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Research Article

Δ^m-Ideal Convergence of Generalized Difference Sequences in Neutrosophic Normed Spaces

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Abstract. The objective of this paper is to introduce the perception of ideal convergence of generalized difference sequences in Neutrosophic Normed Spaces. We defined the concepts of Δ^m-I_N-Cauchy and Δ^m-I_N-completeness for generalized difference sequences in Neutrosophic Normed Spaces (briefly known as N.N.S). Another, closely related concept Δ^m-I_N^*-convergence, Δ^m-I_N^*-Cauchy and Δ^m-I_N^*-completeness in N.N.S are also defined. Later, we establish some relations among these perceptions which shows that this method of convergence is more generalized.

Keywords. Neutrosophic normed spaces, Statistical convergence, Statistical cauchy, Difference sequence, Generalized difference sequence, I-convergence and I-cauchy

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1. Introduction

The most interesting generalization of the concept of classical convergence of sequences was coined by Zygmund [34] and stated as statistical convergence in 1935. Steinhaus [31] and Fast [8] also presented the perception of statistical convergence simultaneously in the same year 1951. The perception was initiated to deal with the theory of series summation and has been studied by various researchers in different spaces such as intutionistic fuzzy normed spaces [20], random 2-normed spaces [21], probabilistic normed spaces [10] etc. It has also been studied for different sequences such as ordinary sequences [28], double sequences [23], triple sequences [27] and multiple sequences [19] by various authors see [3, 4, 26].

Definition 1.1. Assume K be a subset of $\mathbb{N}$. The natural density $\delta(K)$ of the set K is defined as:

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \in K : k \le n\}|,$$

where the vertical bars denotes the order of enclosed set.

If $\{x_{k_i}\}$ is a subsequence of the sequence $x = \{x_k\}_k \in \mathbb{N}$ of $\mathbb{R}$ and $A = \{k_i : i \in \mathbb{N}\}$, then we denote it by $\{x_A\}$. In case $\delta(A) = 0$, $\{x_A\}$ is called a subsequence of natural density zero or a thin subsequence of x. However, if $\{x_A\}$ does not have natural density zero or fails to have natural density then it is named as non-thin subsequence of x.

Definition 1.2 ([8]). A sequence $x = \{x_k\}$ is named as statistically convergent to number L if, for every $\epsilon > 0$, we have $\delta(\{k \le n : |x_k - L| \ge \epsilon\}) = 0$. Symptomatic as $St - \lim x_k = L$ and St is the collection of all the statistically convergent sequences.

The idea is generalized by Kostyrko *et al.* [16] with the help of an admissible ideal I and presented nicely a new method of convergence which he called I-convergence. After his remarkable work, much efforts has been done by different authors to study I-convergence and related concepts. For some pioneer works on I-convergence, we refer [5, 9, 15–18, 24] etc.

On another side, Fuzzy sets were initiated by Zadeh [33] and generalized by Yu and Yuan [32] while observing that Zadeh's idea of fuzzy sets need more modification to handle certain problems in time domain. He called this set as Intuitionistic fuzzy sets. His work is followed by many authors, for instance intuitionistic fuzzy metric spaces by Park [25], Intuitionistic fuzzy topological spaces by Karakus *et al.* [12] etc. In last few years ideas of Statistical convergence and I-convergence have been extended in Intuitionistic fuzzy normed spaces respectively in [11] and [22].

Recently, Smarandache [30], initiated a new generalization of Intuitionistic fuzzy sets and called it Neutrosophic set. This idea is further used to define Neutrosophic metric spaces and Neutrosophic soft linear spaces respectively in [13] and [1]. In another paper, Bera and Mahapatra [2] initiated the concept of Neutrosophic norm and define some sequential concepts like convergence, Cauchy and convexity in these spaces.

Recently, Kirişci and Şimşek [13] extended statistical convergence and study its properties in these spaces. We aim in this paper, to extend and study some concepts: I_N-convergence, I_N-Cauchy, I_N-completeness, I_N^*-convergence, I_N^*-Cauchy and I_N^*-completeness in N.N.S.

In 1981, Kizmaz [14] initiated the perception of difference sequence spaces $X(\Delta)$ for $X = l_\infty, c$, c_0, where $X(\Delta)$ is a Banach space. Further, Et and Çolak [6] generalized the perception of difference sequences using m as a non-negative integer, then the generalized difference operator $\Delta^m x_k$ is defined as

$$\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1},$$

where $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$.

Definition 1.3 ([7]). Assume m be a fixed positive integer. A sequence $x = \{x_k\}$ is named as Δ^m-statistically convergent to L if for every $\epsilon > 0$, we have

$$\delta\left(\{k \le n : |\Delta^m x_k - L| \ge \epsilon\}\right) = \lim_{n\to\infty} \frac{1}{n}|\{k \in \mathbb{N} : |\Delta^m x_k - L| \ge \epsilon\}| = 0.$$

Symptomatic as St-$\lim \Delta^m x_k = L$.

Some basic definitions used for the progression of the perception are given below:

Definition 1.4 ([29]). A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ is named as a continuous trangularnormort-norm if it satisfies the following conditions:

(i) $\odot$ is associative and commutative,

(ii) $\odot$ is continuous,

(iii) $a \odot 1 = a$ for every $a \in [0,1]$ and

(iv) $a \odot b \le c \odot d$ for each a, b, c and $d \in [0,1]$.

Definition 1.5 ([30]). A binary operation $\cdot : [0,1] \times [0,1] \to [0,1]$ is named as a continuous triangular conormort-conorm if it satisfies the following conditions:

(i) $\cdot$ is associative and commutative,

(ii) $\cdot$ is continuous,

(iii) $a \cdot 0 = a$ for every $a \in [0,1]$and

(iv) $a \cdot b \le c \cdot d$ whenever $a \le c$ and $b \le d$ for each a, b, c and $d \in [0,1]$.

Using these definitions, Kirişci and Şimşek [13], recently defined N.N.S and studied statistical convergence in these spaces.

Definition 1.6 ([13]). Assume F be a vector space, $N = \{\zeta, G(\zeta), B(\zeta), Y(\zeta) : \zeta \in F\}$ be a normed space in such a manner, $N : F \times \mathbb{R}^+ \to [0,1]$ and $\odot$, $\cdot$ respectively are continuous t-norm and continuous t-conorm. Then a four touple $V = (F, N, \odot, \cdot)$ is called a Neutrosophic Normed Spaces (N.N.S) if the following conditions are satisfied. For every $u, v \in F$ and $t, \lambda > 0$ and for every $\sigma \neq 0$ we have

(i) $0 \le G(u,t) \le 1$, $0 \le B(u,t) \le 1$, $0 \le Y(u,t) \le 1$ for every $t \in \mathbb{R}^+$;

(ii) $G(u,t) + B(u,t) + Y(u,t) \le 3$ for $t \in \mathbb{R}^+$;

(iii) $G(u,t) = 1$ (for$t > 0$) if and only if $u = 0$;

(iv) $G(\sigma u, t) = G(u, \frac{t}{|\sigma|})$;

(v) $G(u,\lambda) \odot G(v,t) \le G(u+v, \lambda+t)$;

(vi) $G(u,.)$ is continuous non-decreasing function;

(vii) $\lim_{t\to\infty} G(u,t) = 1$;

(viii) $B(u,t) = 0$ (for $t > 0$) if and only if $u = 0$;

(ix) $B(\sigma u, t) = B(u, \frac{t}{|\sigma|})$;

(x) $B(u,\lambda) \cdot B(v,t) \ge B(u+v, \lambda+t)$;

(xi) $B(u,.)$ is continuous non-decreasing function;

(xii) $\lim_{t\to\infty} B(u,t) = 0$;

(xiii) $Y(u,t) = 0$ (for $t > 0$) if and only if $u = 0$;

(xiv) $Y(\sigma u, t) = Y(u, \frac{t}{|\sigma|})$;

(xv) $Y(u,\lambda) \cdot Y(v,t) \ge Y(u+v, \lambda+t)$;

(xvi) $Y(u,.)$ is continuous non-decreasing function;

(xvii) $\lim_{t\to\infty} Y(u,t) = 0$ and

(xviii) $t \le 0$, then $G(u,t) = 0$, $B(u,t) = 1$ and $Y(u,t) = 1$.

