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Research Article

Some Binomial Sums of κ-Jacobsthal and κ-Jacobsthal-Lucas Numbers

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Abstract. In this paper, we formulate some crucial identities containing κ-Jacobsthal and κ-Jacobsthal-Lucas numbers and use these identities to establish some binomial sums of κ-Jacobsthal and κ-Jacobsthal-Lucas numbers.

Keywords. Jacobsthal number, Jacobsthal-Lucas number, κ-Jacobsthal number, κ-Jacobsthal-Lucas number

Mathematics Subject Classification (2020). 11B37, 11B50

1. Introduction

Jacobsthal and Jacobsthal-Lucas numbers are particular examples of generalized Fibonacci numbers. These numbers were first defined in the year 1996 by Horadam [4]. In the previous decade, many authors have studied the various properties of these numbers.

Uygun and Eldogan [8, 9, 11, 12], first defined κ-Jacobsthal and κ-Jacobsthal-Lucas numbers and formulated different properties of these numbers.

Definition 1.1 (Uygun [8]). The κ-Jacobsthal numbers satisfy the recurrence relation $\Phi_{\kappa,n+1} = \kappa\Phi_{\kappa,n} + 2\Phi_{\kappa,n-1}$, for $n \geq 1$ with $\Phi_{\kappa,0} = 0$ and $\Phi_{\kappa,1} = 1$.

Definition 1.2 (Uygun [8]). The κ-Jacobsthal-Lucas numbers satisfy the recurrence relation $\Psi_{\kappa,n+1} = \kappa\Psi_{\kappa,n} + 2\Psi_{\kappa,n-1}$, for $n \geq 1$ with $\Psi_{\kappa,0} = 2$ and $\Psi_{\kappa,1} = \kappa$.

The Binet formula of $\Phi_{\kappa,n}$ and $\Psi_{\kappa,n}$ is (Uygun [8]):

$$\Phi_{\kappa,n} = \frac{\xi_1^n - \xi_2^n}{\xi_1 - \xi_2}, \tag{1.1}$$

$$\Psi_{\kappa,n} = \xi_1^n + \xi_2^n. \tag{1.2}$$

The characteristic roots ξ_1 and ξ_2 appeared in (1.1) and (1.2) satisfy the following relations (Uygun [8]):

$$\xi_1 = \frac{\kappa + \sqrt{\kappa^2+8}}{2}, \tag{1.3}$$

$$\xi_2 = \frac{\kappa - \sqrt{\kappa^2+8}}{2}, \tag{1.4}$$

$$\xi_1 - \xi_2 = \sqrt{\kappa^2+8} = \sqrt{\delta}, \tag{1.5}$$

$$\xi_1 + \xi_2 = \kappa, \tag{1.6}$$

$$\xi_1 \xi_2 = -2, \tag{1.7}$$

$$\xi_1^2 = \kappa \xi_1 + 2, \tag{1.8}$$

$$\xi_2^2 = \kappa \xi_2 + 2. \tag{1.9}$$

In particular, for $\kappa = 1$, the κ-Jacobsthal numbers transform into a famous Jacobsthal number. These numbers are the origins of many interesting properties. In the past few years, many authors have studied the properties of these numbers, see [1, 5–12] and the references cited therein. Some of these are listed below:

Lemma 1.3. *Let $n, m \in \mathbb{Z}^+$. Then (Uygun [8–12]):*

(i) $\Phi_{\kappa,n+1} + 2\Phi_{\kappa,n-1} = \Psi_{\kappa,n}$, (1.10)

(ii) $\Phi_{\kappa,n+1} + 2\Psi_{\kappa,n-1} = \delta\Phi_{\kappa,n}$, (1.11)

(iii) $2\Phi_{\kappa,m+n} = \Phi_{\kappa,m}\Psi_{\kappa,n} + \Phi_{\kappa,n}\Psi_{\kappa,m}$, (1.12)

(iv) $2\Phi_{\kappa,m-n} = (-1)^n(\Phi_{\kappa,m}\Psi_{\kappa,n} - \Phi_{\kappa,n}\Psi_{\kappa,m})$, (1.13)

(v) $\Phi_{\kappa,n-1}\Phi_{\kappa,n+1} - \Phi_{\kappa,n}^2 = -(-2)^{n-1}$. (1.14)

In the year 1970, Carlitz [2] derived various Fibonacci and Lucas identities, Zhang [13] in the year 1997 proved different identities for second order integer sequences. Latest [3] and in this paper, we are inspired by the work of Carlitz and Zhang to develop binomial sums of $\Phi_{\kappa,n}$ and $\Psi_{\kappa,n}$.

2. Binomial Sums involving $\Phi_{\kappa,n}$ and $\Psi_{\kappa,n}$

In this section, we establish binomial sums for $\Phi_{\kappa,n}$ and $\Psi_{\kappa,n}$. Lemma 2.1 plays important role in proving Theorems 2.2 to 2.6.

