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Almost Periodic Dynamic of a Discrete Wazewska-Lasota Model

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Abstract The purpose of this article is to investigate the existence of an almost periodic solution of a discrete Wazewska-Lasota model involving a linear harvesting term

$$x(n+1)-x(n)=-\alpha(n)x(n+1)+\sum_{i=1}^{l}\beta_i(n)e^{-\gamma_i(n)x(n-\tau_i(n))}-H(n)x(n-\sigma(n)),$$

by using the contraction mapping principle, and we also show that the solution of above equation converge exponentially to an almost periodic solution by constructing a luxury Liapunov functional.

1. Introduction and Preliminary

For ordinary difference equations and functional difference equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors. One of the most popular method is to assume the certain stability properties [1, 2, 4, 5, 6, 7] and [8]. Recently, J.O. Alzabut *et al*. [1] have shown the existence of an almost periodic solution for a Nicholson's blowflies difference model involving a linear harvesting term by means of the exponential dichotomy condition.

On the other hand, in 1997, Hamaya [6] has studied the global attractivity of the equilibrium point of the delay difference equation

$$x(n+1)-x(n)=-\alpha x(n)+\beta e^{-\gamma x(n-\tau)},$$

which has been proposed by Wazewska-Czyzewska and Lasota [10] as a model for the survival of red blood cells in an animal (cf. [9]). Here $x(n)$ denotes the density

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of red blood cells at time n, α is the probability of death of a red blood cell, β and γ are positive constants which are related to the production of red blood cells, and τ is the time which is required to produce a red blood cell.

In this paper, we discuss the existence of an almost periodic solution for a discrete Wazewska-Lasota difference model with time delay by using the contraction mapping theorem and Liapunov methods.

To the best of our knowledge, there are no relevant results on an almost periodic solution for a discrete Wazewska-Lasota model by means of our approach. We emphasize that our results extend [1] as a discrete Wazewska-Lasota model with multiple delay case.

In what follows, we denote by R^m real Euclidean m-space, Z is the set of integers, Z^+ is the set of nonnegative integers and $|\cdot|$ will denote the Euclidean norm in R^m. For any discrete interval $I \subset Z := (-\infty, \infty)$, we denote by $BS(I)$ the set of all bounded functions mapping I into R^m, and set $\|\phi\|_I = \sup\{|\phi(s)| : s \in I\}$.

Now, for some integer $r > 0$, for any function $x : [-r, a) \to R^m$ and $n < a$, define a function $x_n : [-r, 0] \to R^m$ by $x_n(s) = x(n+s)$ for $s \in [-r, 0]$.

We introduce an almost periodic function $f(n) : Z \to R^m$.

Definition 1. $f(n)$ is said to be almost periodic sequence in n if for any $\epsilon > 0$, there exists a positive integer $L^*(\epsilon)$ such that any interval of length $L^*(\epsilon)$ contains an integer τ for which

$$|f(n+\tau) - f(n)| \le \epsilon$$

for all $n \in Z$. Such a number τ in above inequality is called an ϵ-translation number of $f(n)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above definition, we discuss the concept of the normality of almost periodic functions. Namely, let $f(n)$ be almost periodic sequence in n. Then, for any sequence $\{h'_k\} \subset Z$, there exists a subsequence $\{h_k\}$ of $\{h'_k\}$ and function $g(n)$ such that

$$f(n+h_k) \to g(n) \tag{1.1}$$

uniformly on Z as $k \to \infty$. There are many properties of the discrete almost periodic functions [3], which are corresponding properties of the continuous almost periodic functions $f(t,x) \in C(R \times D, R^m)$, where D is an open set in R^m [11].

