j)\| < \delta, \quad \|wS_jw^* - S_j\| < \delta$$

for $j = 1,2$. We give an indirect proof of the latter statement. If it were false, then there would exist $\delta > 0$ and a sequence $1 \leq n_1 < n_2 < n_3 < \dots$ such that one of

$$\|w\lambda_{v_{n_{k+1}}}(S_i)w^* - \lambda_{v_{n_k}}(S_i)\|, \quad \|wS_iw^* - S_i\|,$$

$i = 1,2$, is greater than δ for every k and for all unitaries w in F_2. We proceed to show that this will lead to a contradiction.

Choose a free ultrafilter ω on N and consider the relative commutant $O_2' \cap (O_2)_\omega$ inside the ultrapower $(O_2)_\omega$. This C^*-algebra is purely infinite and simple. Consider the unital $*$-homomorphisms $\eta_1, \eta_2 : O_2 \to O_2' \cap (O_2)_\omega$ given by

$$\eta_1(x) = \pi_\omega(\lambda_{v_{n_2}}(x), \lambda_{v_{n_3}}(x), \lambda_{v_{n_4}}(x), \dots),$$
$$\eta_2(x) = \pi_\omega(\lambda_{v_{n_1}}(x), \lambda_{v_{n_2}}(x), \lambda_{v_{n_3}}(x), \dots),$$

$x \in O_2$, where $\pi_\omega : \ell^\infty(O_2) \to (O_2)_\omega$ is the quotient mapping. The images of η_1 and η_2 commute by (3.1). Put

$$u = \eta_2(S_1)\eta_1(S_1)^* + \eta_2(S_2)\eta_1(S_2)^* = \pi_\omega(v_{n_1}v_{n_2}^*, v_{n_2}v_{n_3}^*, v_{n_3}v_{n_4}^*, \dots),$$

and notice that u is a unitary element in $O_2' \cap (F_2)_\omega \subseteq O_2' \cap (O_2)_\omega$. To obtain a sequence $\{w_n\}$ of unitaries in $C^*(\eta_1(F_2), u) \subseteq O_2' \cap (F_2)_\omega$ such that $w_n\eta_1(S_j)w_n^* \to \eta_2(S_j)$ for $j = 1,2$. By [8] there is a single unitary w in $O_2' \cap (F_2)_\omega$ such that $w\eta_1(S_j)w^* = \eta_2(S_j)$ for $j = 1,2$ (and hence such that $w\eta_1(x)w^* = \eta_2(x)$ for all $x \in O_2$).

Each unitary element in the ultrapower $(F_2)_\omega$ lifts to a unitary element in $\ell^\infty(F_2)$, so we can write

$$w = \pi_\omega(w_1, w_2, w_3, \dots),$$

where each w_n is a unitary element in F_2. This establishes the desired contradiction, as

$$\lim_{n\to\omega} \|S_jw_n - w_nS_j\| = 0, \quad \lim_{n\to\omega} \|w_n\lambda_{v_{n_{k+1}}}(S_j)w_n^* - \lambda_{v_{n_k}}(S_j)\| = 0,$$

for $j = 1,2$ and for all k. □

Theorem 3.5. *There is a unitary element* $u \in F_2$ *such that the relative commutant* $\lambda_u(O_2)' \cap O_2$ *contains a unital copy of* O_2.

Proof. Let $\{v_n\}$ and $\{w_n\}$ be as in Proposition 3.4 and define endomorphisms on O_2 by

$$\lambda_n(x) = w_1w_2\dots w_n\lambda_{v_{n+1}}(x)w_n^*\dots w_2^*w_1^*, \quad \rho_n(x) = w_1w_2\dots w_nxw_n^*\dots w_2^*w_1^*,$$

for $x \in O_2$. Then

$$\|\lambda_n(S_j) - \lambda_{n-1}(S_j)\| < 2^{-n}, \quad \|\rho_n(S_j) - \rho_{n-1}(S_j)\| < 2^{-n}$$

for $j = 1,2$, and $\lambda_n(x)\rho_n(y) - \rho_n(y)\lambda_n(x) \to 0$ for all $x, y \in O_2$. Using that

