Communications in Mathematics and Applications
Vol. 14, No. 1, pp. 349–373, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i1.1920

http://www.rgnpublications.com

Research Article

Interconnection Networks via Adjacencies and Their Vertex-Degree Based Graph Invariants

S. Sarah Surya and P. Subbulakshmi*

Department of Mathematics, Stella Maris College (Autonomous) (Affiliated to the University of Madras), Chennai, India

***Corresponding author:** subbulakshmi216@gmail.com

Received: May 30, 2022 **Accepted:** November 5, 2022

Abstract. Interconnection Networks are a boon to human made large computing systems which are often designed based on the need. In this paper, we address the question of obtaining an interconnection network to suit a specific need which is a cumbersome procedure. The methods for constructing infinitely large interconnection networks from any simple, undirected graph G is illustrated and their properties are discussed. Further, we study the behaviour of certain vertex-degree based graph invariants of the constructed networks under the transformations by means of graph operation and obtain explicit relations for energy and some degree-based topological indices. Also, we show that they can be computed in polynomial time for these constructed networks. By doing so, we also obtain new methods of constructing infinite families of integral graphs.

Keywords. Interconnection networks, Energy, Degree based topological indices, Integral graphs

Mathematics Subject Classification (2020). 05C50, 34L16

1. Introduction

Due to the steady increase of computation power being driven with the rapid advancement of technology, very large scale multi-processor systems which exploit parallelism have been put to practical use which employ interconnection networks for their inter-processor communication. Such interconnection networks can be modelled by a graph when the nodes are represented as the set of vertices and the links connecting them constitute the set of edges (Cheng *et al.* [7]). Feng [13] affirm the fact that concurrent processing depends on interconnection networks for communication among processors and memory modules. Interconnection networks are also essential when large quantums of data cannot be handled on a single processor. In such instances, the entire task can be split up into small-scale processes which can be performed simultaneously.

The architecture designed for such purposes often utilizes conventional topologies which are mostly structured, connected and have a fixed way of extending to higher dimensions (Jurczyk *et al*. [22]). But in reality, there is a dire need for unconventional, sometimes disconnected large networks which should satisfy the critical necessities of the real time network like increased fault tolerance, reduced transmission delay, etc. and at the same time continue to carry forward the properties of the original graph in the rapidly expanding and changing dynamic network (Ghasempour *et al*. [14], Fan *et al*. [34], and Zhu *et al*. [54]). This brings in the need of structured interconnection networks which can be derived from the previous dimension by a standardized procedure. The procedures can vary from merely adding edges to compositions which specify few constraints on them. Several such methods for constructing networks by extending the graphs chosen with desired properties have been discussed in (Indulal and Vijaykumar [20], Vaidya and Potal [45, 46]. It is always interesting to study the behaviour of certain invariants when the graphs undergo tranformation by means of a graph operation.

Graph energy was first formally put forward as a mathematical definition in 1978, whose chemical roots traces back to 1940s. After this, for about 20 years, graph energy was a dormant area of research and later it flourished into an enormous field of investigation attracting the interest of large number of researchers. Today, a number of variants of this mathematical quantity are confronted in apparently distinct areas such as analysis, probability, matrix theory, etc. (Li *et al*. [27], and Mihalić *et al*. [35]). Apart from the chemical significance of spectral graph theory, it also finds application in various other fields. Though finding the exact value of energy for large families of graphs is an extremely cumbersome task, it undoubtedly has a lot of advantages. Several bounds have been obtained for the energy of various classes of graphs. To know more about these, one may refer to (Li *et al*. [27], and Liu *et al*. [31]). Rarely, exact formulae of energy have been established for certain family of graphs (Chen and Xie [6], Louis [32], Suntornpoch and Meemark [40], and Surya and Subbulakshmi [41]). Likewise, other spectral properties such as the largest eigenvalue have also been studied for various families of graphs and several bounds have been obtained (Li and Feng [25], and Lovász and Pelikán [33]). These help us in studying some more intricate properties about the spectrum and energy, namely equienergetic graphs, integral graphs and cospectral graphs, construction of new classes of graphs and propose several methods for the same (Le *et al*. [26]).

The concept of integral graphs was introduced by Harary and Schwenk in 1974 [17]. Characterization of integral graphs seems to be very difficult (Adiga *et al*. [1]). Hence, construction of new classes of integral graphs becomes interesting. Further advancements on this topic can be found in (Adiga *et al*. [1], and Balinska *et al*. [4]).

Topological Indices is yet another research topic which has originated from chemistry. They are graph invariants that characterise their topological structure. The motivation of works in this area comes from the wide applications and the vast expanse of works done in this topic (Dehmer [9]).

In this paper, we enumerate different methods for construction of interconnection networks which can also be visualized as a graph undergoing a transformation under a certain graph operation. This enables us to derive exact analytical expressions for energy, spectral radius, least eigenvalue and some degree based topological indices namely, first Zagreb index, first modified Zagreb index, simple topological index and total adjacency index for the constructed interconnection networks in terms of these parameters of the underlying graph. The obtained polynomial time computable expressions have been verified using MATLAB[1]. As a consequence, we also construct six new infinite classes of integral graphs. In addition, the computed spectral parameters and the degree based topological indices of the various interconnection networks constructed are compared both numerically and graphically.

2. Motivation

Interconnection networks have become an integral part in many cutting edge technologies such as High-Performance Computing (HPC) platforms (Yasudo *et al*. [51]), 3D-TTN which is a power efficient cost effective high performance hierarchical interconnection network for the next generation green supercomputer (Faisal *et al*. [11]), DPillar network which can accomodate millions of servers for a future data center (Liao *et al*. [29]), in packet-switched networks of large systems with many nodes (Lüdtke and Tutsch [23]) and in various other applications that help in realizing computers clocked at several GHz (Jain [21]). These applications cover a wide range of different scientific applications, including weather forecasts, physical simulations, molecular modeling, nuclear research, quantum mechanics, and artificial intelligence (Zahn [52]).

The usefulness of the concept of graph energy can be observed in the context of large, complex networks. It has been shown that when graph energies are applied to local egocentric networks and some generative network models, the values of these energies correlate strongly with vertex centrality measures, particularly with the betweenness and the eigencentrality of vertices. As the exact computation of these centrality measures is expensive and requires global processing of a network, this result establishes a significant success in the field of network science [36].

The growth rate of a transportation network respresented by a dichotomous matrix can be described by its spectral radius (Ulanowicz [43]). The spectral radius of the channel graph have been known to bound the channel capacity (Cohn [8]). Further, the maximum entropy of a unifilar Markov information source can be expressed in terms of the largest eigenvalue of its connection matrix (Immink [19]). The least eigenvalue provides information about independence number and chromatic number of the graph and interlacing gives information about its substructures (Fan *et al*. [12], and Tan and Fan [42]).

Topological indices have been found propitious in various instances, to mention a few, to quantify the pattern of interactions of trophic processes (Ulanowicz [44]), to gauge the structural properties of networks which are very beneficial connecting experimental data with a functional interpretation in biological terms (Emmert-Streib and Dehmer [10]), in characterizing networks based on an information theoretic approach (Wilhelm [49]) and as correlation measures to characterize the structure of complex networks (Solé and Valverde [39]).

[1]The Matlab Team, *MATLAB*, The Mathworks, Inc., Natick, Massachusetts, Version 9.3.0.713579 (R2017b) (2017).

3. Preliminaries

Definition 1 ([5]). The *spectrum* of a graph is the list of eigenvalues of the adjacency matrix together with its multiplicities. We write $Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_p \\ m_1 & m_2 & \dots & m_p \end{pmatrix}$, where $\lambda_1, \lambda_2, \dots, \lambda_p$ are the eigenvalues and m_i is the multiplicity of λ_i, $1 \le i \le p$.

Definition 2 ([15]). The *energy* of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. i.e., if A is a $n \times n$ matrix with n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Definition 3 ([12, 47]). If the eigenvalues of the adjacency matrix of a graph G are written in a nonincreasing manner, i.e., $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$, then λ_1 is referred to as the *spectral radius* or (*index*) and λ_n is the *least eigenvalue* of G.

Definition 4 ([30]). A graph whose spectrum consists entirely of integers is called an *integral graph*.

Definition 5 ([48]). The *Kronecker product* of $A = [a_{ij}] \in M_{m,n}(R)$ and $B = [b_{ij}] \in M_{p,q}(R)$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \in M_{mp,nq}(R).$$

Definition 6 ([53]). The first Zagreb index Z_1 is equal to the sum of the squares of the degrees of the vertices of the underlying graph.

Definition 7 ([16]). For a simple connected graph $G(V,E)$, $\sum_{v \in V(G)} \frac{1}{(d(v))^2}$ is called the first modified Zagreb index, denoted by MZ_1.

Definition 8 ([38]). The simple topological index P_G for a graph G is defined as the product of the order $d(G,k)$'s, where $d(G,k)$ is the number of edges which radiate from the kth vertex.

Definition 9 ([3]). The total adjacency index is the sum A' of all matrix elements in the upper triangular adjacency matrix, i.e., $A' = \sum_{i,j=1, i \le j}^{N} a_{ij}$.

Theorem 1 ([18]). *Let A and B be square matrices of order m and n respectively with real entries. If λ is an eigenvalue of A with corresponding eigenvector x and if μ is an eigenvalue of B with corresponding eigenvector y, then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

4. Construction of Interconnection Networks via Adjacencies

Let G be a graph with vertex set $V = \{u_i\}$, $1 \le i \le n$ and $N(u_i)$ denote the set of vertices which are adjacent to u_i. In this section, we construct the following new interconnection networks via their adjacencies.