Theorem 0 ([3],[4]). *If $f, g : Z \to R^m$ are almost periodic sequences in n, then*

(i) *$f(n)$ is bounded on Z.*

(ii) *cf, $f+g$ and fg are almost periodic sequences in n, where c is constant real number.*

(iii) *$F(n)$ is almost periodic sequence in n if and only if $F(n)$ is bounded on z, where $F(n) = \sum_{j=0}^{n-1} f(j)$.*

(iv) *$F \circ f$ is almost periodic sequences in n, whenever $F(\cdot)$ is defined on the value field of $f(n)$.*

We shall denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n) = f(n+\tau), \quad \tau \in Z. \tag{1.2}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (1.2). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and $f(n+n_k) \to g(n)$ uniformly on Z. By (1.1), if $f : Z \to R^m$ is almost periodic sequence in n, so is a function in $\Omega(f)$. The following concept of asymptotic almost periodicity was introduced by Fr chet in the case of continuous function (cf. [11]).

Definition 2. $u(n)$ is said to be asymptotically almost periodic sequence if it is a sum of a almost periodic sequence $p(n)$ and a sequence $q(n)$ defined on $I^* = [a, \infty) \subset Z^+ = [0, \infty)$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n).$$

$u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{n_k\}$ for which $u(n+n_k)$ converges uniformly on n; $a \le n < \infty$.

2. Wazewska-Lasota Model

We consider the following delay difference equation

$$x(n+1) - x(n) = -\alpha(n)x(n+1) + \sum_{i=1}^{l} \beta_i(n)e^{-\gamma_i(n)x(n-\tau_i(n))} - H(n)x(n-\sigma(n)), \tag{2.1}$$

which describes a model of the dynamics of a discrete Wazewska-Lasota model in mathematical biology. In (2.1), we set $\alpha(n)$, $\beta_i(n)$, $\gamma_i(n)$, $\tau_i(n)$, $\sigma(n)$ and $H(n)$ are R^+-valued bounded almost periodic function on Z and $l \ge 1$ is a fixed integer. For a bounded sequence g defined on Z, we define by g^- and g^+ as follows

$$g^- = \min_i \left\{ \liminf_{n\to\infty} g(n) \right\} \quad \text{and} \quad g^+ = \max_i \left\{ \limsup_{n\to\infty} g(n) \right\}.$$

We now make the following assumptions;

(i) $\alpha^- > 0$, $\beta^- > 0$ and $\gamma^- > 0$,

(ii) $r = \max\{\tau^+, \sigma^+\}$,

(iii) there exist two positive constants K_1 and K_2 such that

$$K_1 > K_2, \quad \frac{\beta^+}{\alpha^-} l < K_1 \quad \text{and} \quad \frac{\beta^-}{\alpha^+} l e^{-\gamma^+ K_1} - \frac{H^+ K_1}{\alpha^+} > K_2.$$

Under the above assumptions, it follows that for any $(0,\varphi) \in Z^+ \times BS^+$ there is a unique solution $x(n) = x(n,0,\varphi)$ of equation (2.1) through $(0,\varphi)$, if it remains bounded. Here $BS^+ = \{\varphi \in BS : \varphi \geq 0, \varphi(0) > 0\}$.

We can show the following lemma.

Lemma 1. *If $x(n)$ is a solution of (2.1) through (n_0,ϕ) such that $K_2 < \phi(s) < K_1$ for all $-r \leq s \leq 0$, then we have*

$$K_2 < x(n) < K_1, \quad \textit{for all } n \geq n_0. \tag{2.2}$$

Proof. Let $[n_0, T) \subset [n_0, \infty)$ be an interval such that

$$x(n) > 0, \quad \text{for all } n \in [n_0, T).$$

First, we claim that

$$0 < x(n) < K_1, \quad \text{for all } n \in [n_0, T). \tag{2.3}$$

To do this, we first assume that (2.3) is not true. There exists an $n_1 \in (n_0, T)$ such that

$$x(n_1+1) = K_1 \ \text{ and } \ 0 < x(n) < K_1, \quad \text{for all } n \in [n_0 - r, n_1 + 1). \tag{2.4}$$