Here, $N(G,B,Y)$ is called the Neutrosophic norm.

Some examples of N.N.S can be found in [13]. A sequence $\{x_k\}$ in N.N.S V is named as convergent if for each $\epsilon > 0$ and $t > 0$, $\exists$ a positive integer p and $L \in F$ in such a manner,

$$G(x_k - L,t) > 1-\epsilon,\ B(x_k - L,t) < \epsilon \text{ and } Y(x_k - L,t) < \epsilon\ \forall\ k \ge p.$$

This is equivalent to say

$$\lim_{k\to\infty} G(x_k - L,t) = 1, \lim_{k\to\infty} B(x_k - L,t) = 0 \text{ and } \lim_{k\to\infty} Y(x_k - L,t) = 0.$$

Symptomatic as $N\text{-}\lim_{k\to\infty}\{x_k\} = L$. The sequence $\{x_k\}$ is named as Cauchy if for each $\epsilon > 0$ and $t > 0$, $\exists$ a positive integer s in such a manner,

$$G(x_k - x_s,t) > 1-\epsilon,\ B(x_k - x_s,t) < \epsilon \text{ and } Y(x_k - x_s,t) < \epsilon\ \forall\ k,s \ge n$$

Definition 1.7 ([13])**.** Assume V be an N.N.S; $0 < \epsilon < 1$ and $t > 0$. A sequence $\{x_k\}$ in V is named as statistically convergent if there $\exists\ L \in F$ in such a manner,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : G(x_k - L,t) \le 1-\epsilon \text{ or } B(x_k - L,t) > \epsilon\} \text{ and } Y(x_k - L,t) > \epsilon\}\| = 0$$

equivalently, the natural density of the set

$$A(\epsilon,t) = \{k \in \mathbb{N} : G(x_k - L,t) \le 1-\epsilon \text{ or } B(x_k - L,t) > \epsilon\} \text{ and } Y(x_k - L,t) > \epsilon\}$$

is zero, that is, $\delta(A(\epsilon,t)) = 0$.

Definition 1.8 ([13])**.** Assume V be an N.N.S; $0 < \epsilon < 1$ and $t > 0$. A sequence $\{x_k\}$ in V is named as statistically Cauchy if $\exists s \in \mathbb{N}$ in such a manner,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : G(x_k - x_s,t) \le 1-\epsilon \text{ or } B(x_k - x_s,t) > \epsilon\} \text{ and } Y(x_k - x_s,t) > \epsilon\}| = 0$$

equivalently, the natural density of the set

$$A(\epsilon,t) = \{k \in \mathbb{N} : G(x_k - x_s,t) \le 1-\epsilon \text{ or } B(x_k - x_s,t) > \epsilon\} \text{ and } Y(x_k - x_s,t) > \epsilon\}$$

is zero, that is, $\delta(A(\epsilon,t)) = 0$.

Now we give a brief introduction related to I-convergence and related concepts. For any set X, Assume $\mathscr{P}(X)$ denotes the power set of X.

Definition 1.9 ([16])**.** Assume X be a non empty set. A family of sets $I \subseteq \mathscr{P}(N)$ is called an ideal in X if and only if

(i) $\phi \in I$,

(ii) $A,B \in I \implies A \cup B \in I$ and

(iii) For each $A \in I$ and $B \subseteq A$, we have $B \in I$.

Definition 1.10 ([16])**.** Assume X be a non empty set. A non-empty family of sets $F \subseteq \mathscr{P}(N)$ is called a filter on X iff;

(i) $\phi \notin F$,

(ii) $A,B \in F \implies A \cap B \in F$ and

(iii) For each $A \in F$ and $A \subseteq B$, we have $B \in F$.

An ideal I is called non-trivial if $I \neq \phi$ and $X \neq I$. A non-trivial ideal $I \subseteq \mathscr{P}(N)$ is called an admissible ideal in $X \iff$ it contains all singletons, that is, if it contains $\{\{x\} : x \in X\}$. If $I \subseteq \mathscr{P}(N)$ be a non-trivial ideal. Then the class $F = F(I) = \{A^c \subseteq X : A \in I\}$ is a filter on X. $F = F(I)$ is called the filter associated with the ideal I.

2. Main Results

Definition 2.1. Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. A sequence $x = \{x_k\}$ in V is said to Δ^m-I_N-convergent to $L \in F$ if for every $\epsilon > 0$ and $t > 0$, we have

$$A(\epsilon) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) \leq 1-\epsilon \text{ or } B(\Delta^m x_k - L, t) \geq \epsilon \text{ and } Y(\Delta^m x_k - L, t) \geq \epsilon\} \in I.$$

Symptomatic as $I_N\text{-}\lim \Delta^m x_k = L$ or $\Delta^m x_k \xrightarrow{I_N} L$.

Example 2.1. Assume $(F, \|\cdot\|)$ be a neutrosophic set, I be a non-trivial admissible ideal. For all $u, v \in [0,1]$. Take $u \odot v = uv$ as t-norm and $u \cdot v = \min\{u+v, 1\}$ as triangular co-norm. For all $x \in F$ and every $t > 0$, we assume $G(x,t) = \frac{t}{t+\|x\|}$, $B(x,t) = \frac{x}{t+\|x\|}$ and $Y(x,t) = \frac{\|x\|}{t}$. Then V is an N.N.S, a sequence $x = \{x_k\}$ can be defined in the following:

$$\Delta^m x_k = \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{otherwise.} \end{cases}$$

Consider $A(\epsilon, t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) \leq 1-\epsilon$ or $B(\Delta^m x_k - L, t) \geq \epsilon$ and $Y(\Delta^m x_k - L, t) \geq \epsilon\}$ for every $\epsilon \in (0,1)$ and for arbitrary $t > 0$.
Then,

$$\begin{aligned} A(\epsilon, t) &= \left\{k \in \mathbb{N} : \frac{t}{t+\|\Delta^m x_k\|} \leq 1-\epsilon \text{ or } \frac{\|\Delta^m x_k\|}{t+\|\Delta^m x_k\|} \geq \epsilon \text{ and } \frac{\|\Delta^m x_k\|}{t} \geq \epsilon\right\} \\ &= \left\{k \in \mathbb{N} : \|\Delta^m x_k\| \geq \frac{t\epsilon}{1-\epsilon} \text{ or } \|\Delta^m x_k\| \geq t\epsilon\right\} \\ &= \{k \in \mathbb{N} : \|\Delta^m x_k\| = 1\} \\ &= \{k \in \mathbb{N} : k = n^2 (n \in \mathbb{N})\} \end{aligned}$$

which is a finite set. So, $\delta(A(\epsilon,t)) = 0$. Thus,

$$A(\epsilon, t) \in I \implies I_N\text{-}\lim \Delta^m x_k = 0.$$

Lemma 2.1. *Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S, then for every $\epsilon > 0$ and $t > 0$, the following conditions are equivalent:*

(i) $I_N\text{-}\lim \Delta^m x_k = x$.