Construction 1. The *k-layer enveloping graph* $LE_k(G)$, $k \geq 3$ of G is obtained by taking the union of G with $(k-1)$ duplicated copies of G, denoted by G_j, $1 \leq j \leq k-1$ with $V(G_j) = \{u_{ij}\}$ and such that

(i) $u_{i(k-1)}$ is adjacent to $N(u_i)$,

(ii) $u_{i(k-l-1)}$ is adjacent to $N(u_{il})$, where $1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor$.

See Figure 1.

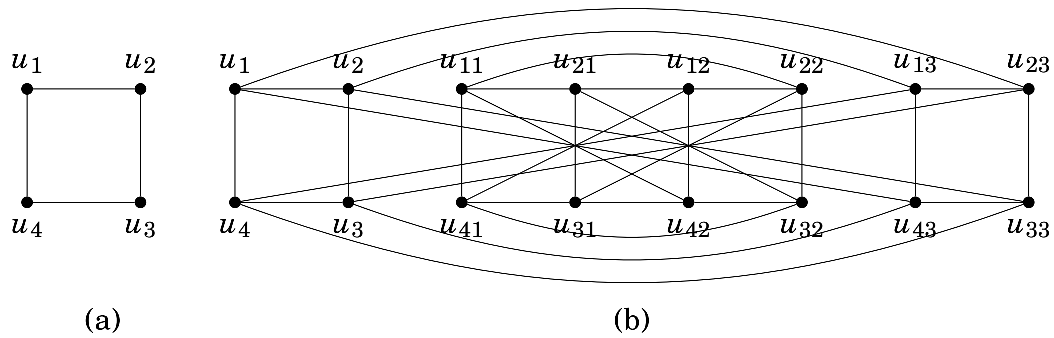

Figure 1. (a) C_4, (b) $LE_4(C_4)$

Construction 2. The *k-replica cleaving graph* $RC_k(G)$, $k \geq 3$ of any graph G is obtained by taking the union of G with $(k-1)$ replicated copies of G and each vertex in these replicated copies is adjacent to all the neighbours of the corresponding vertex in the given graph.

Construction 3. The *k-spin cleaving graph* $SC_k(G)$, $k \geq 3$ for any graph can be obtained by adding to each vertex of G, new $k-1$ vertices $u_{i1}, u_{i2}, \dots, u_{i(k-1)}$ such that

(i) $u_{i(k-1)}$ is adjacent to the neighbours of u_i,

(ii) $u_{i(k-l-1)}$ is adjacent to the neighbours of u_{il}, where $1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor$.

Construction 4. The *k-penultimate cleaving graph* $PC_k(G)$, $k \geq 3$ of G is obtained by adding to each vertex u_i of G, new $k-2$ vertices $u_{i_1}, u_{i_2}, \dots, u_{i_{k-2}}$ such that

(i) $u_{i_1}, u_{i_2}, \dots, u_{i_{k-2}}$ is adjacent to the neighbours of u_i,

(ii) take the union of the above resultant graph with G.

Construction 5. The *k-penultimate enveloping graph* $PE_k(G)$, $k \geq 3$ for any graph G is constructed by taking $k-2$ copies $G_1, G_2, \dots G_{k-2}$ of the graph such that,

(i) every vertex in G_i is adjacent to the neighbours of the corresponding vertex in G_j, $1 \leq i$, $j \leq k-2$,

(ii) take the union of the resultant graph with G.

Construction 6. The *k-semi eveloping graph* $SE_k(G)$ of G can be constructed by taking $\lfloor \frac{k-1}{2} \rfloor$ copies of G and $\lceil \frac{k-1}{2} \rceil$ copies of each vertex of G such that

(i) each vertex in the replicated copies of graph is adjacent to all the neighbours of the corresponding vertex in the given graph and each n^{th} replicated copy is adjacent to all the neighbours of the corresponding vertices in the 1^{st} to $(n-1)^{th}$ copy for $1 \leq n \leq \left\lfloor \frac{k-1}{2} \right\rfloor$,

(ii) each vertex in the replicated copies of the vertices are made adjacent to all the neighbours of the corresponding vertex in the given graph and each vertex in the m^{th} copy of the vertex taken should be made adjacent to the neighbours of the corresponding vertices of the 1st to $\left(\left\lceil\frac{k-1}{2}\right\rceil-k\right)$th copy of the vertex.

Here, we take $k \geq 3$.

Construction 7. The *k-inceptive enveloping graph*, $IE_k(G)$, $k \geq 3$ of G can be constructed by taking one copy of the given graph and $k-2$ copies of its vertices. Further, all the newly added vertices of the copied graph and the copy of the vertices should be made adjacent to the neighbours of the corresponding vertices of the given graph and its copy.

Construction 8. The *k-vertex enveloping graph*, denoted by $VE_k(G)$, $k \geq 3$ for any graph G is obtained by attaching $k-1$ copies of each vertex of G and then joining every copy of the vertex to all the neighbours of the corresponding vertex in all the copies.

Construction 9. The *k-edge depleted enveloping graph of G*, $DE_k(G)$, $k \geq 3$ can be constructed by taking $k-2$ copies of the given graph and one copy of each vertex. To complete the construction, we shall make the vertices in the replicated copies adjacent to all the neighbours of the corresponding vertex in all the copies and the copy of the vertex.

Construction 10. The *k-terminal over enveloping graph*, $TE_k(G)$, $k \geq 3$ of any graph G is obtained by adding $k-1$ copies of the graph and making each of the vertices in these copies adjacent to the neighbours of the corresponding vertex in all the copies except the vertices of $(k-2)^{th}$ copy which is adjacent to the neighbours of the $(k-1)$th copy.

Construction 11. The *k-circumscribed enveloping graph of G*, $CE_k(G)$, $k \geq 3$ is obtained by attaching $k-2$ copies of vertices of the given graph and it's copy once. After which, all the attached vertices are made adjacent to the corresponding neighbours of all the vertices in the newly attached copy of the graph and the copy of the vertices.

5. Realization as Interconnection Networks

This section delibrates the properties of the interconnection networks defined in Section 4 by means of graph operations. We observe that all the graphs constructed by the aforemention methods possess $k|V(G)|$ vertices and attain the status of an interconnection network due to the following reasons as stated in [50].

(1) *Fixed number of interfaces:* For an efficient interconnection network, fixed degree of its vertices is desirable. In all the constructed interconnection networks, the degree of all the vertices can be uniquely determined with the information about the degree of the vertices in the graph which is used to generate it. Through this, the achieved fixed degree of the vertices of all the interconnection networks constructed can also be maximized to any extent based on the dimension of the chosen extension, which is definitely a favourable feature for its successful implementation.

(2) *Minimized Transmission Delay:* The time taken for an information to be transmitted corresponds to the small diameter of the underlying interconnection network. Also, it is well known that the density of any graph is inversely proportional to that of its diameter

and density is the average number of links per node [2, 24]. Hence, interconnection networks with large density are preferred for practical purposes. The density of the interconnection networks defined above are recorded in terms of the density $d(G)$ of the underlying graph G in Table 1.

Table 1. Density of the constructed interconnection networks

Operations	Density
Layer Enveloping	$\left(2-\frac{1}{k}\right)d(G)$
Replica Cleaving	$\left(3-\frac{2}{k}\right)d(G)$
Spin Cleaving	$\left(1+\frac{1}{k}\right)d(G)$
Penultimate Cleaving	$2\left(1-\frac{1}{k}\right)d(G)$
Penultimate Enveloping	$\left(k-2+\frac{2}{k}\right)d(G)$
Semi Enveloping	$\left(\frac{k+1}{2}\right)d(G)$
Inceptive Enveloping	$4\left(1-\frac{1}{k}\right)d(G)$
Vertex Enveloping	$\left(k-1+\frac{1}{k}\right)d(G)$
Edge Depleted Enveloping	$\left(k-\frac{1}{k}\right)d(G)$
Terminal Over Enveloping	$\left(k-\frac{2}{k}\right)d(G)$
Circumscribed Enveloping	$\left(k-1+\frac{2}{k}\right)d(G)$

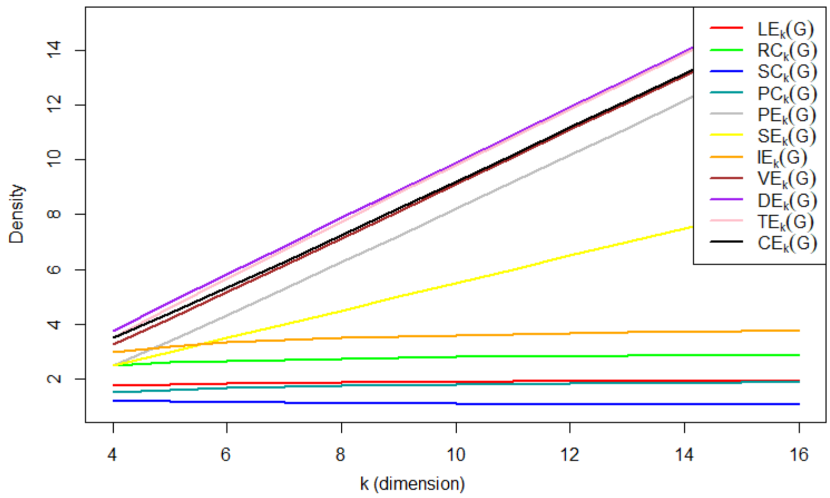

Figure 2. Density of the constructed Interconnection Networks

As we see, most of the networks are of high density which increases with increase in k. In some cases, to achieve a reasonable density, a smaller k should be designated. Figure 2 shows the plot with the ratio of the density of the interconnection network to the density of the underlying graph plotted against its dimension.

