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Research Article

The Karush-Kuhn-Tucker Optimality Condition for *q*-Rung Orthopair Fuzzy Optimization Problem

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Abstract. In this paper, we defined the interval form of α,β level set for triangular q-rung orthopair fuzzy (qROPF) number. To obtain a differentiability notion for qROPF valued functions, Hukuhara differentiability (H-differentiability) and α,β level-wise H-differentiability is defined. Using this KKT optimality condition for qROPF optimization problem with triangular qROPF objective function are formulated.

Keywords. Triangular qROPF number, Hukuhara differentiability, KKT optimality conditions, α,β level set

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1. Introduction

Optimization Problems (OP) are modelled and solved as deterministic OPs, but in the real-world applications, OPs are not purely deterministic, because it contains uncertain information. Therefore, to model such uncertain and imprecise data, fuzzy optimization problems were introduced. Again, through further research, fuzzy set theory is advanced by *Intuitionistic Fuzzy Set Theory* (InFST), then by *PYthagorean Fuzzy Sets* (PYFS) and finally qROPF set theory.

In Intuitionistic Fuzzy set theory, the degree of belongingness and non-belongingness of elements in the set are modelled by degree of membership and degree of non-membership on $[0,1]$ such that, whose sum is less than 1. However, because $0.5+0.6>1$, an analyst cannot choose membership (mbp) and non-membership (n-mbp) degrees of an element such as 0.5 and 0.6. To overcome such situations, the PYFS theory was introduced, and then, in 2016, Yager [15] introduced the qROPF set theory. This theory allows the analyst to choose membership and non-membership degrees such that, sum of the qth power is less than one, for $q \geq 1$.

Motivated by fuzzy optimization problem, we discuss the qROPF optimization problem in this paper by considering a triangular qROPF (T-qROPF) valued objective function with convex linear inequality constraints. Therefore, we introduced α,β level set for triangular qROPF numbers and Hukuhara differentiability for qROPF valued functions. By using them, we formulated the KKT optimality condition for the qROPF optimization problem. This condition allows us to obtain the non-dominated solution for the given qROPF optimization problem.

2. Preliminaries

Definition 2.1 ([2]). Let X be a universe of discourse, an *Intuitionistic Fuzzy Set* (IFS) $\widehat{A}$ on a set A is an object having the form $\widehat{A}=\{\langle \dot{x}, \zeta_{\widehat{A}}(\dot{x}), \eta_{\widehat{A}}(\dot{x})\rangle \mid \dot{x}\in X\}$, where the function $\zeta_{\widehat{A}}(\dot{x}):\widehat{A}\to[0,1]$ is the degree of mbp, and $\eta_{\widehat{A}}(x):\widehat{A}\to[0,1]$ is the degree of n-mbp of the elements in the set $\widehat{A}$ satisfying $0\leq \zeta_{\widehat{A}}(x)+\eta_{\widehat{A}}(x)\leq 1$.

Definition 2.2 ([15]). Let X be a universe of discourse, a *PYthagorean Fuzzy Set* (PYFS) in X is given by $\widehat{P}=\{\langle x, \zeta_{\widehat{P}}(\dot{x}), \eta_{\widehat{P}}(\dot{x})\rangle \mid \dot{x}\in X\}$, where the function $\zeta_{\widehat{P}}(\dot{x}):X\to[0,1]$ is the degree of mbp, and $\eta_{\widehat{P}}(\dot{x}):X\to[0,1]$ is the degree of n-mbp of the elements in the set S satisfying $0\leq (\zeta_{\widehat{P}}(\dot{x}))^2+(\eta_{\widehat{P}}(\dot{x}))^2\leq 1$.