Lemma 2.1. *Let $u = \mu_1$ or μ_2. Then show that*

(a) $u^n = u\Phi_{\kappa,n} + 2\Phi_{\kappa,n-1}$, (2.1)

(b) $u^{2n} = u^n\Psi_{\kappa,n} - (-2)^n$, (2.2)

(c) $u^{tn} = u^n \frac{\Phi_{\kappa,tn}}{\Phi_{\kappa,n}} - (-2)^n \frac{\Phi_{\kappa,(t-1)n}}{\Phi_{\kappa,n}}$, (2.3)

(d) $u^{sn}\Phi_{\kappa,rn} - u^{rn}\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,(r-s)n}$. (2.4)

Proof. (a): We adopt P.M.I. on n to prove this result.
For $n = 2$, we have from (1.8) and (1.9)

$$\xi_1^2 = \xi_1\Phi_{\kappa,2} + \Phi_{\kappa,1},$$
$$\xi_2^2 = \xi_2\Phi_{\kappa,2} + \Phi_{\kappa,1}.$$

Now assume that the result is true for n. Hence, we have

$$\xi_1^n = \xi_1\Phi_{\kappa,n} + \Phi_{\kappa,n-1}, \tag{2.5}$$
$$\xi_2^n = \xi_2\Phi_{\kappa,n} + \Phi_{\kappa,n-1}. \tag{2.6}$$

Furthermore, by using (1.8) and (2.5), we obtain

$$\begin{aligned}
\xi_1^{n+1} &= \xi_1{\xi_1}^n \\
&= \xi_1(\xi_1\Phi_{\kappa,n} + 2\Phi_{\kappa,n-1}) \\
&= {\xi_1}^2\Phi_{\kappa,n} + 2\xi_1\Phi_{\kappa,n-1} \\
&= (\kappa\xi_1 + 2)\Phi_{\kappa,n} + 2\xi_1\Phi_{\kappa,n-1} \\
&= (\kappa\Phi_{\kappa,n} + 2\Phi_{\kappa,n-1})\xi_1 + 2\Phi_{\kappa,n} \\
&= \Phi_{\kappa,n+1}\xi_1 + 2\Phi_{\kappa,n}.
\end{aligned}$$

In similar way, we can show that

$$\xi_2^{n+1} = \xi_2\Phi_{\kappa,n+1} + 2\Phi_{\kappa,n}.$$

(b): From (a), we have

$$\begin{aligned}
u^{2n} &= \Phi_{\kappa,n}u^{n+1} + 2u^n\Phi_{\kappa,n-1} \\
&= \Phi_{\kappa,n}(u\Phi_{\kappa,n+1} + 2\Phi_{\kappa,n}) + 2u^n\Phi_{\kappa,n-1} \\
&= u\Phi_{\kappa,n}\Phi_{\kappa,n+1} + 2\Phi_{\kappa,n-1}u^n + 2\Phi^2_{\kappa,n} \\
&= (u^n - \Phi_{\kappa,n-1})\Phi_{\kappa,n+1} + 2\Phi_{\kappa,n-1}u^n + 2\Phi^2_{\kappa,n} \\
&= u^n(\Phi_{\kappa,n+1} + 2\Phi_{\kappa,n-1}) + 2(\Phi^2_{\kappa,n} - \Phi_{\kappa,n+1}\Phi_{\kappa,n-1}).
\end{aligned}$$

Finally, by using (1.10) and (1.14), we obtain

$$u^{2n} = \Psi_{\kappa,n}u^n - (-2)^n.$$

(c): Let $u = \xi_1$. Then adopting the Binet formula of $\Phi_{\kappa,n}$, we can write

$$\begin{aligned}
{\xi_1}^n \frac{\Phi_{\kappa,tn}}{\Phi_{\kappa,n}} - (-2)^n \frac{\Phi_{\kappa,(t-1)n}}{\Phi_{\kappa,n}} &= \frac{1}{\Phi_{\kappa,n}}\left\{\left(\frac{{\xi_1}^{tn} - {\xi_2}^{tn}}{\xi_1 - \xi_2}\right){\xi_1}^n - (\xi_1\xi_2)^n\left(\frac{{\xi_1}^{(t-1)n} - {\xi_2}^{(t-1)n}}{\xi_1 - \xi_2}\right)\right\} \\
&= \frac{1}{\Phi_{\kappa,n}}\left\{\frac{\xi_1^{tn}\xi_1^n - \xi_2^{tn}\xi_1^n - \xi_2^n\xi_1^{tn} + \xi_1^n\xi_2^{tn}}{\xi_1 - \xi_2}\right\} \\
&= \frac{1}{\Phi_{\kappa,n}}\left\{\frac{\xi_1^{tn}\left(\xi_1^n - \xi_2^n\right)}{\xi_1 - \xi_2}\right\}
\end{aligned}$$

$$= \frac{1}{\Phi_{\kappa,n}}(\xi_1^{tn}\Phi_{\kappa,n})$$
$$= \xi_1^{tn}.$$

Similarly, if $u = \xi_2$ then, we get

$$\xi_2^{tn} = \xi_2^n \frac{\Phi_{\kappa,tn}}{\Phi_{\kappa,n}} - (-2)^n \frac{\Phi_{\kappa,(t-1)n}}{\Phi_{\kappa,n}}.$$