(3) *Maximum Fault Tolerance:* Maximizing fault tolerance in an interconnection network is usually correlated with maximizing its edge connectivity. All the defined operations except,

layer enveloping, spin cleaving, penultimate cleaving and penultimate enveloping generate connected networks when the underlying graph G is connected. The edge connectivity of the constructed interconnection networks are tabulated with reference to the edge connectivity, $\kappa'(G)$ of their underlying connected graphs G in Table 2.

Table 2. Edge Connectivity of the constructed interconnection networks

Operations	Edge Connectivity
Replica Cleaving	2deg(v)
Semi Enveloping	deg(v)
Inceptive Enveloping	2deg(v)
Vertex Enveloping	$(k-1)$deg(v)
Edge Depleted Enveloping	$(k-1)$deg(v)
Terminal Over Enveloping	$(k-1)$deg(v)
Circumscribed Enveloping	$(k-1)$deg(v)

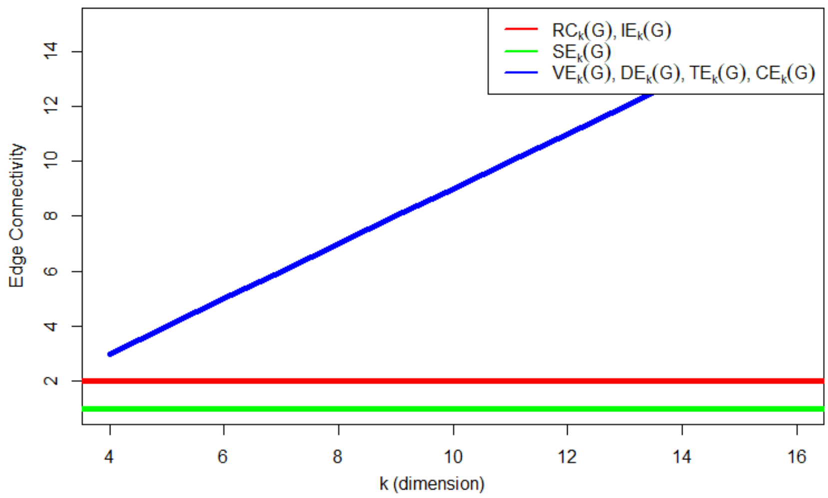

Figure 3. Edge Connectivity of the constructed Interconnection Networks

It is evident that the edge connectivity of the constructed networks depend on the vertex v of G with the smallest degree, denoted by deg(v). Also, it is convincing that as the network becomes larger and larger with increase in k, both the factors deg(v) and $k-1$ increase which widens the scope of attaining interconnection networks with maximum edge connectivity. It is worth noting that disconnected graphs can also be constructed with the operations tabulated in Table 3 if the underlying graph G taken is disconnected. In this case, the number of components in the obtained interconnection network will be equal to that of the underlying graph G. A plot with the ratio of the edge connectivity of constructed interconnection networks to the edge connectivity of the underlying graph plotted against its dimension is displayed in Figure 3.

(4) *Scope of Easy Routing Algorithms:* The construction of interconnection networks that are proposed here enhances the number of vertices and their degree via adjacency relationships which offer a wide range of possibilities for the application of routing algorithms to suit various needs.

(5) *Enhanced Embeddability:* We can easily see that many existing well known interconnection networks can be viewed as a subgraph of the interconnection networks that can be constructed from our methodology. For example, the popular hypercube Q_3 forms a subgraph of $LS_2(C_4)$ and $SS_k(P_2)$ forms the well-known book graph. The extensive applications of hypercube networks and their variants can be found in [28]. Further, when G is a connected graph, one can easily see that

- (i) $RC_k(G)$ can be embedded in $TE_k(G)$,
- (ii) $SE_k(G)$ can be embedded in $DE_k(G)$ and $TE_k(G)$,
- (iii) $IE_k(G)$ can be embedded in $DE_k(G)$ and $TE_k(G)$,
- (iv) $VE_k(G)$ can be embedded in $DE_k(G)$ and $CS_k(G)$.

These extensive embeddability options in the constructed interconnection networks allows us to apply the algorithms designed for one to another which enables their economical implementation.

(6) *Symmetry:* It is easy to see that all the constructed interconnection networks are symmetric about the horizontal axis that pass through the middle. Further, even in the disconnected interconnection networks, we can see that they can be viewed as the union of similar subgraphs. This feature enables us to employ specific algorithms to the identical subgraphs which reduces the cost and toiling involved in designing for the entire interconnection network.

(7) *Infinite Extendability:* All the interconnection networks defined in the above section presents assured infinite extendability of any graph chosen as the existence of infinite number of natural numbers is guaranteed.

(8) *Efficient Layout of VLSI Circuits:* An efficient layout is marked by the maximum number of layers it has, the size of each of its layers and so on. The interconnection networks which can be constructed using our methodology satisfy most of these requisite properties.

Apart from these properties, it is also worthy to note that composition of two or more of the operations may result in another graph operation. Though there are contradictions between the desirable properties, most of the proposed interconnection networks can render useful applications in many fields of science and technology which can be chosen based on the need [37].

6. Spectral Parameters of the Constructed Networks

In this section, we describe the adjacency matrices of all the networks that can be generated through the operations we have defined. As a consequence, we also tabulate the eigenvalues, energy, largest eigenvalue and the least eigenvalue of all the networks obtained through the defined operations.

Theorem 2. Let G be any graph on n vertices whose adjacency matrix is A. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of G, then the energy of the graphs defined above are obtained as in Table 3, where E and k denote the energy of the graph G and dimension of the constructed network respectively.

Table 3. Energy of the constructed interconnection networks

Constructed Networks	Energy
Layer Enveloping	kE
Replica Cleaving	$(k-2+2\sqrt{k-1})E$
Spin Cleaving	$(k-2+\sqrt{5})E$
Penultimate Cleaving	$(1+\sqrt{4k-7})E$
Penultimate Enveloping	kE
Semi Enveloping	kE
Inceptive Enveloping	$(2\sqrt{2k-3})E$
Vertex Enveloping	$(k-2+\sqrt{k^2-2k+5})E$
Edge Depleted Enveloping	$(\sqrt{k^2+2k-3})E$
Terminal Over Enveloping	$(1+\sqrt{k^2+2k-7})E$
Circumscribed Enveloping	$(k-3+\sqrt{k^2-2k+9})E$

Proof. The adjacency matrices of the constructed networks can be realized as in Table 4 and their spectrum can be obtained as a consequence of Theorem 1. Then, the last column of Table 4 establishes the result.

Table 4. Adjacency matrix and Spectrum of the constructed interconnection networks

Operations	Adjacency Matrix	Spectrum
Layer Enveloping	$\begin{bmatrix} A & 0 & 0 & \dots & 0 & A \\ 0 & A & 0 & \dots & A & 0 \\ 0 & 0 & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & A & 0 & \dots & A & 0 \\ A & 0 & 0 & \dots & 0 & A \end{bmatrix}$	$\begin{pmatrix} 0 & \lambda_i & 2\lambda_i \\ n(\frac{k-1}{2}) & 1 & (\frac{k-1}{2}) \end{pmatrix}$, if k is odd $\begin{pmatrix} 0 & 2\lambda_i \\ \frac{nk}{2} & \frac{k}{2} \end{pmatrix}$, if k is even
Replica Cleaving	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & A & 0 & \dots & 0 & 0 \\ A & 0 & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & 0 & 0 & \dots & A & 0 \\ A & 0 & 0 & \dots & 0 & A \end{bmatrix}$	$\begin{pmatrix} \lambda_i & (1+\sqrt{k-1})\lambda_i & (1-\sqrt{k-1})\lambda_i \\ k-2 & 1 & 1 \end{pmatrix}$

Table Contd.

Spin Cleaving	$\begin{bmatrix} A & 0 & 0 & \dots & 0 & A \\ 0 & 0 & 0 & \dots & A & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & A & 0 & \dots & 0 & 0 \\ A & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \lambda_i & -\lambda_i & \frac{1+\sqrt{5}}{2}\lambda_i & \frac{1-\sqrt{5}}{2}\lambda_i \\ (\frac{k-1}{2}) & (\frac{k-3}{2}) & 1 & 1 \end{bmatrix}$, if k is odd $\begin{bmatrix} \lambda_i & -\lambda_i & \frac{1+\sqrt{5}}{2}\lambda_i & \frac{1-\sqrt{5}}{2}\lambda_i \\ (\frac{k}{2}-1) & (\frac{k}{2}-1) & 1 & 1 \end{bmatrix}$, if k is even
Penultimate Cleaving	$\begin{bmatrix} A & A & A & \dots & A & 0 \\ A & 0 & 0 & \dots & 0 & 0 \\ A & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & A \end{bmatrix}$	$\begin{pmatrix} 0 & \lambda_i & (\frac{1+\sqrt{4k-7}}{2})\lambda_i & (\frac{1-\sqrt{4k-7}}{2})\lambda_i \\ n(k-3) & 1 & 1 & 1 \end{pmatrix}$
Penultimate Enveloping	$\begin{bmatrix} A & A & A & \dots & A & 0 \\ A & A & A & \dots & A & 0 \\ A & A & A & \dots & A & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & A & \dots & A & 0 \\ 0 & 0 & 0 & \dots & 0 & A \end{bmatrix}$	$\begin{pmatrix} 0 & \lambda_i & (k-1)\lambda_i \\ n(k-2) & 1 & 1 \end{pmatrix}$
Semi Enveloping	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & A & A & \dots & A & 0 \\ A & A & A & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ A & A & 0 & \dots & 0 & 0 \\ A & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$	$\begin{pmatrix} \lambda_i \\ k \end{pmatrix}$
Inceptive Enveloping	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & A & A & \dots & A & A \\ A & A & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ A & A & 0 & \dots & 0 & 0 \\ A & A & 0 & \dots & 0 & 0 \end{bmatrix}$	$\begin{pmatrix} 0 & (1-\sqrt{2k-3})\lambda_i & (1+\sqrt{2k-3})\lambda_i \\ n(k-2) & 1 & 1 \end{pmatrix}$
Vertex Enveloping	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & 0 & A & \dots & A & A \\ A & A & 0 & \dots & A & A \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ A & A & A & \dots & 0 & A \\ A & A & A & \dots & A & 0 \end{bmatrix}$	$\begin{pmatrix} -\lambda_i & (\frac{(k-1)+\sqrt{k^2-2k+5}}{2})\lambda_i & (\frac{(k-1)-\sqrt{k^2-2k+5}}{2})\lambda_i \\ (k-2) & 1 & 1 \end{pmatrix}$

Table Contd.