Definition 2.3 ([14]). Let X be a universe of discourse, a qROPF set $\widetilde{Q}$ in X is given by

$$\widetilde{Q}=\{\langle \dot{x},\zeta_{\widetilde{Q}}(\dot{x}), \eta_{\widetilde{Q}}(\dot{x})\rangle \mid \dot{x}\in X\}, \tag{2.1}$$

where the function $\zeta_{\widetilde{Q}}(\dot{x}):X\to[0,1]$ is the degree of mbp, and $\eta_{\widetilde{Q}}(\dot{x}):X\to[0,1]$ is the degree of n-mbp of the elements in the set X satisfying $0\leq(\zeta_{\widetilde{Q}}(\dot{x}))^q+(\eta_{\widetilde{Q}}(\dot{x}))^q\leq 1$, $q\geq 1$, for all $\dot{x}\in S$. The degree of Hesitancy of elements in X is denoted by $\Pi_{\widetilde{Q}}(\dot{x})=(1-(\zeta_{\widetilde{Q}}(\dot{x}))^q-(\eta_{\widetilde{Q}}(\dot{x}))^q)^{1/q}$.

Definition 2.4 ([4]). A qROPF relation ${}^qR$ is a qROPF subset of $X\times Y$, which is defined as

$${}^qR=\{\langle(\dot{x},\dot{y}),\zeta_{{}^qR}(\dot{x},\dot{y}),\eta_{{}^qR}(\dot{x},\dot{x})\rangle \mid \dot{x}\in X, \dot{y}\in Y\}, \tag{2.2}$$

where $\zeta_{{}^qR}(\dot{x},\dot{y}):X\times Y\to[0,1]$ and $\eta_{{}^qR}(\dot{x},\dot{y}):X\times Y\to[0,1]$ denote the mbp and n-mbp degrees of $(\dot{x},\dot{y})$ in $X\times Y$ which satisfies the condition $0\leq\zeta^q_{{}^qR}(\dot{x},\dot{y})+\eta^q_{{}^qR}(\dot{x},\dot{y})\leq 1$, for all $(\dot{x},\dot{y})\in X\times Y$. The set of all qROPF relations on $X\times Y$ is denoted qROPF$(X\times Y)$.

Definition 2.5 ([1]). Let $\widehat{A}$ and $\widehat{B}$ be two nonempty subsets of R^n and $c\in R$. The Minkowski addition and scalar multiplication are defined as follows:

$$\widehat{A}+\widehat{B}=\{\widehat{a}+\widehat{b}:\widehat{a}\in\widehat{A} \text{ and } \widehat{b}\in\widehat{B}\},$$
$$\widehat{c}\widehat{A}=\{\widehat{c}\widehat{a}:\widehat{a}\in\widehat{A}\}.$$

Definition 2.6 (Convexity of Fuzzy Mapping [6])**.** A fuzzy valued function $\ddot{f}:\Omega\subseteq\mathbb{R}^n\to\mathbb{F}_\Omega$ is said to be convex on a convex subset $\Omega\subseteq\mathbb{R}^n$, if for each $\alpha\in[0,1]$ both level sets $\ddot{f}(\dot{x},\alpha)^L$ and $\ddot{f}(\dot{x},\alpha)^U$ are convex on Ω. That is for $0\le\lambda'\le 1$, $\dot{x},\dot{y}\in\Omega$

$$\ddot{f}^L_\alpha((1-\lambda')\dot{x}+\lambda'\dot{y})\le(1-\lambda')\ddot{f}^L_\alpha(\dot{x})+\lambda'\ddot{f}^L_\alpha(\dot{y}),$$

$$\ddot{f}^U_\alpha((1-\lambda')\dot{x}+\lambda'\dot{y})\le(1-\lambda')\ddot{f}^U_\alpha(\dot{x})+\lambda'\ddot{f}^U_\alpha(\dot{y}).$$

$\ddot{f}$ is said to be concave if $-\ddot{f}$ is convex.

3. Differentiability of *q*ROPF Valued Function

Definition 3.1 ([9])**.** Let $\ddot{A}=\langle(\dot{a},\dot{b},\dot{c}):M_{\ddot{A}},N_{\ddot{A}}\rangle$ be a Triangular qROPF number. Its mbp function and n-mbp functions are given by

$$\zeta_{\ddot{A}}(\dot{x})=\begin{cases}\frac{(\dot{x}-\dot{a})M_{\ddot{A}}}{\dot{b}-\dot{a}}, & a\le\dot{x}\le\dot{b},\\ \frac{(\dot{c}-\dot{x})M_{\ddot{A}}}{\dot{c}-\dot{b}}, & \dot{b}\le\dot{x}\le\dot{c},\\ 0, & \dot{a}>\dot{x}\text{ or }\dot{x}>\dot{c},\end{cases}\tag{3.1}$$