(ii) $\{k \in \mathbb{N} : G(\Delta^m x_k - L, t) \leq 1-\epsilon\} \in I$; $\{k \in \mathbb{N} : B(\Delta^m x_k - L, t) \geq \epsilon\} \in I$ *and* $\{k \in \mathbb{N} : Y(\Delta^m x_k - L, t) \geq \epsilon\} \in I$.

(iii) $\{k \in \mathbb{N} : G(\Delta^m x_k - L, t) > 1-\epsilon$ *and* $B(\Delta^m x_k - L, t) < \epsilon$, $Y(\Delta^m x_k - L, t) < \epsilon\} \in F(I)$.

(iv) $\{k \in \mathbb{N} : G(\Delta^m x_k - L, t) > 1-\epsilon\} \in F(I)$; $\{k \in \mathbb{N} : B(\Delta^m x_k - L, t) < \epsilon\} \in F(I)$ *and* $\{k \in \mathbb{N} : Y(\Delta^m x_k - L, t) < \epsilon\} \in F(I)$.

(v) $I_N\text{-}\lim G(\Delta^m x_k, t) = 1$ *and* $I_N\text{-}\lim B(\Delta^m x_k, t) = 0$, $I_N\text{-}\lim Y(\Delta^m x_k, t) = 0$.

Now we formulate and prove the following theorem related to uniqueness of I-convergence.

Theorem 2.1. *Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S, $\{x_k\}$ be a sequence in V in such a manner, I_N-$\lim \Delta^m x_k = L_1$ and I_N-$\lim \Delta^m x_k = L_2$, then $L_1 = L_2$.*

Proof. Assume $L_1 \neq L_2$ and $\epsilon > 0$. Select $\lambda > 0$ in such a manner,

$$(1-\epsilon)\odot(1-\epsilon) > 1-\lambda \text{ and } \epsilon\cdot\epsilon < \lambda \tag{2.1}$$

For $t > 0$, we define the following sets:

$$K_{G_1}(\epsilon,t) = \left\{k \in \mathbb{N} : G\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \leq 1-\epsilon\right\},\ K_{G_2}(\epsilon,t) = \left\{k \in \mathbb{N} : G\left(\Delta^m x_k - L_2, \frac{t}{2}\right) \leq 1-\epsilon\right\};$$

$$K_{B_1}(\epsilon,t) = \left\{k \in \mathbb{N} : G\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \geq \epsilon\right\},\ K_{B_2}(\epsilon,t) = \left\{k \in \mathbb{N} : G\left(\Delta^m x_k - L_2, \frac{t}{2}\right) \geq \epsilon\right\};$$

$$K_{Y_1}(\epsilon,t) = \left\{k \in \mathbb{N} : G\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \geq \epsilon\right\} \text{ and } K_{Y_2}(\epsilon,t) = \left\{k \in \mathbb{N} : G\left(\Delta^m x_k - L_2, \frac{t}{2}\right) \geq \epsilon\right\}.$$

Since I_N-$\lim \Delta^m x_k = L_1$ and I_N-$\lim \Delta^m x_k = L_2$ so by Lemma 2.1, we respectively have the sets $K_{G_1}(\epsilon,t)$; $K_{B_1}(\epsilon,t)$; $K_{Y_1}(\epsilon,t)$; $K_{G_2}(\epsilon,t)$; $K_{B_2}(\epsilon,t)$; and $K_{Y_2}(\epsilon,t)$; belongs to I. Define a set $K_N(\epsilon,t)$ by

$$K_N(\epsilon,t) = \{K_{G_1}(\epsilon,t) \cup \{K_{G_2}(\epsilon,t)\} \cap \{K_{B_1}(\epsilon,t) \cup \{K_{B_2}(\epsilon,t)\} \cap \{K_{Y_1}(\epsilon,t) \cup \{K_{Y_2}(\epsilon,t)\};$$

then $K_N(\epsilon,t) \in I$ which implies that its complement $\{\mathbb{N} - K_N(\epsilon,t)\} \in F(I)$. Then $\{\mathbb{N} - K_N(\epsilon,t)\}$ is a nonempty set as otherwise $\{\mathbb{N} - K_N(\epsilon,t)\} \in I$. Assume $k \in \{\mathbb{N} - K_N(\epsilon,t)\}$, then we have the following possibilities:

(i) $k \in \mathbb{N} - \{\{K_{G_1}(\epsilon,t)\} \cup \{K_{G_2}(\epsilon,t)\}\}$,

(ii) $k \in \mathbb{N} - \{\{K_{B_1}(\epsilon,t)\} \cup \{K_{B_2}(\epsilon,t)\}\}$,

(iii) $k \in \mathbb{N} - \{\{K_{Y_1}(\epsilon,t)\} \cup \{K_{Y_2}(\epsilon,t)\}\}$.

Assume (i) holds, then $k \notin \{K_{G_1}(\epsilon,t)\} \cup \{K_{G_2}(\epsilon,t)\}$ which gives $k \notin \{K_{G_1}(\epsilon,t)\}$ and $k \notin \{K_{G_2}(\epsilon,t)\}$. This implies that

$$G\left(\Delta^m x_k - L_1, \frac{t}{2}\right) > 1-\epsilon \text{ and } G\left(\Delta^m x_k - L_2, \frac{t}{2}\right) > 1-\epsilon. \tag{2.2}$$

Using equations (2.1) and (2.2)

$$G(L_1 - L_2, t) \geq G\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \odot G\left(\Delta^m x_k - L_2, \frac{t}{2}\right) > (1-\epsilon)\odot(1-\epsilon) > 1-\lambda. \tag{2.3}$$

Since λ is arbitrary and eq. (2.3) holds for every $t > 0$, it follows that $B(L_1 - L_2, t) = 1$ and therefore $L_1 = L_2$. Now, we assume (ii) holds, then $k \notin \{K_{B_1}(\epsilon,t)\}$ and $k \notin \{K_{B_2}(\epsilon,t)\}$ and therefore we have

$$B\left(\Delta^m x_k - L_1, \frac{t}{2}\right) < \epsilon \text{ and } B\left(\Delta^m x_k - L_2, \frac{t}{2}\right) < \epsilon. \tag{2.4}$$

Using equations (2.1) and (2.4)

$$B(L_1 - L_2, t) \leq B\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \cdot B\left(\Delta^m x_k - L_2, \frac{t}{2}\right) < \epsilon\cdot\epsilon < \lambda. \tag{2.5}$$