Edge Depleted Enveloping	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & A & A & \dots & A & A \\ A & A & A & \dots & A & A \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ A & A & A & \dots & A & A \\ A & A & A & \dots & A & 0 \end{bmatrix}$	$\begin{pmatrix} 0 & (\frac{(k-1)+\sqrt{k^2+2k-3}}{2})\lambda_i & (\frac{(k-1)-\sqrt{k^2+2k-3}}{2})\lambda_i \\ n(k-2) & 1 & 1 \end{pmatrix}$
Terminal Over Enveloping	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & A & A & \dots & A & A \\ A & A & A & \dots & A & A \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ A & A & A & \dots & A & 0 \\ A & A & A & \dots & 0 & A \end{bmatrix}$	$\begin{pmatrix} 0 & \lambda_i & (\frac{(k-1)+\sqrt{k^2+2k-7}}{2})\lambda_i & (\frac{(k-1)-\sqrt{k^2+2k-7}}{2})\lambda_i \\ n(k-3) & 1 & 1 & 1 \end{pmatrix}$
Circumscribed Enveloping	$\begin{bmatrix} A & A & A & \dots & A & A \\ A & 0 & A & \dots & A & A \\ A & A & 0 & \dots & A & A \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ A & A & A & \dots & 0 & A \\ A & A & A & \dots & A & A \end{bmatrix}$	$\begin{pmatrix} 0 & -\lambda_i & (\frac{(k-1)+\sqrt{k^2-2k+9}}{2})\lambda_i & (\frac{(k-1)-\sqrt{k^2-2k+9}}{2})\lambda_i \\ n & (k-3) & 1 & 1 \end{pmatrix}$

□

Corollary 1. *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of G written in a nonincreasing manner, then the spectral radius and the least eigenvalue of the constructed networks are obtained as in Table 5.*

Table 5. Spectral Radius and Least Eigenvalue of the constructed networks

Operations	Spectral Radius	Least Eigenvalue
Layer Enveloping	$2\lambda_1(G)$	$2\lambda_n(G)$
Replica Cleaving	$(1+\sqrt{k-1})\lambda_1$ if $\|\lambda_1\| > \|\lambda_n\|$ $(1-\sqrt{k-1})\lambda_n$ if $\|\lambda_n\| > \|\lambda_1\|$	$(1+\sqrt{k-1})\lambda_n$ if $\|\lambda_n\| > \|\lambda_1\|$ $(1-\sqrt{k-1})\lambda_1$ if $\|\lambda_1\| > \|\lambda_n\|$
Spin Cleaving	$(\frac{1+\sqrt{5}}{2})\lambda_1$ if $\|\lambda_1\| > \|\lambda_n\|$ $-\lambda_n$ if $\|\lambda_n\| > \|\lambda_1\|$	$(\frac{1+\sqrt{5}}{2})\lambda_n$ if $\|\lambda_n\| > \|\lambda_1\|$ $-\lambda_1$ if $\|\lambda_1\| > \|\lambda_n\|$
Penultimate Cleaving	$(\frac{1+\sqrt{4k-7}}{2})\lambda_1$ if $\|\lambda_1\| > \|\lambda_n\|$ $(\frac{1-\sqrt{4k-7}}{2})\lambda_n$ if $\|\lambda_n\| > \|\lambda_1\|$	$(\frac{1+\sqrt{4k-7}}{2})\lambda_n$ if $\|\lambda_n\| > \|\lambda_1\|$ $(\frac{1-\sqrt{4k-7}}{2})\lambda_1$ if $\|\lambda_1\| > \|\lambda_n\|$
Penultimate Enveloping	$(k-1)\lambda_1(G)$	$(k-1)\lambda_n(G)$
Semi Enveloping	$\lambda_1(G)$	$\lambda_n(G)$

Table Contd.

Operations	Spectral Radius	Least Eigenvalue
Inceptive Enveloping	$(1+\sqrt{2k-3})\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$ $(1-\sqrt{2k-3})\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$	$(1+\sqrt{2k-3})\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$ $(1-\sqrt{2k-3})\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$
Vertex Enveloping	$\frac{(k-1+\sqrt{k^2-2k+5})}{2}\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$ $-\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$	$\frac{(k-1+\sqrt{k^2-2k+5})}{2}\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$ $-\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$
Edge Depleted Enveloping	$\frac{(k-1+\sqrt{k^2+2k-3})}{2}\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$ $\frac{(k-1-\sqrt{k^2+2k-3})}{2}\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$	$\frac{(k-1+\sqrt{k^2+2k-3})}{2}\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$ $\frac{(k-1-\sqrt{k^2+2k-3})}{2}\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$
Terminal Over Enveloping	$\frac{(k-1+\sqrt{k^2+2k-7})}{2}\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$ $\frac{(k-1-\sqrt{k^2+2k-7})}{2}\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$	$\frac{(k-1+\sqrt{k^2+2k-7})}{2}\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$ $\frac{(k-1-\sqrt{k^2+2k-7})}{2}\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$
Circumscribed Enveloping	$\frac{(k-1+\sqrt{k^2-2k+9})}{2}\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$ $-\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$	$\frac{(k-1+\sqrt{k^2-2k+9})}{2}\lambda_n$ if $\lvert\lambda_n\rvert > \lvert\lambda_1\rvert$ $-\lambda_1$ if $\lvert\lambda_1\rvert > \lvert\lambda_n\rvert$

Proof. Proof follows from column 3 of Table 4. □

Corollary 2. *The k-layer enveloping graph of any integral graph is integral.*

Corollary 3. *The k-replica cleaving graph of any integral graph is integral if and only if $k = m^2 + 1$, where m is any integer.*

Corollary 4. *The k-penultimate cleaving graph of any integral graph is integral if and only if $k = \frac{(2m-1)^2+7}{4}$, where m is any integer.*

Corollary 5. *The k-penultimate enveloping graph of any integral graph is integral.*

Corollary 6. *The k-semi enveloping graph of any integral graph is integral.*

Corollary 7. *The k-inceptive enveloping graph of any integral graph is integral if and only if $k = \frac{m^2+3}{2}$, where m is any integer.*

7. Degree-Based Topological Indices of the Interconnection Networks

This section comprises some findings about four degree-based topological indices of the connected interconnection networks which have been defined.

Theorem 3. *The interconnection network $RC_k(G)$ embodies the following degree-based topological indices, when G is connected.*

(i) $Z_1(RC_k(G)) = (k^2 + 4k - 4)Z_1(G)$,

(ii) $MZ_1(RC_k(G)) = (\frac{k}{4} - \frac{1}{4} + \frac{1}{k^2})MZ_1(G)$,

(iii) $P_{RC_k(G)} = (k^{|V(G)|}2^{(k-1)|V(G)|})(P_G)^k$,

(iv) $A'(RC_k(G)) = (4k-2)A'(G)$.

Proof.

(i)
$$\begin{aligned}
Z_1(RC_k(G)) &= \sum_{v\in G}[kd_v]^2 + \sum_{v\in \text{ all } (k-1) \text{ copies of } G}[2d_v]^2 \\
&= k^2\sum_{v\in G}[d_v]^2 + 4\sum_{v\in \text{ all } (k-1) \text{ copies of } G}[d_v]^2 \\
&= (k^2+4(k-1))\sum_{v\in V(G)}[d_v]^2 \\
&= (k^2+4k-4)Z_1(G),
\end{aligned}$$

(ii)
$$\begin{aligned}
MZ_1(RC_k(G)) &= \sum_{v\in G}\frac{1}{[kd_v]^2} + \sum_{v\in \text{ all } (k-1) \text{ copies of } G}\frac{1}{[2d_v]^2} \\
&= \frac{1}{k^2}\sum_{v\in G}\frac{1}{[d_v]^2} + \frac{1}{4}\sum_{v\in \text{ all } (k-1) \text{ copies of } G}\frac{1}{[d_v]^2} \\
&= \left(\frac{1}{k^2}+\frac{k-1}{4}\right)\sum_{v\in V(G)}\frac{1}{[d_v]^2} \\
&= \left(\frac{k}{4}-\frac{1}{4}+\frac{1}{k^2}\right)MZ_1(G),
\end{aligned}$$

(iii)
$$\begin{aligned}
P_{RC_k(G)} &= kd_{v_1}\times kd_{v_2}\times\ldots\times kd_{v_n}\times 2d_{v_{1i}}\times 2d_{v_{2i}}\times\ldots\times 2d_{v_{ni}} \\
&= k^{|V(G)|}P_G\times 2^{(k-1)|V(G)|}(P_G)^{k-1} \\
&= [k^{|V(G)|}2^{(k-1)|V(G)|}](P_G)^k,
\end{aligned}$$

(iii)
$$A'(RC_k(G)) = [k+(k-1)]2A'(G) = (4k-2)A'(G).$$ □

Along similar lines, we can prove the following results.