Then, it follows from equation (2.1) and (2.4) that

$$\begin{aligned}
&0 < x(n_1+1) - x(n_1) \\
&\ = -\alpha(n_1)x(n_1+1) + \sum_{i=1}^{l} \beta_i(n) e^{-\gamma_i(n_1)x(n_1-\tau_i(n_1))} \\
&\ - H(n_1)x(n_1 - \sigma(n_1)) \\
&\ \leq -\alpha^- x(n_1+1) + \beta^+ l \\
&\ = \alpha^- \left[-K_1 + \frac{\beta^+}{\alpha^-} l \right] \\
&\ < 0,
\end{aligned}$$

which is a contradiction and this implies that (2.3) holds. We next show that

$$x(n) > K_2, \quad \text{for all } n \in [n_0, T). \tag{2.5}$$

On the contrary, we assume that there exists $n_2 \in (n_0, T)$ such that

$$x(n_2+1) = K_2 \ \text{ and } \ x(n) > K_2, \quad \text{for all } n \in [n_0 - r, n_2 + 1). \tag{2.6}$$

In virtue of (2.3), we obtain

$$K_2 < x(n) < K_1, \quad \text{for all } n \in [n_0 - r, n_2 + 1). \tag{2.7}$$

In view of (2.1), assumption (iii), (2.6) and (2.7), we have

$$
\begin{aligned}
0 &> x(n_2+1)-x(n_2)\\
&= -\alpha(n_2)x(n_2+1)+\sum_{i=1}^{l}\beta_i(n)e^{-\gamma_i(n_2)x(n_2-\tau_i(n_2))}\\
&\quad -H(n_2)x(n_2-\sigma(n_2))\\
&\geq -\alpha^+K_2+\beta^- l e^{-\gamma^+K_1}-H^+K_1\\
&\geq \alpha^+\left[-K_2+\frac{\beta^-}{\alpha^+}le^{-\gamma^+K_1}-\frac{H^+K_1}{\alpha^+}\right]\\
&> 0,
\end{aligned}
$$

which is a contradiction and this implies that (2.5) holds. Since (2.3) and (2.5), it follows that relation (2.2) is true. This proof is complete. □

We now consider the linear system

$$x(n+1)=A(n)x(n), \quad n\in Z, \tag{2.8}$$

where $A(n)$ is an $m\times m$ invertible matrix sequence for each $n\in Z$. We denote by $\|\cdot\|$ any convenient norm either of a vector or of a matrix.

In what follows, we need the following definition of dichotomy.

Definition 3. System (2.8) is said to possess an exponential dichotomy on Z if there exists a projection P, that is, an $l\times l$ matrix P such that $P^2=P$, and constants $G>0$, $\nu>0$ such that

$$
\begin{aligned}
\|X(t)PX^{-1}(s+1)\| &\leq G\left(\frac{1}{1+\nu}\right)^{(t-s-1)}, \quad t\geq s,\\
\|X(t)(I-P)X^{-1}(s+1)\| &\leq G\left(\frac{1}{1+\nu}\right)^{(s-t+1)}, \quad s\geq t,
\end{aligned}
$$

where $X(t)$ is the fundamental solution matrix of (2.8) such that $X(0)=I$ and $t,s\in Z$.

We also consider the following almost periodic system

$$x(n+1)=A(n)x(n)+f(n), \quad n\in Z, \tag{2.9}$$

where $A(n)$ is an invertible $m\times m$ almost periodic matrix sequence for each $n\in Z$ and $f(n)$ is an almost periodic sequence from $Z\to R^m$.

Theorem A ([1, 5, 12]). *If the linear system* (2.8) *admits exponential dichotomy, then system* (2.9) *has a bounded solution* $x(n)$ *in the form*

$$x(n)=\sum_{m=-\infty}^{n-1}X(n)PX^{-1}(m+1)f(m)-\sum_{m=n}^{\infty}X(n)(I-P)X^{-1}(m+1)f(m),$$

where $X(n)$ *is the fundamental solution matrix of* (2.8).