As λ is arbitrary and equation (2.5) holds for every $t > 0$, we must have $B(L_1 - L_2, t) = 0$, which gives $L_1 = L_2$. Finally, assume that (iii) holds. It follows that $k \notin \{K_{Y_1}(\epsilon,t)\}$ and $k \notin \{K_{Y_2}(\epsilon,t)\}$ and therefore, we have

$$Y\left(\Delta^m x_k - L_1, \frac{t}{2}\right) < \epsilon \text{ and } Y\left(\Delta^m x_k - L_2, \frac{t}{2}\right) < \epsilon. \tag{2.6}$$

Now,

$$Y(L_1-L_2,t)\le Y\left(\Delta^m x_k-L_1,\frac{t}{2}\right)\cdot Y\left(\Delta^m x_k-L_2,\frac{t}{2}\right)<\epsilon\cdot\epsilon<\lambda. \tag{2.7}$$

Since λ is arbitrary and equation (2.6) holds for every $\lambda>0$, we must have $Y(L_1-L_2,t)=0$ and therefore we have $L_1=L_2$. This completes the proof. □

Theorem 2.2. *Assume $I\subseteq\mathscr{P}(N)$ is an admissible ideal, V be an N.N.S, $\{x_k\}$ be a sequence in V in such a manner, N-$\lim\Delta^m x_k=L$, then I_N-$\lim\Delta^m x_k=L$.*

Proof. Assume N-$\lim\Delta^m x_k=L$, then for each $\epsilon>0$ and $t>0$ $\exists$ a positive integer p in such a manner,

$$\begin{aligned}&G(\Delta^m x_k-L,t)>1-\epsilon,\ B(\Delta^m x_k-L,t)<\epsilon\}\ \text{ and }\ Y(\Delta^m x_k-L,t)<\epsilon\}\ \ \forall\ \ k\ge p\\ \implies\ &\{k\in\mathbb{N}:G(\Delta^m x_k-L,t)\le 1-\epsilon \text{ or } B(\Delta^m x_k-L,t)>\epsilon\} \text{ and } Y(\Delta^m x_k-L,t)>\epsilon\}\end{aligned}$$

is a finite set and therefore belongs to I.
Hence I_N-$\lim\Delta^m x_k=L$. □

Theorem 2.3. *Assume $I\subseteq\mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. Assume $\{x_k\}$ and $\{y_k\}$ be two sequences in V in such a manner, I_N-$\lim\Delta^m x_k=L_1$ and I_N-$\lim\Delta^m y_k=L_2$, then*

(a) I_N-$\lim(\Delta^m x_k+\Delta^m y_k)=L_1+L_2$

(b) I_N-$\lim(\beta\Delta^m x_k)=\beta L_1$ *for* $\beta\ne 0$

Proof. (a) I_N-$\lim\Delta^m x_k=L_1$ and I_N-$\lim\Delta^m y_k=L_2$. For $\epsilon>0$, select $\lambda>0$ in such a manner,

$$(1-\epsilon)\odot(1-\epsilon)>1-\lambda \text{ and } \epsilon\cdot\epsilon<\lambda \tag{2.8}$$

For $t>0$, we define the following sets:

$$\begin{aligned}&K_{G_1}(\epsilon,t)=\left\{k\in\mathbb{N}:G\left(\Delta^m x_k-L_1,\frac{t}{2}\right)\le 1-\epsilon\right\},\ K_{G_2}(\epsilon,t)=\left\{k\in\mathbb{N}:G\left(\Delta^m x_k-L_2,\frac{t}{2}\right)\le 1-\epsilon\right\};\\ &K_{B_1}(\epsilon,t)=\left\{k\in\mathbb{N}:G\left(\Delta^m x_k-L_1,\frac{t}{2}\right)\ge\epsilon\right\},\ K_{B_2}(\epsilon,t)=\left\{k\in\mathbb{N}:G\left(\Delta^m x_k-L_2,\frac{t}{2}\right)\ge\epsilon\right\};\\ &K_{Y_1}(\epsilon,t)=\left\{k\in\mathbb{N}:G\left(\Delta^m x_k-L_1,\frac{t}{2}\right)\ge\epsilon\right\} \text{ and } K_{Y_2}(\epsilon,t)=\left\{k\in\mathbb{N}:G\left(\Delta^m x_k-L_2,\frac{t}{2}\right)\ge\epsilon\right\}.\end{aligned}$$

Since I_N-$\lim\Delta^m x_k=L_1$, we have $K_{G_1}(\epsilon,t)$, $K_{B_1}(\epsilon,t)$ and $K_{Y_1}(\epsilon,t)\in I$.

Further, using I_N-$\lim\Delta^m y_k=L_2$, $K_{G_2}(\epsilon,t)$, $K_{B_2}(\epsilon,t)$ and $K_{Y_2}(\epsilon,t)\in I$.

Now, assume $K_N(\epsilon,t)=\{K_{G_1}(\epsilon,t)\cup\{K_{G_2}(\epsilon,t)\}\cap\{K_{B_1}(\epsilon,t)\cup\{K_{B_2}(\epsilon,t)\}\cap\{K_{Y_1}(\epsilon,t)\cup\{K_{Y_2}(\epsilon,t)\}$.

Then $K_N(\epsilon,t)\in I$, which implies that $\phi\notin K_N^c(\epsilon,t)\in F(I)$.

Now, its sufficient to prove that

$K_N^c(\epsilon,t)\subset\{k\in\mathbb{N}:G((\Delta^m x_k+\Delta^m y_k)-(L_1+L_2),t)>1-\lambda$ and $B((\Delta^m x_k+\Delta^m y_k)-(L_1+L_2),t)<\lambda, Y((\Delta^m x_k+\Delta^m y_k)-(L_1+L_2),t)<\lambda\}$.

If $k\in K_N^c(\epsilon,t)$, then we get

$$\begin{aligned}&G\left(\Delta^m x_k-L_1,\frac{t}{2}\right)>1-\epsilon,\ \ G\left(\Delta^m y_k-L_2,\frac{t}{2}\right)>1-\epsilon,\\ &B\left(\Delta^m x_k-L_1,\frac{t}{2}\right)<\epsilon,\ \ B\left(\Delta^m y_k-L_2,\frac{t}{2}\right)<\epsilon,\end{aligned}$$

$$Y\left(\Delta^m x_k - L_1, \frac{t}{2}\right) > 1-\epsilon \text{ and } Y\left(\Delta^m y_k - L_2, \frac{t}{2}\right) > 1-\epsilon.$$

Therefore,

$$\begin{aligned} G((\Delta^m x_k + \Delta^m y_k) - (L_1+L_2), t) &\geq G\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \odot G\left(\Delta^m y_k - L_2, \frac{t}{2}\right) \\ &> (1-\epsilon)\odot(1-\epsilon) > 1-\lambda \end{aligned}$$

and

$$B((\Delta^m x_k + \Delta^m y_k) - (L_1+L_2), t) \leq B\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \cdot B\left(\Delta^m y_k - L_2, \frac{t}{2}\right) \leq \epsilon\cdot\epsilon < \lambda,$$

$$Y((\Delta^m x_k + \Delta^m y_k) - (L_1+L_2), t) \leq Y\left(\Delta^m x_k - L_1, \frac{t}{2}\right) \cdot Y\left(\Delta^m y_k - L_2, \frac{t}{2}\right) \leq \epsilon\cdot\epsilon < \lambda.$$

This shows that $K_N^c(\epsilon,t) \subset \{k \in \mathbb{N} : G((\Delta^m x_k + \Delta^m y_k) - (L_1+L_2), t) > 1-\lambda$ and $B((\Delta^m x_k + \Delta^m y_k) - (L_1+L_2), t) < \lambda, Y((\Delta^m x_k + \Delta^m y_k) - (L_1+L_2), t) < \lambda\}$.
Since $K_N^c(\epsilon,t) \in F(I)$, I_N-$\lim(\Delta^m x_k + \Delta^m y_k) = L_1 + L_2$.