$$\eta_{\ddot{A}}(\dot{x})=\begin{cases}\frac{\dot{b}-\dot{x}+N_{\ddot{A}}(\dot{x}-\dot{a})}{\dot{b}-\dot{a}}, & \dot{a}\le\dot{x}\le\dot{b},\\ \frac{\dot{x}-\dot{b}+N_{\ddot{A}}(\dot{c}-\dot{x})}{\dot{c}-\dot{b}}, & \dot{b}\le\dot{x}\le\dot{c},\\ 1, & \dot{a}>\dot{x}\text{ or }\dot{x}>\dot{c},\end{cases}\tag{3.2}$$

where $\zeta_{\ddot{A}}(\dot{x})$ and $\eta_{\ddot{A}}(\dot{x})$ denote the degree of mbp and n-mbp of $\ddot{A}$, $M_{\ddot{A}}$ denote maximum degree of mbp and $N_{\ddot{A}}$ denote minimum degree of n-mbp, $M_{\ddot{A}},N_{\ddot{A}}\in[0,1]$, and $0\le M^q_{\ddot{A}}+N^q_{\ddot{A}}\le 1$. $a,b,c\in[0,1]$ and $\zeta_{\ddot{A}}(\dot{x})^q+\eta_{\ddot{A}}(\dot{x})^q\le 1$.

Definition 3.2. The (α,β) level set of the triangular qROPF number $\ddot{A}=\langle(\dot{a},\dot{b},\dot{c}):M_{\ddot{A}},N_{\ddot{A}}\rangle$ is the set of all x such that whose degree of mbp greater than equal to α and degree of n-mbp is less than or equal to β, i.e.,

$$\ddot{A}_{\alpha,\beta}=\{\dot{x}\in X:\zeta_{\ddot{A}}(\dot{x})\ge\alpha,\eta_{\ddot{A}}(\dot{x})\le\beta,\alpha^q+\beta^q\le 1\}\tag{3.3}$$

which is represented in the interval form $\ddot{A}_{\alpha,\beta}=[\ddot{A}^L,\ddot{A}^U]$, where

$$\ddot{A}^L=\max\left\{\dot{a}+\frac{\alpha(\dot{b}-\dot{a})}{M_{\ddot{A}}},\frac{\dot{b}(1-\beta)+\dot{a}(\beta-N_{\ddot{A}})}{1-N_{\ddot{A}}}\right\},\tag{3.4}$$

$$\ddot{A}^U=\min\left\{\dot{c}-\frac{\alpha(\dot{c}-\dot{b})}{M_{\ddot{A}}},\frac{\dot{b}(1-\beta)+\dot{c}(\beta-N_{\ddot{A}})}{1-N_{\ddot{A}}}\right\}.\tag{3.5}$$

Definition 3.3. Let $\widehat{X}$ and $\widehat{Y}$ be two sets, a qROP fuzzy valued function from X to qROPF set of $\widehat{Y}$ is defined as

$$\varphi:\widehat{X}\to qROPFS(\widehat{Y})\text{ such that }x'\to\varphi(x'),$$

where $\varphi(x')=\langle y',\zeta^q_{\varphi(x')}(y'),\ \eta^q_{\varphi(x')}(y'),\ y'\in\widehat{Y}\rangle$, where for $x'\in\widehat{X},\zeta^q_{\varphi(x')}(y'):\widehat{Y}\to[0,1]$ and $\eta^q_{\varphi(x')}(y'):\widehat{Y}\to[0,1]$ denote the mbp and n-mbp of $y'=\varphi(x')$, such that $0\le\zeta^q_{\varphi(x')}(y')+\eta^q_{\varphi(x')}(y')\le 1$.

Let G be an open subset of R^n and $\mathbb{Q}^n$ be the set of all qROPF numbers. Let $\ddot{f}: G \to \mathbb{Q}^1$ be a qROPF valued function defined on G.