Theorem B ([1, 5, 12]). *If $\alpha(n)$ is an almost periodic sequence on Z, where $1-\frac{\alpha(n)}{1+\alpha(n)}>0$ for all $n\in Z$ and $\inf_{n\in Z}\frac{\alpha(n)}{1+\alpha(n)}>0$, then the linear system*

$$x(n+1)-x(n)=-\alpha(n)x(n+1)$$

admits an exponential dichotomy on Z.

3. Main Theorems

We set

$$K=\{\phi\mid\phi\text{ is an almost periodic sequence on }Z\}.$$

We define the norm $\|\phi\|_K=\sup_{n\in Z}|\phi(n)|$, for any $\phi\in K$, then one can easily deduce that K is a Banach space. Now we shall see that the existence of a strictly positive almost periodic solution of equation (2.1) can be obtained under assumptions (i), (ii), (iii) and

(iv) $$\frac{\beta^+ l}{\alpha^- e}+\frac{H^+}{\alpha^-}<1.$$

Theorem 1. *We assume conditions* (i), (ii), (iii) *and* (iv). *Then equation* (2.1) *has a unique positive almost periodic solution, $\phi\in K^*$, in the set*

$$K^*:=\{\phi\in K\mid K_2<\phi(n)<K_1,\ \textit{for all } n\in Z\}.$$

Proof. For any $\phi\in K$, we consider an equation

$$x(n+1)-x(n)=-\alpha(n)x(n+1)+\sum_{i=1}^{l}\beta_i(n)e^{-\gamma_i(n)\phi(n-\tau_i(n))}-H(n)\phi(n-\sigma(n)),$$

Since $\inf_{n\in Z}\frac{\alpha(n)}{1+\alpha(n)}>0$, it follows from Theorem B that the linear system

$$x(n+1)-x(n)=-\alpha(n)x(n+1)$$

admits an exponential dichotomy on Z. By Theorem A, we deduce that equation (2.1) has a bounded solution

$$x^\phi(n)=\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha(r)}\right)\left[\sum_{i=1}^{l}\beta_i(m)e^{-\gamma_i(m)\phi(m-\tau_i(m))}-H(m)\phi(m-\sigma(m))\right].$$

In virtue of Theorem A and the almost periodicity of parameters of equation (2.1), we deduce that x^ϕ is also almost periodic. Define a mapping $T:K\to K$ by setting

$$T(\phi(n))=x^\phi(n),\quad\text{for all }\phi\in K.$$

It is easy to see that K^* is a closed subset of K. For any $\phi\in K^*$, we have

$$\begin{aligned}x^\phi(n)&\le\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha(r)}\right)\sum_{i=1}^{l}\beta_i(m)e^{-\gamma_i(m)\phi(m-\tau_i(m))}\\&\le\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha^-}\right)\beta^+ l.\end{aligned}$$

Using that $\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha^-}\right)=\frac{1}{\alpha^-}$ we end up with

$$x^{\phi}(n)\le\frac{\beta^+l}{\alpha^-}<K_1,\quad \text{for all } n\in Z.$$

On the other hand, we have

$$\begin{aligned}x^{\phi}(n)&=\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha(r)}\right)\left[\sum_{i=1}^{l}\beta_i(m)e^{-\gamma_i(m)\phi(m-\tau_i(m))}-H(m)\phi(m-\sigma(m))\right]\\&\ge\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha(r)}\right)\left[\sum_{i=1}^{l}\beta_i(m)e^{-\gamma^+\phi(m-\tau_i(m))}-H^+K_1\right].\end{aligned}$$

Thus, we obtain

$$x^{\phi}(n)\ge\frac{\beta^-l}{\alpha^+}e^{-\gamma^+K_1}-\frac{H^+K_1}{\alpha^+}>K_2,\quad \text{for all } n\in Z.$$