(b) This is true for $\beta = 0$. Assume $\beta \neq 0$. Then, for given $\epsilon > 0$ and $t > 0$,

$$B(\lambda) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) > 1-\lambda \text{ and } B(\Delta^m x_k - L, t) < \lambda, Y(\Delta^m x_k - L, t) < \lambda\} \in F(I). \tag{2.9}$$

It is adequate to show that, for each $\lambda > 0$ and $t > 0$,

$$B(\lambda) \subset \{k \in \mathbb{N} : G(\beta\Delta^m x_k - \beta L, t) > 1-\lambda \text{ and } B(\beta\Delta^m x_k - \beta L, t) < \lambda, Y(\beta\Delta^m x_k - \beta L, t) < \lambda\}.$$

Assume $k \in B(\lambda)$. Then, we have

$$\begin{aligned} & G(\Delta^m x_k - L, t) > 1-\lambda \text{ and } B(\Delta^m x_k - L, t) < \lambda, Y(\Delta^m x_k - L, t) < \lambda \\ \implies \quad & G(\beta\Delta^m x_k - \beta L, t) = G\left(\beta\Delta^m x_k - L, \frac{t}{|\beta|}\right) \\ & \qquad \geq G(\Delta^m x_k - L, t) \odot G\left(0, \frac{t}{|\beta|} - t\right) \\ & \qquad = G(\Delta^m x_k - L, t) \odot 1 \\ & \qquad = G(\Delta^m x_k - L, t) \\ & \qquad \geq 1-\lambda \end{aligned}$$

and

$$\begin{aligned} B(\beta\Delta^m x_k - \beta L, t) &= B\left(\beta\Delta^m x_k - L, \frac{t}{|\beta|}\right) \\ &\leq B(\Delta^m x_k - L, t) \cdot B\left(0, \frac{t}{|\beta|} - t\right) \\ &= B(\Delta^m x_k - L, t) \cdot 0 \\ &= B(\Delta^m x_k - L, t) < \lambda. \end{aligned}$$

Also,

$$\begin{aligned} Y(\beta\Delta^m x_k - \beta L, t) &= Y\left(\beta\Delta^m x_k - L, \frac{t}{|\beta|}\right) \\ &\leq Y(\Delta^m x_k - L, t) \cdot Y\left(0, \frac{t}{|\beta|} - t\right) \end{aligned}$$

$$= Y(\Delta^m x_k - L, t) \cdot 0$$
$$= Y(\Delta^m x_k - L, t) < \lambda$$

$\implies B(\lambda) \subset \{k \in \mathbb{N} : G(\beta\Delta^m x_k - \beta L, t) > 1-\lambda$ and $B(\beta\Delta^m x_k - \beta L, t) < \lambda, Y(\beta\Delta^m x_k - \beta L, t) < \lambda\}$.

From equation (2.9) we get, I_N-$\lim \beta\Delta^m x_k = \beta L$. □

Definition 2.2. Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. A sequence $\{x_k\}$ in V is said to Δ^m-I_N^*-convergent $\iff$ $\exists$ a subset $J = \{j_1 < j_2 < j_3 < \ldots\} \subset \mathbb{N}$ in such a manner, $J \in F(I)$ and N-$\lim \Delta^m x_{j_n} = L$. Symptomatic as I_N^*-$\lim \Delta^m x_k = L$.

Now we establish relationship between the two convergence that is Δ^m-I-convergence and Δ^m-I^*-convergence in N.N.S.

Theorem 2.4. *Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S and a sequence $\{x_k\}$ in V, in such a manner, $I_N^* - \lim \Delta^m x_k = L$, then $I_N - \lim \Delta^m x_k = L$.*

Proof. Since I_N^*-$\lim \Delta^m x_k = L$, then $\exists$ a subset $J = j_1 < j_2 < j_3 < \ldots \subset \mathbb{N}$ in such a manner, $J \in F(I)$ and N-$\lim \Delta^m x_{j_n} = L$.
For each $\epsilon > 0$ and $t > 0$ $\exists$ a positive integer p,for every $n \geq p$ in such a manner,

$$G(\Delta^m x_k - L, t) > 1-\epsilon,\ B(\Delta^m x_k - L, t) > \epsilon \text{ and } Y(\Delta^m x_k - L, t) > \epsilon.$$

If we take a set $P = N - J$, then $P \in J$ and therefore we have the containment

$$A(\epsilon, t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) \leq 1-\epsilon \text{ or } B(\Delta^m x_k - L, t) \geq \epsilon$$
$$\text{and } Y(\Delta^m x_k - L, t) \geq \epsilon\} \subseteq P \cup \{j_1 < j_2 < j_3 < \ldots < j_p\}.$$

Since $P \in I$ and $\{j_1 < j_2 < j_3 < \ldots < j_p\}$ is a finite set so their union must be I.
Hence I_N-$\lim \Delta^m x_k = L$. □

The converse of the above theorem need not be true, which is described in the example described below:

Example 2.2. Assume $(\mathbb{R}, \|\cdot\|)$ be a real normed space. For all $u, v \in [0,1]$. Take $u \odot v = uv$ as t-norm and $u \cdot v = \min\{u+v, 1\}$ as triangular co-norm. For all $x \in F$ and every $t > 0$, we assume $G(x,t) = \frac{t}{t+\|x\|}$, $B(x,t) = \frac{x}{t+\|x\|}$ and $Y(x,t) = \frac{\|x\|}{t}$. Then $(\mathbb{R}, N, \odot, cdot)$ is an N.N.S. Assume $\mathbb{N} = \cup_i D_i$ be a decomposition of $\mathbb{N}$ in such a manner,, for any $k \in \mathbb{N}$, each D_i contains many $i's$, where $i \geq k$ and $D_i \cap D_k = \phi$ for $i \neq k$. Define a sequence $x_k = \frac{1}{i}$ for $k \in D_i$. Then

$$G(\Delta^m x_k, t) = \frac{t}{t + \|\Delta^m x\|} \to 1$$

and

$$B(\Delta^m x_k, t) = \frac{\Delta^m x}{t + \|\Delta^m x\|} \to 0, \quad Y(\Delta^m x_k, t) = \frac{\|\Delta^m x\|}{t}.$$

As k tends to $\infty \implies I_N$-$\lim \Delta^m x_k = 0$. Now, assume that I_N^*-$\lim \Delta^m x_k = 0$, then $\exists$ a subset $J = \{j_1 < j_2 < j_3 < \ldots\} \subset \mathbb{N}$ in such a manner, $J \in F(I)$ and N-$\lim \Delta^m x_{k_j} = 0$. Because $J \in F(I)$, then $J^c = M \in F(I)$ then there exists $p \in \mathbb{N}$ in such a manner, $M \subset \cup_i^p = D_i$. But then $D_p + 1 \subset M$. So, $x_{k_j} = \frac{1}{p+1} > 0$ for infinitely many k_j's from J which contradicts N-$\lim \Delta^m x_{k_j} = 0$. Hence the supposition of I_N^*-$\lim \Delta^m x_k = 0$ is wrong.