Definition 3.4. For two qROPF numbers q_1, q_2, $d = q_1 \boxminus q_2$ means the Hukuhara difference of q_1 and q_2, if there exist qROPF number d such that $d \boxminus q_2 = q_1$.

Definition 3.5. Let P and Q be two qROPF sets then the Hausdorff distance between P and Q is defined by $\mathbb{HD}(P,Q) = \min(\mathbb{HD}(q_i, q_j))$, where $q_i \in P$ and $q_j \in Q$.

Definition 3.6. Let $q_1 = (\zeta^q_{q_1}, \eta^q_{q_1})$ and $q_2 = (\zeta^q_{q_2}, \eta^q_{q_2})$ be two qROPF numbers, the Hausdorff distance between A and B is given by

$$\mathbb{HD}(q_1, q_2) = \max(|\zeta^q_{q_1} - \zeta^q_{q_2}|, |\eta^q_{q_1} - \eta^q_{q_2}|). \tag{3.6}$$

Theorem 3.1. *Let $\tilde{q}_i$ and $\tilde{q}_j$ be two qROPF numbers in $\mathbb{Q}^n$, such that $\tilde{q}_i = (\zeta^q_{\tilde{q}_i}, \eta^q_{\tilde{q}_i})$ and $\tilde{q}_j = (\zeta^q_{\tilde{q}_j}, \eta^q_{\tilde{q}_j})$. The function $\mathbb{HD}(\tilde{q}_i, \tilde{q}_j) = \max(|\zeta^q_{\tilde{q}_i} - \zeta^q_{\tilde{q}_j}|, |\eta^q_{\tilde{q}_i} - \eta^q_{\tilde{q}_j}|)$, is a metric on $\mathbb{Q}^n$.*

Proof. We are going to show that $\mathbb{HD}$ satisfies all the axioms of a metric

- M_1: $\mathbb{HD}(\tilde{q}_i, \tilde{q}_j) = 0 \Leftrightarrow \max(|\zeta^q_{\tilde{q}_i} - \zeta^q_{\tilde{q}_j}|, |\eta^q_{\tilde{q}_i} - \eta^q_{\tilde{q}_j}|) = 0 \Leftrightarrow \zeta^q_{\tilde{q}_i} = \zeta^q_{\tilde{q}_j}, \eta^q_{\tilde{q}_i} = \eta^q_{\tilde{q}_j}$.
  Therefore, $\Leftrightarrow \tilde{q}_i = \tilde{q}_j$.
- M_2: $\mathbb{HD}(\tilde{q}_i, \tilde{q}_j) = \max(|\zeta^q_{\tilde{q}_i} - \zeta^q_{\tilde{q}_j} \eta^q_{\tilde{q}_i} - \eta^q_{\tilde{q}_j}|) = \max(|\zeta^q_{\tilde{q}_j} - \zeta^q_{\tilde{q}_i}|, |\eta^q_{\tilde{q}_j} - \eta^q_{\tilde{q}_i}|) = \mathbb{HD}(\tilde{q}_j, \tilde{q}_i)$.
- M_3: Let $\tilde{q}_i, \tilde{q}_j$ and $\tilde{q}_k \in \mathbb{Q}^n$. It is obvious that $\mathbb{HD}(\tilde{q}_i, \tilde{q}_j) \le \mathbb{HD}(\tilde{q}_i, \tilde{q}_k) + \mathbb{HD}(\tilde{q}_k, \tilde{q}_j)$, by applying $\max\{x, y\} = \frac{x+y+|x-y|}{2}$.

From the axioms M_1, M_2 and M_3 we get $\mathbb{HD}$ is a metric on $\mathbb{Q}^n$. Therefore, $(\mathbb{Q}^n, \mathbb{HD})$ is a metric space. □

Definition 3.7. A qROPF valued function $\ddot{f}: [a,b] \subseteq R \to \mathbb{Q}^n$ is said to be continuous at $x'_0 \in [a,b]$ if for every $\varepsilon > 0$, $\delta > 0$ such that for all $x' \in [a,b]$ with $|x' - x'_0| < \delta \Rightarrow \mathbb{HD}(\ddot{f}(x'), \ddot{f}(x'_0)) < \varepsilon$.