This tells that the mapping T is a self-mapping from K^* to K^*. Let $\phi,\psi\in K^*$. Then,

$$\begin{aligned}&\|T(\phi)-T(\psi)\|_{K^*}\\&\quad=\sup_{n\in Z}|T(\phi(n))-T(\psi(n))|\\&\quad=\sup_{n\in Z}\Bigg|\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha(r)}\right)\Bigg[\sum_{i=1}^{l}\beta_i(m)\{e^{-\gamma_i(m)\phi(m-\tau_i(m))}-e^{-\gamma_i(m)\psi(m-\tau_i(m))}\}\\&\qquad-H(m)\{\phi(m-\sigma(m))-\psi(m-\sigma(m))\}\Bigg]\Bigg|.\end{aligned}$$

Since $\sup\limits_{u\ge1}\frac{1}{e^u}=\frac{1}{e}$, we obtain

$$\begin{aligned}|e^{-x}-e^{-y}|&=\left|\frac{1}{e^{x+\theta(y-x)}}\right||x-y|\\&\le\frac{1}{e}|x-y|,\quad x,y\in[1,\infty),\ 0<\theta<1.\end{aligned}\tag{3.1}$$

Therefore, by (2.1), assumption (i) and (3.1), we have

$$\begin{aligned}&\|T(\phi)-T(\psi)\|_{K^*}\\&\quad\le\sup_{n\in Z}\Bigg|\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\left(\frac{1}{1+\alpha^-}\right)\Bigg[\frac{1}{e}\sum_{i=1}^{l}\beta_i(m)|\phi(m-\tau_i(m))-\psi(m-\tau_i(m))|\\&\qquad+H(m)|\phi(m-\sigma(m))-\psi(m-\sigma(m))|\Bigg]\Bigg|\\&\quad\le\left(\sup_{n\in Z}\sum_{m=-\infty}^{n-1}\prod_{r=m}^{n-1}\frac{1}{1+\alpha^-}\right)\left(\frac{1}{e}\sum_{i=1}^{l}\beta_i(m)+H^+\right)\|\phi-\psi\|.\end{aligned}$$

Thus, we end up with

$$\|T(\phi)-T(\psi)\|_{K^*} \leq \left(\frac{\beta^+ l}{\alpha^- e} + \frac{H^+}{\alpha^-}\right)\|\phi-\psi\|,$$

which implies by assumption (iv) that the mapping T is contractive on K^*. Therefore, the mapping T possesses a unique fixed point $\phi^* \in K^*$ such that $T\phi^* = \phi^*$. Thus, ϕ^* is an almost periodic solution of (2.1) in the set K^*. The proof of this theorem is complete. □

Let $x^*(n)$ be a positive almost periodic solution of (2.1) in the set K^*. We assume that

(v) $$\alpha^- > 1 + \frac{\beta^+ l}{e} + H^+.$$

Theorem 2. *If we assume conditions* (i), (ii), (iii) *and* (v), *then the solution* $x(n)$ *of equation* (2.1) *with* $\phi \in K^*$ *converges exponentially to* $x^*(n)$ *as* $n \to \infty$.

Proof. Set $x(n) = x(n, n_0, \phi)$ and $y(n) = x(n) - x^*(n)$, where $n \in [n_0 - r, \infty)$. Then,

$$\begin{aligned} y(n+1)-y(n) = -\alpha(n)y(n+1) + \sum_{i=1}^{l} \beta_i(n)[e^{-\gamma_i(n)x(n-\tau_i(n))} - e^{-\gamma_i(n)x^*(n-\tau_i(n))}] \\ - H(n)y(n-\sigma(n)), \end{aligned} \tag{3.2}$$

From the result of Lemma 1, $x(n)$ is positive and bounded on $[n_0, \infty)$, and

$$K_2 < x(n) < K_1, \quad \text{for all } n \in [n_0 - r, \infty).$$

Define a function $\Phi(u)$ by setting

$$\Phi(u) = e^u - \alpha^- + \frac{\beta^+ l}{e} e^{u(r+1)} + H^+ e^{u(r+1)}, \quad u \in [0,1].$$