Remark 2.1. The above example denotes that Δ^m-I_N^*-convergence implies Δ^m-I_N-convergence in N.N.S but not conversely, but the converse holds under the property (AP). For this, the lemma is described below:

Lemma 2.2. *Assume $\{A_i\}_{i=1}^{\infty}$ be a countable collection of subsets of N in such a manner, $A_i \in I$ for each i where $F(I)$ is a filter associate with an admissible ideal I with property (AP). Then $\exists$ a set $A \subseteq \mathbb{N}$ in such a manner, $A \in F(I)$ and the set $A - A_i$ is finite for each i.*

Theorem 2.5. *If $I \subseteq \mathscr{P}(N)$ is an admissible ideal with property (AP) then the concepts of Δ^m-I_N-convergence and Δ^m-I_N^*-convergence in N.N.S coincide.*

Proof. To prove the result it is sufficient to show that if I is an admissible ideal with property (AP) then Δ^m-I_N-convergence implies Δ^m-I_N^*-convergence in N.N.S. Assume $\{\Delta^m - x_k\}$ be a sequences in V in such a manner, I_N-$\lim \Delta^m x_k = L$. By definition, for every $\epsilon > 0$ and $t > 0$ we have

$$A(\epsilon,t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L,t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - L,t) \ge \epsilon \text{ and } Y(\Delta^m x_k - L,t) \ge \epsilon\} \in I.$$

For $\lambda \in \mathbb{N}$, we define sets $E(\lambda,t)$and $F(\lambda,t)$ by

$$E(\lambda,t) = \left\{k \in \mathbb{N} : G(\Delta^m x_k - L,t) > 1-\frac{1}{\lambda} \text{ and } B(\Delta^m x_k - L,t) < \frac{1}{\lambda} \text{ or } Y(\Delta^m x_k - L,t) < \frac{1}{\lambda}\right\}, \tag{2.10}$$

$$F(\lambda,t) = \left\{k \in \mathbb{N} : G(\Delta^m x_k - L,t) \le 1-\frac{1}{\lambda} \text{ or } B(\Delta^m x_k - L,t) > \frac{1}{\lambda} \text{ and } Y(\Delta^m x_k - L,t) > \frac{1}{\lambda}\right\}. \tag{2.11}$$

Since I_N-$\lim \Delta^m x_k = L$, so for $t > 0$ and $\lambda \in \mathbb{N}$, that is $\lambda = 1,2,3,\dots$, $E(\lambda,t) \in I$ which implies $E(\lambda,t) \in F(I)$. Thus $E(1,t), E(2,t), E(3,t), \dots,$ is a sequence of sets in $F(I)$. As the ideal satisfy the property (AP) so by Lemma 2.2, there exists a set $E \in \mathbb{N}$ in such a manner, $E = \{e_1, e_2, e_3, \dots\} \in F(I)$ and the set $E - E(\lambda,t)$ is finite for $\lambda = 1,2,3,\dots$.
Now to prove the result it is sufficient to show that I_N-$\lim \Delta^m x_{k_j} = L$.
Suppose that Δ^m-$\lim \Delta^m x_{k_j} \ne L$. Then there is some $\epsilon_1 > 0$ and a positive integer p in such a manner, for all $j \ge p$

$$G(\Delta^m x_{k_j} - L,t) \le 1-\epsilon_1 \text{ or } B(\Delta^m x_{k_j} - L,t) \ge \epsilon_1 \text{ and } Y(\Delta^m x_{k_j} - L,t) \ge \epsilon_1$$

which implies that the set

$$\{k_j \in \mathbb{N} : G(\Delta^m x_{k_j} - L,t) > 1-\epsilon_1 \text{ and } B(\Delta^m x_{k_j} - L,t) < \epsilon_1 \text{ or } Y(\Delta^m x_{k_j} - L,t) < \epsilon_1\}$$

is a finite set and must be in I and therefore we obtain a contradiction to equations (2.10) and (2.11) as it belongs to $F(I)$.
Hence, N-$\lim \Delta^m x_{k_j} = L$ and this completes the proof. □

Next, we introduce the concepts of Δ^m-I-Cauchy and I^*-Cauchy for sequences and define Δ^m-I-completeness and Δ^m-I^*-completeness in N.N.S.

Definition 2.3. Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. A sequence $\{x_k\}$ in V is said to Δ^m-I_N-Cauchy if and only if for every $\epsilon > 0$ and $\lambda > 0$ $\exists$ a positive integer m in such a manner, $\{k \in \mathbb{N} : G(\Delta^m x_k - L,t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - L,t) \ge \epsilon\}$ and $Y(\Delta^m x_k - L,t) \ge \epsilon\} \in I$.

Definition 2.4. Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. A sequence $x = \{x_k\}$ in V is Δ^m-I_N^*-Cauchy $\iff$ $\exists$ a subset $J = \{j_1 < j_2 < j_3 < \ldots\} \subset \mathbb{N}$ and $J \in F(I)$ in such a manner, the subsequence $x = \{x_{j_n}\}$ is Δ^m-I_N-Cauchy.

Next, we discuss Cauchy convergence criteria in N.N.S.

Theorem 2.6. *Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. A sequence $\{x_k\}$ in V in such a manner, Δ^m-$I_N - \lim \Delta^m x_k = L$, then $\{x_k\}$ is Δ^m-I_N-Cauchy.*

Proof. Assume $I_N - \lim \Delta^m x_k = L$, For every $\epsilon > 0$ and $t > 0$. select $\lambda > 0$ in such a manner,

$$(1-\epsilon)\odot(1-\epsilon) > 1-\lambda \text{ and } \epsilon\cdot\epsilon < \lambda. \tag{2.12}$$

Then,

$$A(\epsilon,t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - L, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - L, t) \ge \epsilon\} \in I$$

and

$$A^c(\epsilon,t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) > 1-\epsilon,\ B(\Delta^m x_k - L, t) < \epsilon \text{ or } Y(\Delta^m x_k - L, t) < \epsilon\} \in F(I) \tag{2.13}$$

and therefore is a non empty set. Assume $s \in A^c(\epsilon,\lambda)$, then we have

$$G(\Delta^m x_s - L, t) > 1-\epsilon,\ B(\Delta^m x_s - L, t) < \epsilon \text{ or } Y(\Delta^m x_k - L, t) < \epsilon.$$

Assume

$$B(\lambda,t) = \{G(\Delta^m x_k - \Delta^m x_s, t) \le 1-\lambda \textit{ or } B(\Delta^m x_k - \Delta^m x_s, t) \ge \lambda \text{ and } Y(\Delta^m x_k - \Delta^m x_s, t) \ge \lambda\} \in I.$$