Definition 3.8. A qROPF valued function $\ddot{f}: (a,b) \subseteq R \to \mathbb{Q}^1$ is said to be Hukuhara-differentiable or H-differentiable at $t'_0 \in (a,b)$, if the both limits

$$\lim_{\Delta t' \to 0^+} \frac{\ddot{f}(t'_0 + \Delta t') \boxminus \ddot{f}(t'_0)}{\Delta t'} = \lim_{\Delta t' \to 0^+} \frac{\ddot{f}(t'_0) \boxminus \ddot{f}(t'_0 + \Delta t')}{\Delta t'} \tag{3.7}$$

exist and is denoted by $\ddot{f}'(t'_0)$.

Definition 3.9. Let X be a set, a qROP fuzzy valued function $\ddot{f}: X \to \mathbb{Q}$ is said to be α, β level-wise H-differentiable at $\dot{x} \in X$, if and only if both $\ddot{f}^L_{\alpha,\beta}$ and $\ddot{f}^U_{\alpha,\beta}$-differentiable at $\dot{x}_0 \in X$, for all $\alpha, \beta \in [a,b]$ with $0 \le \alpha^q + \beta^q \le 1$. That is the limits

$$\lim_{h \to 0^+} \frac{\ddot{f}^L_{\alpha,\beta}(\dot{x}_0 + h) \boxminus \ddot{f}^L_{\alpha,\beta}(\dot{x}_0)}{h}, \quad \lim_{h \to 0^+} \frac{\ddot{f}^U_{\alpha,\beta}(\dot{x}_0) \boxminus \ddot{f}^U_{\alpha,\beta}(\dot{x}_0 + h)}{h} \tag{3.8}$$

exists.

Theorem 3.2. *If a qROPF valued function $\ddot{f}: X \to \mathbb{Q}^1$ is said to be H-differentiable, then the interval valued function $\ddot{f}_{\alpha,\beta}$ are also H-differentiable. In particular $\ddot{f}^L_{\alpha,\beta}(\bar{X})$, and $\ddot{f}^U_{\alpha,\beta}(\bar{X})$, are also H-differentiable for every $\alpha,\beta \in [0,1]$ with $0 \le \alpha^q + \beta^q \le 1$.*

Proof. Proof is obvious from the definition of $\ddot{f}_\alpha$ and $\ddot{f}_\beta$.
Let G be an open subset of R^n and for $\bar{X} = (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)$, $\ddot{f}(\bar{X}) \in \mathbb{Q}$ and

$$\ddot{f}^L_{\alpha,\beta}(\bar{X}) = \ddot{f}^L_{\alpha,\beta}(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n) = \ddot{f}(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)|^L_{\alpha,\beta},$$
$$\ddot{f}^U_{\alpha,\beta}(\bar{X}) = \ddot{f}^U_{\alpha,\beta}(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n) = \ddot{f}(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)|^U_{\alpha,\beta}.$$

Definition 3.10. The Gradient of the continuous qROPF valued function $\ddot{f}$ denoted by $\triangledown \ddot{f}(\bar{X})$ and is defined as $\nabla \ddot{f}(\bar{X}) = \left(\frac{\partial \ddot{f}(\bar{X})}{\partial \dot{x}_1}, \frac{\partial \ddot{f}(\bar{X})}{\partial \dot{x}_2}, \dots, \frac{\partial \ddot{f}(\bar{X})}{\partial \dot{x}_n}\right)$.

4. KKT Optimality Condition for *q*ROPF Optimization Problem

Consider the following linear optimization problem with qROPF valued objective function and inequality constraints

$$\text{(qFP)} \quad \max/\min \ddot{f}(X) = \ddot{f}(x_1, x_2, \dots, x_n), \quad i = 1,2,\dots,k$$
$$\text{subject to } \widetilde{h}_j(X) \ge \text{ or } \le 0, \text{ for } j = 1,2,\dots,m,$$

where X belongs to an open set $G \subseteq R^n$, $\ddot{f}(X)$ and $\widetilde{h}_j \in \mathbb{Q}^n$.
For convenience, consider a minimization problem

$$\text{(qFoP)} \quad \min \ddot{f}(X) = \ddot{f}(x_1, x_2, \dots, x_n), \quad i = 1,2,\dots,k$$
$$\text{subject to } \widetilde{h}_j(X) \le 0, \text{ for } j = 1,2,\dots,m,$$

where $\ddot{f}, \widetilde{h}_j$, where $j = 1$ to m are convex, continuously H-differentiable functions defined on $\mathbb{R}^n$.