Theorem 4. *The degree-based topological indices of the interconnection network $SE_k(G)$ can be expressed as follows when G is connected.*

(i) $Z_1(SE_k(G)) = (\frac{(2k^3+3k^2+k)}{6})Z_1(G)$,

(ii) $MZ_1(SE_k(G)) = (\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{k^2})MZ_1(G)$,

(iii) $P_{SE_k(G)} = (k!^{|V(G)|})(P_G)^k$,

(iv) $A'(SE_k(G)) = \begin{cases} 2(\lceil\frac{k}{2}\rceil)^2A'(G) & \text{if } k \text{ is odd}, \\ k(\frac{k}{2}+1)A'(G) & \text{if } k \text{ is even}. \end{cases}$

Theorem 5. *For the interconnection network $IE_k(G)$, the degree-based topological indices are obtained in the following way when G is connected.*

(i) $Z_1(IE_k(G)) = (2k^2+4k-8)Z_1(G)$,

(ii) $MZ_1(IE_k(G)) = (\frac{k}{4}-\frac{1}{2}+\frac{2}{k^2})MZ_1(G)$,

(iii) $P_{IE_k(G)} = (k^{2|V(G)|}2^{(k-2)|V(G)|})(P_G)^k$,

(iv) $A'(IE_k(G)) = (4k-2)A'(G)$.

Theorem 6. *When G is a connected graph, the constructed interconnection network $VE_k(G)$ has the following degree-based topological indices.*

(i) $Z_1(VE_k(G)) = (k^3-2k^2+3k-1)Z_1(G)$,

(ii) $MZ_1(VE_k(G)) = (\frac{k^2+k-1}{k^3-k^2})MZ_1(G)$,

(iii) $P_{VE_k(G)} = (k^{|V(G)|}(k-1)^{(k-1)|V(G)|})(P_G)^k$,

(iv) $A'(VE_k(G)) = (k^2-k+2)A'(G)$.

Theorem 7. *The interconnection network $DE_k(G)$ procures the following degree-based topological indices, when G is connected.*

(i) $Z_1(DE_k(G)) = (k^3-2k+1)Z_1(G)$,

(ii) $MZ_1(DE_k(G)) = (\frac{k^3-3k^2+3k-1}{k^4-2k^3+k^2})MZ_1(G)$,

(iii) $P_{DE_k(G)} = (k^{(k-1)|V(G)|}(k-1)^{|V(G)|})(P_G)^k$,

(iv) $A'(DE_k(G)) = (k^2+k-2)A'(G)$.

Theorem 8. *The following expressions are derived for the degree-based topological indices of the interconnection network $TE_k(G)$ when a connected graph G is its underlying graph:*

(i) $Z_1(TE_k(G)) = (k^3-4k+2)Z_1(G)$,

(ii) $MZ_1(TE_k(G)) = (\frac{k^3-2k^2+5k-2}{k^4-2k^3+k^2})MZ_1(G)$,

(iii) $P_{TE_k(G)} = (k^{(k-2)|V(G)|}(k-1)^{2|V(G)|})(P_G)^k$,

(iv) $A'(TE_k(G)) = (k^2+k-2)A'(G)$.

Theorem 9. *The degree-based topological indices for the constructed interconnection network $CE_k(G)$ can be expressed in the following manner when G is a connected graph.*

(i) $Z_1(CE_k(G)) = (k^3-2k^2+5k-2)Z_1(G)$,

(ii) $MZ_1(CE_k(G)) = (\frac{k^3-4k+2}{k^4-2k^3+k^2})MZ_1(G)$,

(iii) $P_{CE_k(G)} = (k^{2|V(G)|}(k-1)^{(k-2)|V(G)|})(P_G)^k$,

(iv) $A'(CE_k(G)) = (k^2-k+4)A'(G)$.

As we observe that for any dimension, the computed spectral parameters and the degree-based topological indices of all the defined interconnection networks are expressed as a scalar multiple of the corresponding parameter, it follows that

Corollary 8. *The graph invariants Graph energy, Spectral radius, Least eigenvalue, first Zagreb index, first modified Zagreb index, simple topological index and the total adjacency index are polynomial time solvable for all the defined interconnection networks.*

8. Numerical Results and Interpretations

This section includes the graphical representations of the scalar multiples by which the parameters of the networks are magnifying across their dimension. Also, the numerical values are recorded for better visualization of the trends in the descriptors. These representations plotted using R[2] would enable the researchers to compare the behaviour of the parameters for the different networks in order to choose the optimal network of their choice to suit the intended purpose.

[2] R Core Team, *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria (2020).

Table 6. Energy of the constructed interconnection networks

k	LE	RC	SC	PC	PE	SE	IE	VE	DE	TE	CE
4	4	5.4641	4.2361	4	4	4	4.4721	5.6056	4.5826	5.1231	5.1231
5	5	7	5.2361	4.6056	5	5	5.2915	7.4721	5.6569	6.2915	6.8990
6	6	8.4721	6.2361	5.1231	6	6	6	9.3852	6.7082	7.4031	8.7446
7	7	9.8990	7.2361	5.5826	7	7	6.6332	11.3246	7.7460	8.4833	10.6332
8	8	11.2915	8.2361	6	8	8	7.2111	13.2801	8.7750	9.5440	12.5498
9	9	12.6569	9.2361	6.3852	9	9	7.7460	15.2462	9.7980	10.5917	14.4853
10	10	14	10.2361	6.7446	10	10	8.2462	17.2195	10.8167	11.6301	16.4340
11	11	15.3246	11.2361	7.0828	11	11	8.7178	19.1980	11.8322	12.6619	18.3923
12	12	16.6332	12.2361	7.4031	12	12	9.1652	21.1803	12.8452	13.6886	20.3578
13	13	17.9282	13.2361	7.7082	13	13	9.5917	23.1655	13.8564	14.7113	22.3288

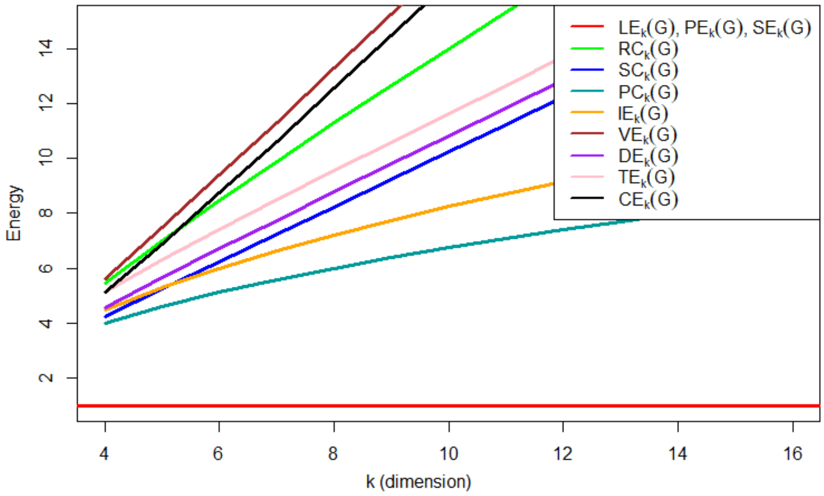

Figure 4. Energy of the constructed Interconnection Networks

Table 7. Spectral Radius and Least Eigenvalue of the constructed Interconnection Networks when $|\lambda_1| > |\lambda_n|$ and $|\lambda_n| > |\lambda_1|$ respectively

k	LE	RC	SC	PC	PE	SE	IE	VE	DE	TE	CE
4	2	2.7321	1.618	2	3	1	3.2361	3.3028	3.7913	3.5616	3.5616
5	2	3	1.618	2.3028	4	1	3.6458	4.2361	4.8284	4.6458	4.4495
6	2	3.2361	1.618	2.5616	5	1	4	5.1926	5.8541	5.7016	5.3723
7	2	3.4495	1.618	2.7913	6	1	4.3166	6.1623	6.873	6.7417	6.3166
8	2	3.6458	1.618	3	7	1	4.6056	7.1401	7.8875	7.772	7.2749
9	2	3.8284	1.618	3.1926	8	1	4.873	8.1231	8.899	8.7958	8.2426
10	2	4	1.618	3.3723	9	1	5.1231	9.1098	9.9083	9.8151	9.217
11	2	4.1623	1.618	3.5414	10	1	5.3589	10.099	10.9161	10.831	10.1962
12	2	4.3166	1.618	3.7016	11	1	5.5826	11.0902	11.9226	11.8443	11.1789
13	2	4.4641	1.618	3.8541	12	1	5.7958	12.0828	12.9282	12.8557	12.1644

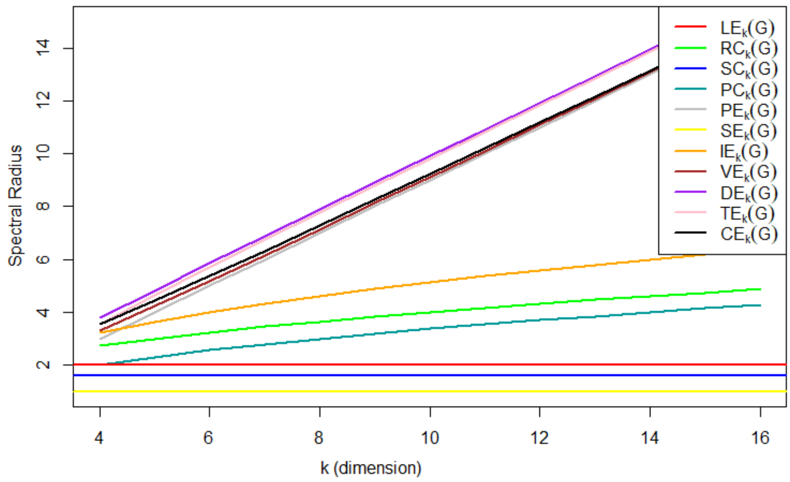

Figure 5. Spectral Radius and Least Eigenvalue of the constructed Interconnection Networks when $|\lambda_1| > |\lambda_n|$ and $|\lambda_n| > |\lambda_1|$ respectively