It is clear that Φ is continuous on $[0,1]$. Then, by assumption (v), we have

$$\Phi(0) = 1 - \alpha^- + \frac{\beta^+ l}{e} + H^+ < 0,$$

which implies that there exist two constants $\eta > 0$ and $0 < \lambda \leq 1$ such that

$$\Phi(\lambda) = e^\lambda - \alpha^- + \frac{\beta^+ l}{e} e^{\lambda(r+1)} + H^+ e^{\lambda(r+1)} < -\eta < 0. \tag{3.3}$$

We consider the discrete Lyapunov functional

$$V(n) = |y(n)|e^{\lambda n}.$$

Calculating the difference of $V(n)$ along the solution $y(n)$ of (3.2), we have

$$\begin{aligned}
V(n+1)-V(n) &= \Delta V(n) = \Delta(|y(n)|e^{\lambda n}) \\
&= \Delta|y(n)|e^{\lambda(n+1)} + |y(n)|\Delta e^{\lambda n} \\
&\leq -\alpha(n)|y(n+1)|e^{\lambda(n+1)} + \sum_{i=1}^{l}\beta_i(n)|e^{-\gamma_i(n)x(n-\tau_i(n))} \\
&\quad - e^{-\gamma_i(n)x^*(n-\tau_i(n))}|e^{\lambda(n+1)} + H(n)|y(n-\sigma(n))|e^{\lambda(n+1)} \\
&\quad + |y(n)|(e^{\lambda(n+1)} - e^{\lambda n}) \\
&\leq |y(n)|e^{\lambda(n+1)} - \alpha(n)|y(n+1)|e^{\lambda(n+1)} \\
&\quad + \Bigg[\sum_{i=1}^{l}\beta_i(n)|e^{-\gamma_i(n)x(n-\tau_i(n))} - e^{-\gamma_i(n)x^*(n-\tau_i(n))}| \\
&\qquad + H(n)|y(n-\sigma(n))|\Bigg]e^{\lambda(n+1)}, \quad \text{for all } n \geq n_0. \quad (3.4)
\end{aligned}$$

Let

$$M := e^{\lambda n_0}\Big(\max_{n\in[n_0,\infty)}|\varphi(n)-x^*(n)|+1\Big), \quad \text{for all } n \geq n_0.$$

Then, we claim that

$$V(n) = |y(n)|e^{\lambda n} < M, \quad \text{for all } n \geq n_0. \tag{3.5}$$

We assume, on the contrarily, that there exists $n^* > n_0$ such that

$$V(n^*) = M \;\text{ and }\; V(n) < M, \quad \text{for all } n \in [n_0 - r, n^*), \tag{3.6}$$

which implies that

$$V(n^*) - M = 0 \;\text{ and }\; V(n) - M < 0, \quad \text{for all } n \in [n_0 - r, n^*).$$

Since (3.1), (3.4) and (3.6), we obtain

$$\begin{aligned}
0 &\leq \Delta(V(n^*) - M) \\
&= \Delta V(n^*) \\
&\leq |y(n^*)|e^{\lambda(n^*+1)} - \alpha(n^*)|y(n^*+1)|e^{\lambda(n^*+1)} \\
&\quad + \Bigg[\sum_{i=1}^{l}\beta_i(n^*)|e^{-\gamma_i(n^*)x(n^*-\tau_i(n^*))} - e^{-\gamma_i(n^*)x^*(n^*-\tau_i(n^*))}|\Bigg]e^{\lambda(n^*+1)} \\
&\quad + H(n^*)|y(n^* - \sigma(n^*))|e^{\lambda(n^*+1)}
\end{aligned}$$

$$\begin{aligned}
&\leq |y(n^*)|e^{\lambda(n^*+1)} - \alpha(n^*)|y(n^*+1)|e^{\lambda(n^*+1)} \\
&\quad + \frac{1}{e}\sum_{i=1}^{l}\beta_i(n^*)|y(n^*-\tau(n^*))|e^{\lambda(n^*+1-\tau_i(n^*))}e^{\lambda\tau_i(n^*)} \\
&\quad + H(n^*)|y(n^*-\sigma(n^*))|e^{\lambda(n^*+1-\sigma(n^*))}e^{\lambda\sigma(n^*)} \\
&\leq (e^{\lambda}-\alpha^-)M + \frac{\beta^+ l}{e}Me^{\lambda(r+1)} + H^+Me^{\lambda(r+1)} \\
&= \left[e^{\lambda} - \alpha^- + \frac{\beta^+}{e}le^{\lambda(r+1)} + H^+e^{\lambda(r+1)}\right]M.
\end{aligned}$$