$$w\lambda_u(x)w^* = \lambda_{wu\varphi(w)^*}(x)$$

whenever w is a unitary in O_2 and $x \in O_2$, we see that $\lambda_n = \lambda_{u_n}$ for some unitary u_n in F_2. It follows from the estimates above that the sequences $\{\lambda_n(S_j)\}$ and $\{\rho_n(S_j)\}$, $j = 1,2$, and hence also the sequence $\{u_n\}$, are Cauchy and therefore convergent. Let $\lambda : O_2 \to O_2$ and $\rho : O_2 \to O_2$ be the (pointwise-norm) limits of the sequences $\{\lambda_n\}$ and $\{\rho_n\}$, respectively, and let $u \in F_2$ be the limit of the sequence $\{u_n\}$. Then $\lambda = \lambda_u$ and the images of λ and ρ commute. □

Corollary 3.6. *There is a unital $*$-homomorphism $\sigma : O_2 \otimes O_2 \to O_2$ such that $\sigma(F_2 \otimes F_2) \subseteq F_2$.*

Proof. Take $\lambda : O_2 \to O_2$ and $\rho : O_2 \to O_2$. Recall that λ and ρ have commuting images and that $\lambda(F_2) \subseteq F_2$ and $\rho(F_2) \subseteq F_2$. We can therefore define a $*$-homomorphism $\sigma : O_2 \otimes O_2 \to O_2$ by

$$\sigma(x \otimes y) = \lambda(x)\rho(y), x, y \in O_2.$$

Then

$$\sigma(F_2 \otimes F_2) = \lambda(F_2)\rho(F_2) \subseteq F_2.$$

We know that $O_2 \otimes O_2$ and O_2 are isomorphic, but we do not know if one can find an isomorphism $\sigma : O_2 \otimes O_2 \to O_2$ such that $\sigma(F_2 \otimes F_2)$ is contained in (or better, equal to) F_2.

Below, ϕ denotes the standard left inverse of φ, i.e., the unital, completely positive map given by $\phi(x) := \frac{1}{n}\sum S_i^* x S_i$, $x \in O_n$. □

Theorem 3.7. *Let $u \in U(O_n)$, then the following conditions are equivalent:*

(i) *$\phi(u) \in U(O_n)$,*

(ii) *$u \in \varphi(O_n)$,*

(iii) *$S_i^* u S_i = S_j^* u S_j \in U(O_n)$, for all $i, j \in \{1, \ldots, n\}$.*

Proof. (i)⇒(ii): It follows from (i) that u lies in the multiplicative domain of ϕ and therefore, by Choi's theorem, $\phi(S_i u) = \phi(S_i)\phi(u)$, that is $uS_i = S_i\phi(u)$ for all $i = 1, \ldots, n$. Thus, $u = \varphi(\phi(u))$. The implications (ii)⇒(iii) and (iii)⇒(i) are obvious. □

Proposition 3.8. *Let $w \in U(O_n)$ be such that $\lambda_w t(F_n^1) \subseteq F_n$. Then the unitary α-cocycle $z_t^{(1)} := \phi(w^*\alpha_t(w))$ is a coboundary, i.e., there exists a unitary z such that $z_t^{(1)} = z\alpha_t(z^*)$ for all $t \in R$.*

Proof. Indeed, since $\lambda_w(F_n^1) \subseteq F_n$ there exists a unitary $u \in F_n$ such that λ_w and λ_u coincide on F_n^1. In fact, we could take as λ_u an inner automorphism implemented by a unitary in F_n. Then w^*u commutes with F_n^1, and thus there exists a unitary z such that $w^*u = \varphi(z)$. Now we have $\varphi(z\alpha_t(z^*)) = w^*u\alpha_t(u^*)\alpha_t(w) = w^*\alpha_t(w)$, since $\alpha_t(u^*) = u^*$. □

Proposition 3.9. *Let $w \in S_n$ be such that $\lambda_w(D_n) = D_n$ or, more generally, such that $D_n \subseteq \lambda_w(F_n)$. Then $\lambda_w(F_n) \subseteq F_n$ if and only if $w \in P_n$.*