Table 8. Spectral Radius and Least Eigenvalue of the constructed interconnection networks when $|\lambda_n| > |\lambda_1|$ and $|\lambda_1| > |\lambda_n|$ respectively

k	LE	RC	SC	PC	PE	SE	IE	VE	DE	TE	CE
4	2	-0.7321	-1	-1	3	1	-1.2361	-1	-0.7913	-0.5616	-1
5	2	-1	-1	-1.3028	4	1	-1.6458	-1	-0.8284	-0.6458	-1
6	2	-1.2361	-1	-1.5616	5	1	-2	-1	-0.8541	-0.7016	-1
7	2	-1.4495	-1	-1.7913	6	1	-2.3166	-1	-0.8730	-0.7417	-1
8	2	-1.6458	-1	-2	7	1	-2.6056	-1	-0.8875	-0.7720	-1
9	2	-1.8284	-1	-2.1926	8	1	-2.8730	-1	-0.8990	-0.7958	-1
10	2	-2	-1	-2.3723	9	1	-3.1231	-1	-0.9083	-0.8151	-1
11	2	-2.1623	-1	-2.5414	10	1	-3.3589	-1	-0.9161	-0.8310	-1
12	2	-2.3166	-1	-2.7016	11	1	-3.5826	-1	-0.9226	-0.8443	-1
13	2	-2.4641	-1	-2.8541	12	1	-3.7958	-1	-0.9282	-0.8557	-1

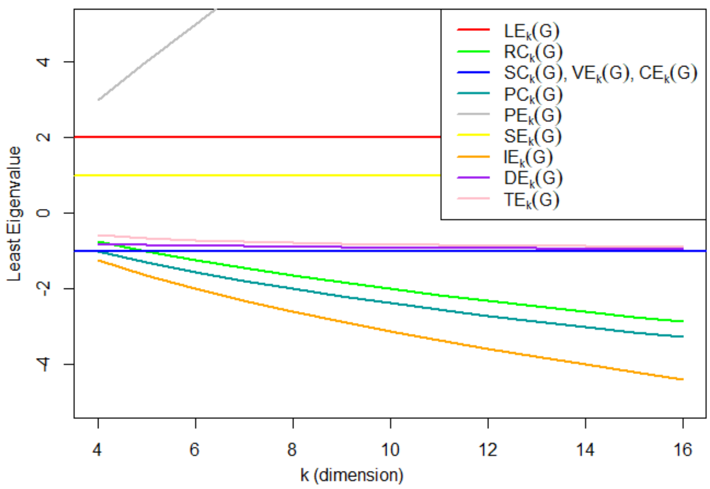

Figure 6. Spectral Radius and Least Eigenvalue of the constructed interconnection networks when $|\lambda_n| > |\lambda_1|$ and $|\lambda_1| > |\lambda_n|$ respectively

Table 9. First Zagreb Index of the constructed Interconnection Networks

Dimension	$RC_k(G)$	$SE_k(G)$	$IE_k(G)$	$VE_k(G)$	$DE_k(G)$	$TE_k(G)$	$CE_k(G)$
4	28	30	40	43	57	50	50
5	41	55	62	89	116	107	98
6	56	91	88	161	205	194	172
7	73	140	118	265	330	317	278
8	92	204	152	407	497	482	422
9	113	285	190	593	712	695	610
10	136	385	232	829	981	962	848
11	161	506	278	1121	1310	1289	1142
12	188	650	328	1475	1705	1682	1498
13	217	819	382	1897	2172	2147	1922

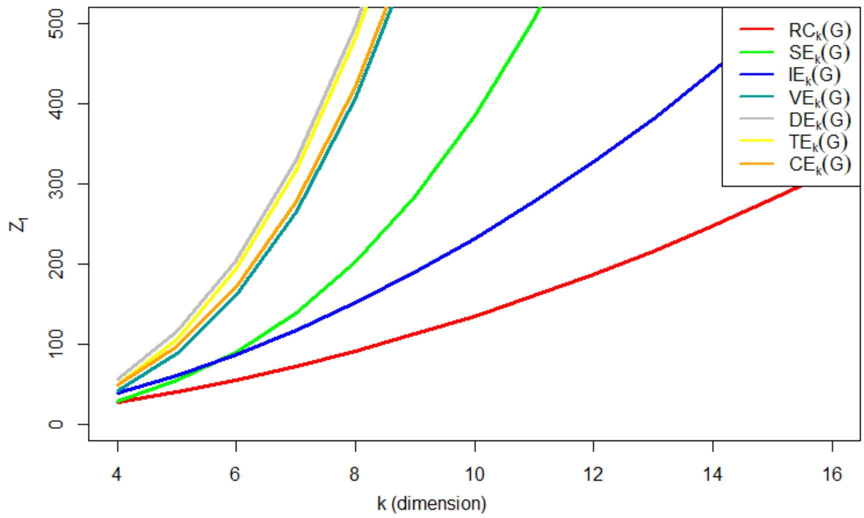

Figure 7. First Zagreb Index of the constructed Interconnection Networks

Table 10. First Modified Zagreb Index of the constructed Interconnection Networks

Dimension	$RC_k(G)$	$SE_k(G)$	$IE_k(G)$	$VE_k(G)$	$DE_k(G)$	$TE_k(G)$	$CE_k(G)$
4	0.8125	1.4236	0.625	0.3958	0.1875	0.3472	0.3472
5	1.04	1.4636	0.83	0.29	0.16	0.245	0.2675
6	1.2778	1.4914	1.0556	0.2278	0.1389	0.1911	0.2156
7	1.5204	1.5118	1.2908	0.1870	0.1224	0.1576	0.17971
8	1.7656	1.5274	1.5313	0.1585	0.1094	0.1346	0.1537
9	2.0123	1.5398	1.7747	0.1373	0.0988	0.1177	0.1341
10	2.26	1.5498	2.02	0.1211	0.09	0.1047	0.1188
11	2.5083	1.5580	2.2665	0.1083	0.0826	0.0944	0.1065
12	2.7569	1.5650	2.5139	0.0979	0.07639	0.0859	0.0965
13	3.0059	1.5709	2.7618	0.0893	0.0710	0.0790	0.0882

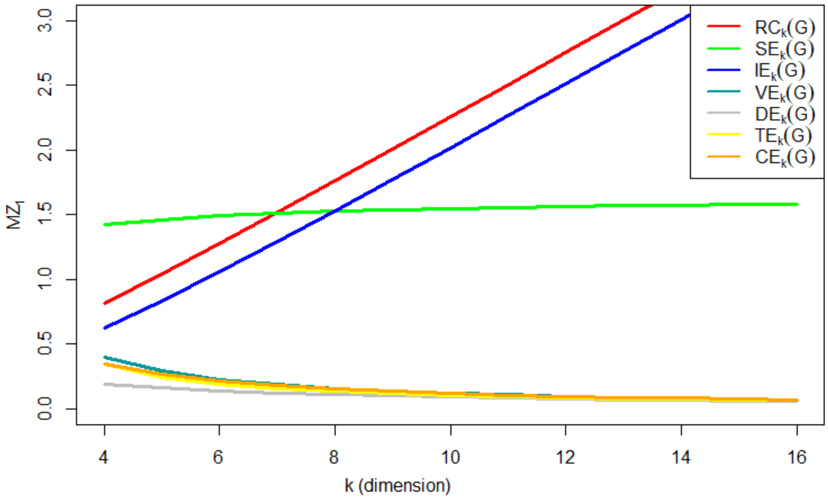

Figure 8. First Modified Zagreb Index of the constructed interconnection networks

Table 11. Total Adjacency Index of the constructed Interconnection Networks

Dimension	$RC_k(G)$	$SE_k(G)$	$IE_k(G)$	$VE_k(G)$	$DE_k(G)$	$TE_k(G)$	$CE_k(G)$
4	14	12	14	14	18	18	16
5	18	18	18	22	28	28	24
6	22	24	22	32	40	40	34
7	26	32	26	44	54	54	46
8	30	40	30	58	70	70	60
9	34	50	34	74	88	88	76
10	38	60	38	92	108	108	94
11	42	72	42	112	130	130	114
12	46	84	46	134	154	154	136
13	50	98	50	158	180	180	160