Definition 4.1. A solution X° is said to be a non-dominated solution to the qROPF optimization problem, if there exist no other feasible solution $X^\star$ such that $\ddot{f}(X^\star) < \ddot{f}(X^\circ)$.

Theorem 4.1. *Let $X = \{\ddot{X} \in R^n : \widetilde{h}_j(\ddot{X}) \le 0, j = 1,2\dots,n\}$, be a feasible set of the constraints $\widetilde{h}_j : \mathbb{R}^n \to \mathbb{R}$, assume that which is convex and continuously H-differentiable at $\ddot{X}^\circ \in X$ for $j = 1,2,\dots,n$. Suppose that the objective function $\ddot{f} : \mathbb{R}^n \to \mathbb{Q}$ is convex and α,β level-wise continuously H-differentiable at $\ddot{X}^\circ$. If there exist non-negative real valued functions (Lagrange function multipliers) ψ_j for $j = 1,2,\dots,m$ defined on $[0,1]$ such that,*

$$\text{(i)} \quad \nabla \ddot{f}^L_{\alpha,\beta}(\ddot{X}^\circ) + \nabla \ddot{f}^U_{\alpha,\beta}(\ddot{X}^\circ) + \sum_{j=1}^{m} \psi_j(\alpha,\beta) \nabla \widetilde{h}_j(\ddot{X}^\circ) = 0, \quad \textit{for all } \alpha,\beta \in [0,1], \tag{4.1}$$

$$\text{(ii)} \quad \psi_j(\alpha,\beta) \widetilde{h}_j(\ddot{X}^\circ) = 0, \quad \textit{for all } \alpha,\beta \in [0,1], \tag{4.2}$$

then $\ddot{X}^\circ$ is a non-dominated solution of the problem.

Proof. To prove the theorem we assume the contradiction that, there will no such non-dominated solution to the qRoP problem satisfying the assumptions and conditions (i) and (ii), i.e., there exist $\bar{X} \in X$ such that $\ddot{f}(\bar{X}) < \ddot{f}(X^{\circ})$. Hence,

$$\begin{cases} \ddot{f}^L_{\alpha,\beta}(\bar{X}) < \ddot{f}^L_{\alpha,\beta}(\ddot{X}^{\circ}) \\ \ddot{f}^U_{\alpha,\beta}(\bar{X}) \le \ddot{f}^U_{\alpha,\beta}(\ddot{X}^{\circ}) \end{cases} \quad \text{or} \quad \begin{cases} \ddot{f}^L_{\alpha,\beta}(\bar{X}) \le \ddot{f}^L_{\alpha,\beta}(\ddot{X}^{\circ}) \\ \ddot{f}^U_{\alpha,\beta}(\bar{X}) < \ddot{f}^U_{\alpha,\beta}(\ddot{X}^{\circ}) \end{cases} \quad \text{or} \quad \begin{cases} \ddot{f}^L_{\alpha,\beta}(\bar{X}) < \ddot{f}^L_{\alpha,\beta}(\ddot{X}^{\circ}) \\ \ddot{f}^U_{\alpha,\beta}(\bar{X}) < \ddot{f}^U_{\alpha,\beta}(\ddot{X}^{\circ}) \end{cases} \tag{4.3}$$

Now, define a function

$$\dot{f}(X) = \ddot{f}^L_{\alpha,\beta}(X) + \ddot{f}^U_{\alpha,\beta}(X). \tag{4.4}$$