We shall show that $B(\lambda,t) \subseteq A(\lambda,t)$. For this, let $k_0 \in B(\lambda,t) - A(\lambda,t)$, then we have

$$G(\Delta^m x_{k_0} - \Delta^m x_s, t) \ge 1-\lambda\} \text{ and } G\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) > 1-\lambda.$$

In particular,

$$G\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) > 1-\epsilon.$$

Now,

$$1-\lambda \ge G(\Delta^m x_{k_0} - \Delta^m x_s, t) \ge G\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) \odot G\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) > 1-\lambda$$

which is not possible. If $B(\Delta^m x_{k_0} - \Delta^m x_s, t) \ge \lambda\}$ and $B\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) < \lambda$.
In particular, $B\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) < \epsilon$.
Now,

$$\lambda \le B(\Delta^m x_{k_0} - \Delta^m x_s, t) \le B\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) \odot B\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) < \epsilon \odot \epsilon < \lambda$$

which is not possible.
Finally, if $Y(\Delta^m x_{k_0} - \Delta^m x_s, t) \ge \lambda\}$ and $Y\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) < \lambda$.
In particular, $Y\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) < \epsilon$.
Now,

$$\lambda \le Y(\Delta^m x_{k_0} - \Delta^m x_s, t) \le Y\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) \cdot Y\left(\Delta^m x_{k_0} - L, \frac{t}{2}\right) < \epsilon \cdot \epsilon < \lambda$$

which is not possible.
Thus, in every case, $B(\lambda,t) \subseteq A(\lambda,t)$.

Since $A(\lambda,t)\in I$, therefore $B(\lambda,t)\in I$ as $B(\lambda,t)\subseteq A(\lambda,t)$.
Hence, $\{x_k\}$ is Δ^m-I_N-Cauchy sequence. $\square$

Theorem 2.7. *Assume $I\subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. If a sequence $x=\{x_k\}$ in V is Δ^m-I_N^*-Cauchy, then it is Δ^m-I_N-Cauchy.*

Proof. Assume $x=\{x_k\}$ be a Δ^m-I_N^*-Cauchy sequence in V. Then by definition $\exists$ a subset $J\subset\{j_1<j_2<j_3<\ldots<j_p<\ldots\}\subset\mathbb{N}$ and $J\in F(I)$ in such a manner, for every $\lambda>0$, $\exists$ a $j_s\in J$ in such a manner,

$$|\{k\in\mathbb{N}: G(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\le 1-\epsilon \text{ or } B(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon \text{ and } Y(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon\}|=0,$$

Then for any $\lambda>0$ we have:

$$|\{k\in\mathbb{N}: G(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\le 1-\epsilon \text{ or } B(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon \text{ and } Y(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon\}|\ge\lambda\in I,$$

Since I is an admissible ideal, Assume $H=\mathbb{N}|J$. Its clear that $H\in I$ and
$|\{k\in\mathbb{N}: G(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\le 1-\epsilon$ or $B(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon$ and $Y(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon\}|\ge\lambda\in I$ $\subset H\cup|\{k\in\mathbb{N}: G(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\le 1-\epsilon$ or $B(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon$ and $Y(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon\}|\ge\lambda\in I$.
$\Longrightarrow$ $\{x_k\}$ is Δ^m-I_N-Cauchy sequence in V. $\square$

But the converse is not true which is described in the example given below:

Example 2.3. Assume $(\mathbb{R},\|\cdot\|)$ be a real normed space. $(\mathbb{R},N,\odot,\cdot)$ is an N.N.S. Assume $\mathbb{N}=\cup_{i=1}^{\infty}D_i$ be a decomposition of $\mathbb{N}$ in such a manner, each D_i is infinite and $D_i\cap D_n=\phi$ for $i\ne n$. Assume I be the class of all those subsets Q of $\mathbb{N}$ that can intersect only finite number of D_i's. Then, I is a non-trivial admissible ideal of $\mathbb{N}$. Define a sequence $x_k=\frac{1}{i}$ for $k\in D_i$.
Assume $\epsilon>0$ be given, then $\exists$ a $p\in\mathbb{N}$ for $k,s\ge p$ in such a manner,
$\{k\in\mathbb{N}: G(\Delta^m x_k-\Delta^m x_s,t)\le 1-\epsilon$ or $B(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon$ and $Y(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon\}\subset D_1\cup D_2\cup D_3\cup\ldots\cup D_p\in I$.
Now, $\{k\in\mathbb{N}: G(\Delta^m x_p-\Delta^m x_s,t)\le 1-\epsilon$ or $B(\Delta^m x_p-\Delta^m x_s,t)\ge\epsilon$ and $Y(\Delta^m x_p-\Delta^m x_s,t)\ge\epsilon\}$ $\le\{k\in\mathbb{N}: G(\Delta^m x_k-\Delta^m x_s,t)\le 1-\epsilon$ or $B(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon$ and $Y(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon\}$.
So for any $\lambda>0$,
$|\{k\in\mathbb{N}: G(\Delta^m x_p-\Delta^m x_s,t)\le 1-\epsilon$ or $B(\Delta^m x_p-\Delta^m x_s,t)\ge\epsilon$ and $Y(\Delta^m x_p-\Delta^m x_s,t)\ge\epsilon\}|\ge\lambda$ $\subseteq\{k\in\mathbb{N}: G(\Delta^m x_k-\Delta^m x_s,t)\le 1-\epsilon$ or $B(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon$ and $Y(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon\}\in I$.
Therefore, $\{x_k\}$ is Δ^m-I_N-Cauchy sequence in V. Assume if possible $\{x_k\}$ is Δ^m-I_N^*-Cauchy sequence in V. Then $\exists$ $Q\in F(I)$ in such a manner, $\{x_k\}_{k\in Q}$ is Cauchy sequence in V. Since $\mathbb{N}|Q\in I$ $\exists$ $p\in\mathbb{N}$ in such a manner, $\mathbb{N}|Q\subset D_1\cup D_2\cup\ldots\cup D_p$. Therefore, $D_p+1\cup D_p+2\subset Q$. Now, if $\{j_k\}\in D_p+1$ and $\{j_s\}\in D_p+2$ then $\{j_k\},\{j_s\}\in Q$ and $\{j_k\in\mathbb{N}: G(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\le 1-\epsilon_0$ or $B(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon_0$ and $Y(\Delta^m x_{j_k}-\Delta^m x_{j_s},t)\ge\epsilon_0\}|=2^{(-p+1)}>0$,
where $\epsilon_0=\frac{1}{3(c+1)(c+2)}>0$.
This contradicts that $\{x_k\}$ is Δ^m-I_N-Cauchy sequence in V.