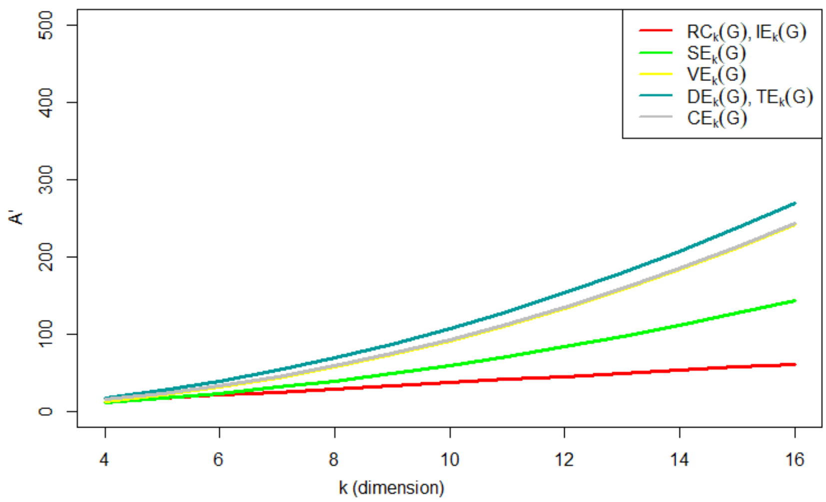

Figure 9. Total Adjacency Index of the constructed interconnection networks

Table 12. Simple Topological Index of the constructed Interconnection Networks

Dimension	$RC_k(G)$	$SE_k(G)$	$IE_k(G)$	$VE_k(G)$	$DE_k(G)$	$TE_k(G)$	$CE_k(G)$
4	4×10^{13}	3×10^{12}	2×10^{16}	2×10^{18}	4×10^{20}	3×10^{19}	3×10^{19}
5	1×10^{17}	5×10^{18}	5×10^{20}	9×10^{27}	4×10^{30}	5×10^{29}	7×10^{28}
6	4×10^{20}	5×10^{25}	7×10^{24}	3×10^{38}	2×10^{41}	4×10^{40}	1×10^{39}
7	7×10^{23}	2×10^{33}	6×10^{28}	4×10^{49}	4×10^{52}	1×10^{52}	2×10^{50}
8	1×10^{27}	3×10^{41}	3×10^{32}	2×10^{61}	3×10^{64}	1×10^{64}	8×10^{61}
9	2×10^{30}	1×10^{50}	1×10^{36}	4×10^{73}	7×10^{76}	2×10^{76}	1×10^{74}
10	2×10^{33}	1×10^{59}	5×10^{39}	2×10^{86}	4×10^{89}	2×10^{89}	5×10^{86}
11	3×10^{36}	3×10^{68}	1×10^{43}	2×10^{99}	5×10^{102}	2×10^{102}	6×10^{99}
12	3×10^{39}	1×10^{78}	3×10^{46}	6×10^{112}	2×10^{116}	7×10^{115}	1×10^{113}
13	3×10^{42}	1×10^{88}	7×10^{49}	4×10^{126}	1×10^{130}	5×10^{129}	8×10^{126}

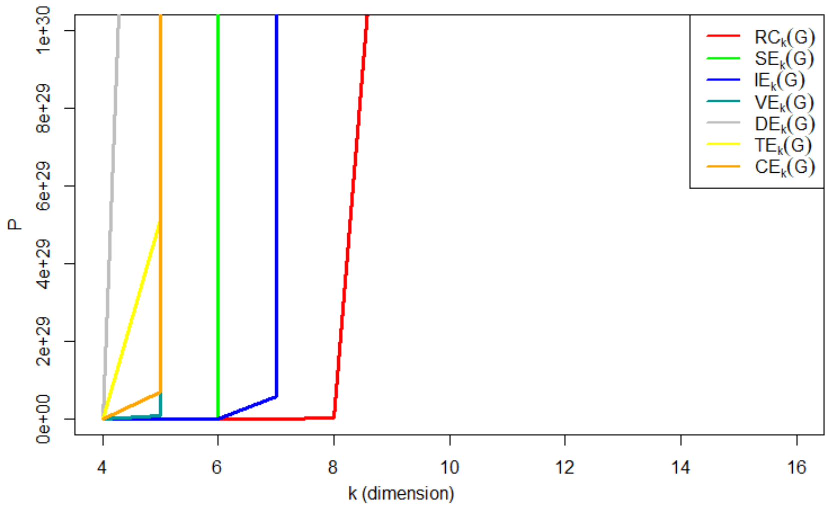

Figure 10. Simple Topological Indices of the constructed Interconnection Networks

9. Computer Realization

The energy, spectral radius, least eigenvalue and the degree based topological indices discussed here have been implemented using MATLAB[1]. The programs can be accessed using the link: https://drive.google.com/drive/folders/130gaThvK4-UgBVI67P5U0efOntIY6He-?usp=sharing.

Once the program is run, it would prompt for the input of the corresponding parameter of the underlying graph. Entering this would enable the user to acquire the desired degree based graph invariant for the chosen interconnection network. Further, one can easily design appropriate programs for the required specific interconnection networks.

10. Concluding Remarks and Future Scope

The demand for interconnection networks which are equipped with destined properties is still evolving. Though there are numerous papers which employ similar techniques in the construction of interconnection networks, we have begun to design an unique approach. Here, the authors provide several novel methods for the construction of interconnection networks by establishing various adjacency schemes. We notice that all the defined interconnection networks satisfy all the existing notions of evaluative measures. Modelling interconnection networks as graphs enabled us to throw light on its various features. We have shown that compositions of networks also result in the same kind of interconnection networks. Further, the procurement of exact analytical expressions for graph energy, spectral radius, least eigenvalue and certain degree-based topological indices as a scalar multiple of the corresponding parameters of the underlying graph has been achieved and that they can be computed in polynomial time has been proved. As a result of these findings, six new infinite classes of integral graphs have been identified. Also, numerical and graphical comparison of the constructed interconnection networks have been presented with some discussions.

Our constructed interconnection networks would be immensely remunerative when we need the setting of networks with clusters. In such an instance, one may regard the underlying graphs or set of vertices as a cluster and foster the connections between them as suggested in our methods. We are not sure whether the future of computing will accelerate towards multiprocessor supercomputers, optical interconnection networks, wireless networks or networks in the cloud. In any instance, we strongly believe that the need for better interconnection networks will persist and their topological structures will always be crucial though the set of preferable properties may be subject to change. In addition, they will continue to be promising applications to complex networks, data center networks and all the aforementioned networks. This ascertains the fact that this will remain an expanding area of fertile research enthralling researchers from graph theory, data science, computer science and engineering. Thus, we would like to focus on research pertaining to the analysis of matrix representations of interconnection networks which would definitely be beneficial.

Acknowledgement

We thank DST (FIST 2015) MATLAB R2017b which was used for computation.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Adiga, B. R. Rakshith and K. N. S. Krishna, Spectra of some new graph operations and some new classes of integral graphs, *Iranian Journal of Mathematical Sciences and Informatics* **13**(1) (2018), 51 – 65, DOI: 10.7508/ijmsi.2018.1.005.

[2] B. S. Anderson, C. Butts and K. Carley, The interaction of size and density with graph-level indices, *Social Networks* **21**(3) (1999), 239 – 267, DOI: 10.1016/s0378-8733(99)00011-8.

[3] A. T. Balaban, I. Motoc, D. Bonchev and O. Mekenyan, Topological indices for structure-activity correlations, in: *Steric Effects in Drug Design*, Topics in Current Chemistry, Vol. **114**, Springer, Berlin — Heidelberg (1983), DOI: 10.1007/bfb0111212.

[4] K. T. Balinska, D. M. Cvetković, Z. S. Radosavljević, S. K. Simić and D. P. Stevanović, A survey on integral graphs, *Publikacije Elektrotehnickog Fakulteta – Serija: Matematika* **13** (2002), 42 – 65, DOI: 10.2298/PETF0213042B.

[5] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, xiv + 250 pages (2011), DOI: 10.1007/978-1-4614-1939-6.

[6] X. Chen and W. Xie, Energy of a hypercube and its complement, *International Journal of Algebra* **6**(16) (2012), 799 – 805.

[7] E. Cheng, K. Qiu and Z. Shen, Diagnosability of interconnection networks: past, present and future, *International Journal of Parallel, Emergent and Distributed Systems* **35**(1) (2020), 2 – 8, DOI: 10.1080/17445760.2019.1655742.

[8] M. Cohn, On the channel capacity of read/write isolated memory, *Discrete Applied Mathematics* **56**(1) (1995), 1 – 8, DOI: 10.1016/0166-218x(93)e0130-q.

[9] M. Dehmer, F. Emmert-Streib and Y. Shi, Interrelations of graph distance measures based on topological indices, *PLOS ONE* **9**(4) (2014), e94985, DOI: 10.1371/journal.pone.0094985.

[10] F. Emmert-Streib and M. Dehmer, Networks for systems biology: Conceptual connection of data and function, *IET Systems Biology* **5**(3) (2011), 185 – 207, DOI: 10.1049/iet-syb.2010.0025.

[11] F. Al Faisal, M. M. H. Rahman and Y. Inoguchi, 3D-TTN: a power efficient cost effective high performance hierarchical interconnection network for next generation green supercomputer, *Cluster Computing* **24** (2021), 2897 – 2908, DOI: 10.1007/s10586-021-03297-1.

[12] Y.-Z. Fan, G.-D. Yu and Y. Wang, The chromatic number and the least eigenvalue of a graph, *The Electronic Journal of Combinatorics* **19**(1) (2012), Article number: P39, DOI: 10.37236/2043.

[13] T.-Y. Feng, A survey of interconnection networks, *Computer* **14**(12) (1981), 12 – 27, DOI: 10.1109/c-m.1981.220290.

[14] M. Ghasempour, J. Heathcote, J. Navaridas, L. A. Plana, J. Garside and M. Luján, Analysis of software and hardware-accelerated approaches to the simulation of unconventional interconnection networks, *Simulation Modelling Practice and Theory* **103** (2020), 102088, DOI: 10.1016/j.simpat.2020.102088.

[15] I. Gutman, The energy of a graph: Old and new results, in: *Algebraic Combinatorics and Applications*, A. Betten, A. Kohnert, R. Laue and A. Wassermann (eds), Springer, Berlin — Heidelberg (2001), DOI: https://doi.org/10.1007/978-3-642-59448-9_13.

[16] J. Hao, Theorems about Zagreb indices and modified Zagreb indices, *MATCH Communications in Mathematical and in Computer Chemistry* **65** (2011), 659 – 670, URL: https://match.pmf.kg.ac.rs/electronic_versions/Match65/n3/match65n3_659-670.pdf.