Therefore,

$$\begin{aligned} \dot{f}(\bar{X}) &= \ddot{f}^L_{\alpha,\beta}(\bar{X}) + \ddot{f}^U_{\alpha,\beta}(\bar{X}) \\ &< \ddot{f}^L_{\alpha,\beta}(\ddot{X}^{\circ}) + \ddot{f}^U_{\alpha,\beta}(\ddot{X}^{\circ}) \quad \text{(from (4.3))} \\ &< \dot{f}(\ddot{X}^{\circ}), \end{aligned}$$

i.e.,

$$\dot{f}(\bar{X}) < \dot{f}(\ddot{X}^{\circ}). \tag{4.5}$$

Since $\ddot{f}$ is continuously differentiable and convex at $\ddot{X}^{\circ}$, $\ddot{f}$ is level-wise continuously differentiable and convex at $\ddot{X}^{\circ}$, i.e., $\ddot{f}^L_{\alpha,\beta}$ and $\ddot{f}^U_{\alpha,\beta}$ is convex and differentiable

$$\nabla\dot{f}(X) = \nabla\ddot{f}^L_{\alpha,\beta}(X) + \nabla\ddot{f}^U_{\alpha,\beta}(X), \quad \text{for all } \alpha,\beta \in [0,1].$$

Hence for any fixed $\alpha',\beta' \in [0,1]$

$$\begin{aligned} & \nabla\ddot{f}^L_{\alpha',\beta'}(\ddot{X}^{\circ}) + \nabla\ddot{f}^U_{\alpha',\beta'}(\ddot{X}^{\circ}) + \sum_{j=1}^{m} \psi_j(\alpha',\beta')\nabla\widetilde{h}_j(\ddot{X}^{\circ}) = 0 \\ \implies \quad & \nabla f(X) + \sum_{j=1}^{m} \psi_j(\alpha',\beta')\nabla\widetilde{h}_j(\ddot{X}^{\circ}) = 0. \end{aligned} \tag{4.6}$$

Also,

$$\psi_j(\alpha',\beta')\widetilde{h}_j(\ddot{X}^{\circ}) = 0, \quad \text{for all } j = 1,2,\dots,m. \tag{4.7}$$

Now consider a new qROPF valued optimization problem

$$\begin{aligned} & \min \dot{f}(\dot{X}) \\ & \text{subject to } \widetilde{h}_j(\dot{X}) \le 0, \quad \text{for all } j = 1,2,\dots,m. \end{aligned}$$

Clearly from (4.6) and (4.7), the above optimization problem satisfies the conditions (i) and (ii) of the hypothesis. Therefore, $\dot{f}(\ddot{X}^{\circ})$ is an optimal solution to the problem.
But this contradicts equation (4.5).
Therefore, our assumption is wrong that is $\ddot{X}^{\circ}$ is a non-dominated solution to the problem. □

Illustrative Example. Consider the following qROPF optimization problem

$$\ddot{f} = \langle(4,5,7);0.7,0.3\rangle \otimes \dot{x}_1 \oplus \langle(5,6,9);0.3,0.8\rangle \otimes \dot{x}_2 \oplus \langle(6,7,8);0.4,0.5\rangle \otimes \dot{x}_3 \oplus \langle(1,2,3);0.2,0.6\rangle \dot{x}_4$$

$$\begin{aligned} \text{subject to } & h_1(\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4) = \dot{x}_1 + \dot{x}_2 + 3\dot{x}_3 + 2\dot{x}_4 \le 35 \\ & h_2(\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4) = 3\dot{x}_1 + 5\dot{x}_2 + 4\dot{x}_3 + 2\dot{x}_4 \le 40 \\ & 4\dot{x}_1 + 3\dot{x}_3 \le 15 \end{aligned}$$

$$\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4 \geq 0$$

$$\ddot{f} = \langle(4,5,7);0.7,0.3\rangle \otimes \dot{x}_1 \oplus \langle(5,6,9);0.3,0.8\rangle \otimes \dot{x}_2 \oplus \langle(6,7,8);0.4,0.5\rangle \otimes \dot{x}_3 \oplus \langle(1,2,3);0.2,0.6\rangle \dot{x}_4$$

subject to $h_1(\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4) = \dot{x}_1 + \dot{x}_2 + 3\dot{x}_3 + 2\dot{x}_4 - 35$