Thus,

$$e^{\lambda} - \alpha^- + \frac{\beta^+}{e}le^{\lambda(r+1)} + H^+e^{\lambda(r+1)} \geq 0,$$

which contradicts (3.3). Hence (3.5) holds. It follows that $|y(n)| < Me^{-\lambda n}$ for all $n \geq n_0$. This proof is complete. □

4. Example

For simplicity, we consider the following discrete Wazewska-Lasota model with 2-delays and linear harvesting term of the form

$$\begin{aligned}
x(n+1) - x(n) = &-(18+\cos^2 n)x(n+1) \\
&+ (e-1)(10+0.003|\sin\sqrt{5}n|)e^{-\frac{19}{22^2}x(n-e^{|\cos\sqrt{3}n|})} \\
&+ (e-1)(10+0.005|\sin\sqrt{7}n|)e^{-\frac{19}{22^2}x(n-e^{|\cos\sqrt{2}n|})} \\
&- \frac{(18+\sin^2 n)|\sin n|}{10000}x(n-e^{|\cos\sqrt{5}n|}),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&\alpha(n) = 18+\cos^2 n, \\
&\beta_1(n) = (e-1)(10+0.003|\sin\sqrt{5}n|), \quad \beta_2(n) = (e-1)(10+0.005|\sin\sqrt{7}n|), \\
&\tau_1(n) = e^{|\cos\sqrt{3}n|}, \quad \tau_2(n) = e^{|\cos\sqrt{2}n|}, \quad \gamma_1(n) = \gamma_2(n) = \frac{19}{22^2}, \quad l = 2,
\end{aligned}$$

and

$$\sigma(n) = e^{|\cos\sqrt{5}n|}, \quad H(n) = \frac{(18+\sin^2 n)|\sin n|}{10000}.$$

It is clear that

$$\alpha^- = 18,\ \alpha^+ = 19, \quad \beta^- = 10(e-1), \quad \beta^+ = 10.005(e-1), \quad \gamma^- = \gamma^+ = \frac{19}{22^2},$$

and

$$\tau^+ = \sigma^+ = e,\ H^- = 0 \quad \text{and} \quad H^+ = \frac{19}{10000}.$$

Thus, assumptions (i) and (ii) hold. Let $K_1 = \frac{22}{19}(e-1)$ and $K_2 = 1$. Then, we have

$$K_1 = \frac{22}{19}(e-1) > 1 = K_2, \quad \frac{\beta^+}{\alpha^-} l \approx 1.91, \quad \frac{\beta^-}{\alpha^+} l e^{-\gamma^+ K_1} - \frac{H^+ K_1}{\alpha^+} \approx 1.80 > 1,$$

and this shows that assumption (iii) is satisfied. It remains to check assumptions (iv) and (v). However, one can see the validity of these assumptions since

$$\frac{\beta^+ l}{\alpha^- e} + \frac{H^+}{\alpha^-} \approx 0.70 < 1 \quad \text{and} \quad 1 + \frac{\beta^+ l}{e} + H^+ \approx 13.65 < 18.$$

Therefore, we conclude that all assumptions of Theorem 1 and Theorem 2 are fulfilled. Hence, equation (4.1) has a positive almost periodic solution $x^* \in K^* = \{\phi \in K | 1 < \phi(n) < \frac{22}{19}(e-1), \text{ for all } n \in Z\}$. Moreover, if $\phi \in K^*$, then solution $x(n)$ converges exponentially to $x^*(n)$ as $n \to \infty$.

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