[17] F. Harary and A. J. Schwenk, Which graphs have integral spectra?, in: *Graphs and Combinatorics: Proceedings of the Capital Conference on Graph Theory and Combinatorics*, George Washington University, June 18-22, 1973, Springer, Berlin — Heidelberg (1974).

[18] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York (1991), DOI: 10.1017/cbo9780511840371.

[19] K. A. S. Immink, *Codes for Mass Data Storage Systems*, Shannon Foundation Publishers, The Netherlands (2004).

[20] G. Indulal and A. Vijayakumar, Energies of some non-regular graphs, *Journal of Mathematical Chemistry* **42** (2007), 377 – 386, DOI: 10.1007/s10910-006-9108-7.

[21] T. Jain, *Nonblocking On-chip Interconnection Networks*, Doctoral dissertation, Technischen Universität Kaiserslautern, Germany (2020), URL: https://kluedo.ub.rptu.de/frontdoor/deliver/index/docId/5976/file/Nonblocking_On-Chip_Interconnection_Networks.pdf.

[22] M. Jurczyk, H. J. Siegel and C. Stunkel, Interconnection networks for parallel computers, *Wiley Encyclopedia of Computer Science and Engineering*, B. W. Wah (Ed.), Wiley, pp. 1613 – 1623 (2007), DOI: 10.1002/9780470050118.ecse197.

[23] D. Lüdtke and D. Tutsch, The modeling power of CINSim: Performance evaluation of interconnection networks, *Computer Networks* **53**(8) (2009), 1274 – 1288, DOI: 10.1016/j.comnet.2009.02.013.

[24] J. Leskovec, J. Kleinberg and C. Faloutsos, Graphs over time: Densification laws, shrinking diameters and possible explanations, *KDD'05: Proceedings of the Eleventh ACM SIGKDD International Conference on Knowledge Discovery in Data Mining*, Association for Computing Machinery, New York, USA (2005), pp. 177 – 187, DOI: 10.1145/1081870.1081893.

[25] Q. Li and K. Q. Feng, On the largest eigenvalue of graphs, *Acta Mathematicae Applicatae Sinica* **2** (1979), 167 – 175.

[26] J. Li, W. C. Shiu, W. H. Chan and A. Chang, On the spectral radius of graphs with connectivity at most k, *Journal of Mathematical Chemistry* **46** (2009), 340 – 346, DOI: 10.1007/s10910-008-9465-5.

[27] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York (2012), DOI: 10.1007/978-1-4614-4220-2.

[28] J. Liang and Q. Zhang, The t/s-diagnosability of hypercube networks under the PMC and comparison models, *IEEE Access* **5** (2017), 5340 – 5346, DOI: 10.1109/access.2017.2672602.

[29] Y. Liao, J. Yin, D. Yin and L. Gao, DPillar: Dual-port server interconnection network for large scale data centers, *Computer Networks* **56**(8) (2012), 2132 – 2147, DOI: 10.1016/j.comnet.2012.02.016.

[30] L. S. de Lima, A. Mohammadian and C. S. Oliveira, On integral graphs with at most two vertices of degree larger than two, *Linear Algebra and its Applications* **584** (2020), 164 – 184, DOI: 10.1016/j.laa.2019.09.019.

[31] H. Liu, M. Lu and F. Tian, Some upper bounds for the energy of graphs, *Journal of Mathematical Chemistry* **41** (2007), 45 – 57, DOI: 10.1007/s10910-006-9183-9.

[32] J. Louis, A formula for the energy of circulant graphs with two generators, *Journal of Applied Mathematics* **2016** (2016), Article ID 1793978, DOI: 10.1155/2016/1793978.

[33] L. Lovász and J. Pelikán, On the eigenvalues of trees, *Periodica Mathematica Hungarica* **3** (1973), 175 – 182, DOI: 10.1007/bf02018473.

[34] M. Lv, J. Fan, J. Zhou, B. Cheng and X. Jia, The extra connectivity and extra diagnosability of regular interconnection networks, *Theoretical Computer Science* **809** (2020), 88 – 102, DOI: 10.1016/j.tcs.2019.12.001.

[35] Z. Mihalić, D. Veljan, D. Amić, S. Nikolić, D. Plavšić and N. Trinajstić, The distance matrix in chemistry, *Journal of Mathematical Chemistry* **11** (1992), 223 – 258, DOI: 10.1007/bf01164206.

[36] M. Morzy and T. Kajdanowicz, Graph energies of egocentric networks and their correlation with vertex centrality measures, *Entropy* **20**(12) (2018), 916, DOI: 10.3390/e20120916.

[37] S. M. Nabavinejad, M. Baharloo, K.-C. Chen, M. Palesi, T. Kogel and M. Ebrahimi, An overview of efficient interconnection networks for deep neural network accelerators, *IEEE Journal on Emerging and Selected Topics in Circuits and Systems* **10**(3) (2020), 268 – 282, DOI: 10.1109/jetcas.2020.3022920.

[38] H. Narumi and M. Katayama, Simple topological index: A newly devised index characterizing the topological nature of structural isomers of saturated hydrocarbons, *Memoirs of the Faculty of Engineering, Hokkaido University* **16**(3) (1984), 209 – 214, URL: https://eprints.lib.hokudai.ac.jp/dspace/bitstream/2115/38010/1/16(3)_209-214.pdf .

[39] R. V. Solé and S. Valverde, Information theory of complex networks: on evolution and architectural constraints, in: *Complex Networks*, E. Ben-Naim, H. Frauenfelder and Z. Toroczkai (eds.), *Lecture Notes in Physics*, Vol. 650, Springer, Berlin — Heidelberg, DOI: 10.1007/978-3-540-44485-5_9.

[40] B. Suntornpoch and Y. Meemark, Cayley graphs over a finite chain ring and GCD-graphs, *Bulletin of the Australian Mathematical Society* **93**(3) (2016), 353 – 363, DOI: 10.1017/s0004972715001380.

[41] S. S. Surya and P. Subbulakshmi, On the spectral parameters of certain cartesian products of graphs with P_2, in: *Mathematical Analysis and Computing – ICMAC 2019*, R. N. Mohapatra, S. Yugesh, G. Kalpana and C. Kalaivani (eds.), Springer Proceedings in Mathematics & Statistics, Vol. **344**, Springer, Singapore (2021), DOI: 10.1007/978-981-33-4646-8_31.

[42] Y.-Y. Tan and Y.-Z. Fan, The vertex (edge) independence number, vertex (edge) cover number and the least eigenvalue of a graph, *Linear Algebra and its Applications* **433**(4) (2010), 790 – 795, DOI: 10.1016/j.laa.2010.04.009.

[43] K. J. Tinkler, The physical interpretation of eigenfunctions of dichotomous matrices, *Transactions of the Institute of British Geographers* **55** (1972), 17 – 46, DOI: 10.2307/621721.

[44] R. E. Ulanowicz, Information theory in ecology, *Computers and Chemistry* **25**(4) (2001), 393 – 399, DOI: 10.1016/s0097-8485(01)00073-0.

[45] S. K. Vaidya and K. M. Popat, Energy of m-splitting and m-shadow graphs, *Far East Journal of Mathematical Sciences* **102**(8) (2017), 1571 – 1578, DOI: 10.17654/ms102081571.

[46] S. K. Vaidya and K. M. Popat, Some new results on energy of graphs, *MATCH Communications in Mathematical and in Computer Chemistry* **77** (2017), 589 – 594, URL: https://match.pmf.kg.ac.rs/electronic_versions/Match77/n3/match77n3_589-594.pdf.

[47] T. Wang, L. Jia and F. Sun, Bounds of the spectral radius and the nordhaus-gaddum type of the graphs, *The Scientific World Journal* **2013** (2013), Article ID 472956, DOI: 10.1155/2013/472956.

[48] P. M. Weichsel, The Kronecker product of graphs, *Proceedings of the American Mathematical Society* **13** (1962), 47 – 52, DOI: 10.1090/s0002-9939-1962-0133816-6.

[49] T. Wilhelm and J. Hollunder, Information theoretic description of networks, *Physica A: Statistical Mechanics and its Applications* **385**(1) (2007), 385 – 396, DOI: 10.1016/j.physa.2007.06.029.

[50] J. Xu, *Topological Structure and Analysis of Interconnection Networks*, Network Theory and Applications series (NETA, Vol. **7**), Springer, New York (2013), DOI: 10.1007/978-1-4757-3387-7.

[51] R. Yasudo, K. Nakano, M. Koibuchi, H. Matsutani and H. Amano, Designing low-diameter interconnection networks with multi-ported host-switch graphs, *Concurrency and Computation: Practice and Experience* **35**(11) (2020), e6115, DOI: 10.1002/cpe.6115.

[52] F. Zahn, *Energy-Efficient Interconnection Networks for High-Performance Computing*, Doctoral dissertation, Ruprecht–Karls University, Heidelberg, German (2020), DOI: 10.11588/heidok.00029177.

[53] B. Zhou and I. Gutman, Further properties of Zagreb indices, *MATCH Communications in Mathematical and in Computer Chemistry* **54** (2005), 233 – 239, URL https://match.pmf.kg.ac.rs/electronic_versions/Match54/n1/match54n1_233-239.pdf.

[54] Q. Zhu, K. Thulasiraman, M. Xu and S. Radhakrishnan, Hybrid PMC (HPMC) fault model and diagnosability of interconnection networks, *AKCE International Journal of Graphs and Combinatorics* **17**(3) (2020), 755 – 760, DOI: 10.1016/j.akcej.2019.12.008.