$$h_2(\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4) = 3\dot{x}_1 + 5\dot{x}_2 + 4\dot{x}_3 + 2\dot{x}_4 - 40$$

$$4\dot{x}_1 + 3\dot{x}_3 - 15$$

$$\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4 \geq 0$$

For $\alpha, \beta \in [0,1]$ with $0 \leq \alpha^q + \beta^q \leq 1$

$$\ddot{f}^L_{\alpha,\beta} = \dot{x}_1\left(\frac{3.8-\beta}{0.7}\right) + \dot{x}_2\left(\frac{2-\beta}{0.2}\right) + \dot{x}_3\left(6 + \frac{\alpha}{0.4}\right) + \dot{x}_4\left(1 + \frac{\alpha}{0.6}\right),$$

$$\ddot{f}^U_{\alpha,\beta} = \dot{x}_1\left(\frac{2.9+2\beta}{0.7}\right) + \dot{x}_2\left(\frac{3\beta-1.2}{0.2}\right) + \dot{x}_3\left(8 - \frac{\alpha}{0.4}\right) + \dot{x}_4\left(3 - \frac{\alpha}{0.6}\right).$$

For $\alpha, \beta \in [0,1]$ with $0 \leq \alpha^q + \beta^q \leq 1$, we can also obtain

$$\nabla \ddot{f}^L_{\alpha,\beta} = \begin{bmatrix} \frac{3.8-\beta}{0.7} \\ \frac{2-\beta}{0.2} \\ 6 + \frac{\alpha}{0.4} \\ 1 + \frac{\alpha}{0.4} \end{bmatrix}, \quad \nabla \ddot{f}^U_{\alpha,\beta} = \begin{bmatrix} \frac{2.9+2\beta}{0.7} \\ \frac{3\beta-1.2}{0.2} \\ 8 - \frac{\alpha}{0.4} \\ 3 - \frac{\alpha}{0.6} \end{bmatrix}, \quad \nabla \widetilde{h}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \quad \nabla \widetilde{h}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \\ 2 \end{bmatrix}, \quad \nabla \widetilde{h}_3 = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Solve $\widetilde{h}_1 = 0$ and $\widetilde{h}_2 = 0$, obtain the solution $x^* = (0,0,5,10)$. Solved using LINGO 18.0.

Applying the conditions (4.1) and (4.2)

$$\begin{cases} \left(\frac{3.8-\beta}{0.7}\right) + \left(\frac{2.9+2\beta}{0.7}\right) + \psi_1(\alpha,\beta) + 3\psi_2(\alpha,\beta) + 4\psi_3(\alpha,\beta) - \psi_4 = 0, \\ \left(\frac{2-\beta}{0.2}\right) + \left(\frac{3\beta-1.2}{0.2}\right) + \psi_1(\alpha,\beta) + 5\psi_2(\alpha,\beta) = 0, \\ \left(6 + \frac{\alpha}{0.4}\right) + \left(8 - \frac{\alpha}{0.4}\right) + 3\psi_1(\alpha,\beta) + 4\psi_2(\alpha,\beta) + 3\psi_3(\alpha,\beta) = 0, \\ \left(1 + \frac{\alpha}{0.4}\right) + \left(3 - \frac{\alpha}{0.6}\right) + 2\psi_1(\alpha,\beta) + 2\psi_2(\alpha,\beta) - \psi_8 = 0. \end{cases} \tag{4.8}$$

The above system is feasible and we get non-negative values for $\psi_1, \psi_2, \psi_3, \psi_8, \psi_4$ is arbitrary, therefore x^* become a solution to the given optimization problem.

5. Conclusion

The Hukuhara difference for qROPF sets and Hukuhara differentiability or H-differentiability for qROPF valued functions are defined. The α,β level set $[\ddot{A}^L, \ddot{A}^U]$ is defined for triangular qROPF number $\ddot{A}$, using this we formulated the Karush-Kuhn-Tucker optimality condition for qROPF optimization problem. Also, a qROPF optimization problem is illustrated using the proposed KKT optimality condition.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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