Theorem 2.8. *Assume $I\subseteq\mathscr{P}(N)$ is an admissible ideal with property (AP) then the concepts of Δ^m-I_N-Cauchy and Δ^m-I_N^*-Cauchy in N.N.S coincide.*

Proof. Assume $\{x_k\}$ be an Δ^m-I_N-Cauchy sequence in V. Then by definition, for every $\epsilon > 0$ and $\lambda > 0$ $\exists$ an s in such a manner,

$$|\{k \in \mathbb{N} : G(\Delta^m x_k - \Delta^m x_s, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon\}| \ge \lambda \in I.$$

Assume $A_i = \{|\{k \in \mathbb{N} : G(\Delta^m x_k - \Delta^m x_s, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon\}| < \frac{1}{i}\}, i = 1,2,3,\dots$. It is clear that $A_i \in F(I)$ for $i = 1,2,3,\dots$. Since, I has the (AP) property, using lemma: Assume $\{A_i\}_{i=1}^{\infty}$ be a countable collection of subsets of N in such a manner, $A_i \in I$ for each i where $F(I)$ is a filter associate with an admissible ideal I with property (AP). Then $\exists$ a set $A \subseteq \mathbb{N}$ in such a manner, $A \in F(I)$ and the set $A|A_i$ is finite for each i.
If $p \in A_i$ then $A|A_p$ is finite, then $\exists p_0 = p_0(i)$ in such a manner, $p \in A_i$ $\forall p > p_0$

$$|\{k \in \mathbb{N} : G(\Delta^m x_k - \Delta^m x_s, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon\}| < \frac{1}{i}.$$

Therefore, $\{k \in \mathbb{N} : G(\Delta^m x_{j_k} - \Delta^m x_{s_i}, t) \le 1-\epsilon \text{ or } B(\Delta^m x_{j_k} - \Delta^m x_{s_i}, t) \ge \epsilon \text{ and } Y(\Delta^m x_{j_k} - \Delta^m x_{s_i}, t) \ge \epsilon\} \subset |\{k \in \mathbb{N} : G(\Delta^m x_k - \Delta^m x_{s_i}, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - \Delta^m x_{s_i}, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - \Delta^m x_{s_i}, t) \ge \epsilon\}| < \frac{1}{i}$
Now, for large values of i, we have:

$$\{j_k \in \mathbb{N} : G(\Delta^m x_{j_k} - \Delta^m x_{s_i}, t) \le 1-\epsilon \text{ or } B(\Delta^m x_{j_k} - \Delta^m x_{s_i}, t) \ge \epsilon \text{ and } Y(\Delta^m x_{j_k} - \Delta^m x_{s_i}, t) \ge \epsilon\}| = 0$$

$\Longrightarrow$ $\{x_k\}$ is an Δ^m-I_N-Cauchy sequence in V.
Hence this completes the proof of this theorem. □

Definition 2.5. Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. V is named as ideal complete or Δ^m-I_N-complete if every Δ^m-I_N-Cauchy sequence in V is Δ^m-I_N-convergent.

Definition 2.6. Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. V is named as Δ^m-I_N^*-complete if every Δ^m-I_N^*-Cauchy sequence in V is Δ^m-I_N^*-convergent.

Theorem 2.9. *Assume $I \subseteq \mathscr{P}(N)$ is an admissible ideal, an N.N.S V is Δ^m-I-complete.*

Proof. Assume $\{x_k\}$ be a generalized difference Δ^m-I_N-Cauchy sequence in V. To prove the result we have to prove that $\{x_k\}$ is Δ^m-I_N-convergent in V. Suppose that $\{x_k\}$ is not Δ^m-I_N-convergent in V.
Assume $\epsilon > 0$ and $t > 0$. select $\lambda > 0$ in such a manner,

$$(1-\epsilon) \odot (1-\epsilon) > 1-\lambda \text{ and } \epsilon \cdot \epsilon < \lambda \tag{2.14}$$

Since, $\{x_k\}$ is Δ^m-I-Cauchy so, $\exists$ a positive integer s in such a manner,

$$A(\epsilon, t) = \{k \in \mathbb{N} : G(\Delta^m x_k - \Delta^m x_s, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - \Delta^m x_s, t) \ge \epsilon\} \in I$$

and

$$\phi \ne A^c(\epsilon, t) = \{k \in \mathbb{N} : G(\Delta^m x_k - \Delta^m x_s, t) > 1-\epsilon,\ B(\Delta^m x_k - \Delta^m x_s, t) < \epsilon \text{ or } Y(\Delta^m x_k - \Delta^m x_s, t) < \epsilon\} \in I. \tag{2.15}$$

Assume $B(\epsilon, t)$ be define by

$$B(\epsilon, t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) \le 1-\epsilon \text{ or } B(\Delta^m x_k - L, t) \ge \epsilon \text{ and } Y(\Delta^m x_k - L, t) \ge \epsilon\}.$$

Since, $\{x_k\}$ is not Δ^m-I_N-convergent in V. So, $B(\epsilon, t) \notin I$ and therefore in $F(I)$,

$$\Longrightarrow \quad B^c(\epsilon, t) = \{k \in \mathbb{N} : G(\Delta^m x_k - L, t) > 1-\epsilon,\ B(\Delta^m x_k - L, t) < \epsilon \text{ or } Y(\Delta^m x_k - L, t) < \epsilon\} \in I.$$

Now we shall show that $A^c(\epsilon,t)\subseteq B^c(\epsilon,t)$. Assume $k_0\in A^c(\epsilon,t)$, then $G(\Delta^m x_{k_0}-\Delta^m x_s,t)>1-\epsilon$, $B(\Delta^m x_{k_0}-\Delta^m x_s,t)<\epsilon$ or $Y(\Delta^m x_{k_0}-\Delta^m x_s,t)<\epsilon\}$. From Theorem 2.6, we have

$$G(\Delta^m x_{k_0}-\Delta^m x_s,t)\ge G\left(\Delta^m x_{k_0}-\Delta^m x_s,\frac{t}{2}\right)\odot G\left(\Delta^m x_{k_0}-\Delta^m x_s,\frac{t}{2}\right)>(1-\epsilon)\odot(1-\epsilon)>1-\lambda,$$

$$B(\Delta^m x_{k_0}-\Delta^m x_s,t)\le B\left(\Delta^m x_{k_0}-\Delta^m x_s,\frac{t}{2}\right)\cdot B\left(\Delta^m x_{k_0}-\Delta^m x_s,\frac{t}{2}\right)<\epsilon\cdot\epsilon<\lambda,\quad\text{and}$$

$$Y(\Delta^m x_{k_0}-\Delta^m x_s,t)\le Y\left(\Delta^m x_{k_0}-\Delta^m x_s,\frac{t}{2}\right)\cdot Y\left(\Delta^m x_{k_0}-\Delta^m x_s,\frac{t}{2}\right)<\epsilon\cdot\epsilon<\lambda.$$

This shows that $k_0\in B^c(\lambda,t)$ and therefore we have $A^c(\epsilon,t)\subseteq B^c(\epsilon,t)$. Since $B^c(\epsilon,t)\in I$ so $A^c(\epsilon,t)\in I$, which contradict equation (2.15). Hence, $\{x_k\}$is Δ^m-I-convergent in V and therefore V is Δ^m-I_N-complete. □

Theorem 2.10. *Assume $I\subseteq\mathscr{P}(N)$ is an admissible ideal, V be an N.N.S. A sequence $x=\{x_k\}$ in V is named as Δ^m-I_N^*-convergent iff it is Δ^m-I_N^*-Cauchy in N.N.S.*

3. Conclusion

We have presented the perception of Δ^m-ideal convergence of sequences in N.N.S. We have defined some examples which shows that this method of convergence is more generalized. We have also defined the perceptions of Δ^m-I_N-Cauchy, Δ^m-I_N-completeness, Δ^m-I_N^*-convergence, Δ^m-I_N^*-Cauchy and Δ^m-I_N^*-completeness for these types of sequences in N.N.S.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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