Proof. An element of S_n normalizes D_n and thus satisfies the first assumption in the previous corollary. Then, the only nontrivial assertion follows from the fact that an endomorphism λ_w of O_n such that $\lambda_w(F_n) \supseteq D_n$ is necessarily irreducible in restriction to F_n by Szymański [25] an argument similar to using the facts that D_n in F_n is simple. □

Corollary 3.10. *Let D be a unital C^*-algebra, and suppose that $\eta_n, \eta_{n+1} : O_{n+1} \to D$ are unital $*$-homomorphisms with commuting images. There is a sequence $\{w_n\}$ of unitaries in the sub-C^*-algebra $D_0 = C^*(\eta_n(F_{n+1}), u)$, where*

$$u = \eta_{n+1}(S_n)\eta_n(S_n)^* + \eta_{n+1}(S_{n+1})\eta_n(S_{n+1})^*,$$

such that $w_n \eta_n(x) w_n^* \to \eta_{n+1}(x)$, for all $x \in O_{n+1}$.

Proof. The $*$-homomorphisms η_n and η_{n+1} induce a $*$-homomorphism $\eta : O_{n+1} \otimes O_{n+1} \to D$ given by

$$\eta(x \otimes y) = \eta_n(x)\eta_{n+1}(y), \quad x, y \in O_{n+1}.$$

In the notation of Proposition 3.8 we have

$$\eta(u_{n-1}) = u, \;\; \eta(F_{n+1} \otimes 1) = \eta_n(F_{n+1}), \;\; \eta(1 \otimes F_{n+1}) = \eta_{n+1}(F_{n+1}).$$

It follows from Proposition 3.8 and its proof that $1 \otimes F_{n+1}$ is contained in the C^*-algebra generated by $\{E_{ij}^{(0)}\}$ and u_{n-1} and hence is contained in $C^*(F_{n+1} \otimes 1, u_{n-1})$ (Cuntz [8]). The C^*-algebra B from that proposition is therefore generated by $F_{n+1} \otimes 1$ and u_{n-1}, which shows that $\eta(b) = D_{n-1}$.

Let $\{z_n\}$ be as in Proposition 3.8 and put $w_n = \eta(z_n) \in D_{n-1}$. Then

$$w_n \eta_n(x) w_n^* = \eta(z_n(x \otimes 1) z_n^*) \to \eta(1 \otimes x) = \eta_{n+1}(x),$$

for all $x \in O_{n+1}$. □

Corollary 3.11. *For $v, w \in U(O_n)$ the following three conditions are equivalent:*

(i) *endomorphisms λ_v and λ_w coincide on F_n,*

(ii) *for each $\varepsilon \geq 0$ we have $w_{1+\varepsilon}^* v_{1+\varepsilon} \in \varphi^{1+\varepsilon}(O_n)$,*

(iii) *there exists a sequence of unitaries $z_{1+\varepsilon} \in U(O_n)$ such that $z_1 = \phi(w^* v)$ and $z_{2+\varepsilon} = \phi(w^* z_{1+\varepsilon} v)$ for all $\varepsilon \geq 0$.*

Proof. The endomorphisms λ_v and λ_w coincide on F_n if and only if they coincide on each $F_n^{1+\varepsilon}$. Now if α and β are two multi-indices of length $1+\varepsilon$ then $\lambda_v(S_\alpha S_\beta^*) = v_{1+\varepsilon} S_\alpha S_\beta^* v_{1+\varepsilon}^*$ and $\lambda_w(S_\alpha S_\beta^*) = w_{1+\varepsilon} S_\alpha S_\beta^* w_{1+\varepsilon}^*$. Thus $\lambda_v(S_\alpha S_\beta^*) = \lambda_w(S_\alpha S_\beta^*)$ for all such α, β if and only if $w_{1+\varepsilon}^* v_{1+\varepsilon}$ is in the commutant of $F_n^{1+\varepsilon}$, that is when $w_{1+\varepsilon}^* v_{1+\varepsilon} \in \varphi^{1+\varepsilon}(O_n)$. Now it easily follows that this holds for all $1+\varepsilon$ if and only if condition (iii) above is satisfied. □

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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