(d): Let $u = \xi_1$. Consider

$$\begin{aligned}
\xi_1^{sn}\Phi_{\kappa,rn} - \xi_1^{rn}\Phi_{\kappa,sn} &= \xi_1^{sn}\left(\frac{\xi_1^{rn}-\xi_2^{rn}}{\xi_1-\xi_2}\right) - \xi_1^{rn}\left(\frac{\xi_1^{sn}-\xi_2^{sn}}{\xi_1-\xi_2}\right)\\
&= \left(\frac{\xi_1^{sn}\xi_1^{rn} - \xi_1^{sn}\xi_2^{rn} - \xi_1^{rn}\xi_1^{sn} + \xi_1^{rn}\xi_2^{sn}}{\xi_1-\xi_2}\right)\\
&= \left(\frac{\xi_1^{rn}\xi_2^{sn}\xi_1^{-sn}\xi_1^{sn} - \xi_1^{sn}\xi_2^{rn}\xi_2^{sn}\xi_2^{-sn}}{\xi_1-\xi_2}\right)\\
&= (\xi_1\xi_2)^{sn}\left(\frac{\xi_1^{(r-s)n}-\xi_2^{(r-s)n}}{\xi_1-\xi_2}\right)\\
&= (-2)^{sn}\Phi_{\kappa,(r-s)n}.
\end{aligned}$$

Furthermore, if $u = \xi_2$ then, we get

$$\xi_2^{sn}\Phi_{\kappa,rn} - \xi_2^{rn}\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,(r-s)n}.$$ □

Theorem 2.2. *Let $n,r,s,t \in \mathbb{Z}^+$ with $t \geq 1$. Then prove that*

(i) $\Phi_{\kappa,n+t} = \Phi_{\kappa,n}\Phi_{\kappa,t+1} + 2\Phi_{\kappa,n-1}\Phi_{\kappa,t}$, (2.7)

(ii) $\Psi_{\kappa,n+t} = \Phi_{\kappa,n}\Psi_{\kappa,t+1} + 2\Phi_{\kappa,n-1}\Psi_{\kappa,t}$, (2.8)

(iii) $\Phi_{\kappa,2n+t} = \Psi_{\kappa,n}\Phi_{\kappa,n+t} - (-2)^n\Phi_{\kappa,t}$, (2.9)

(iv) $\Psi_{\kappa,2n+t} = \Psi_{\kappa,n}\Psi_{\kappa,n+t} - (-2)^n\Psi_{\kappa,t}$, (2.10)

(v) $\Phi_{\kappa,sn+t}\Phi_{\kappa,n} = \Phi_{\kappa,sn}\Phi_{\kappa,n+t} - (-2)^n\Phi_{\kappa,(s-1)n}\Phi_{\kappa,t}$, (2.11)

(vi) $\Psi_{\kappa,sn+t}\Phi_{\kappa,n} = \Phi_{\kappa,sn}\Psi_{\kappa,n+t} - (-2)^n\Phi_{\kappa,(s-1)n}\Psi_{\kappa,t}$, (2.12)

(vii) $\Phi_{\kappa,sn+t}\Phi_{\kappa,rn} - \Phi_{\kappa,rn+t}\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,t}\Phi_{\kappa,(r-s)n}$, (2.13)

(viii) $\Psi_{\kappa,sn+t}\Phi_{\kappa,rn} - \Psi_{\kappa,rn+t}\Phi_{\kappa,sn} = (-2)^{sn}\Psi_{\kappa,t}\Phi_{\kappa,(r-s)n}$. (2.14)

Proof. (i): Using Lemma 2.1(a), we have

$$\xi_1^n = \Phi_{\kappa,n}\xi_1 + 2\Phi_{\kappa,n-1}, \tag{2.15}$$
$$\xi_2^n = \Phi_{\kappa,n}\xi_2 + 2\Phi_{\kappa,n-1}. \tag{2.16}$$

By multiplying (2.15) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and (2.16) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we obtain

$$\frac{\xi_1^{n+t}-\xi_2^{n+t}}{\xi_1-\xi_2} = \Phi_{\kappa,n}\left(\frac{\xi_1^{t+1}-\xi_2^{t+1}}{\xi_1-\xi_2}\right) + 2\Phi_{\kappa,n-1}\left(\frac{\xi_1^t-\xi_2^t}{\xi_1-\xi_2}\right).$$

Now using the Binet formula of κ-Jacobsthal number, we get

$$\Phi_{\kappa,n+t} = \Phi_{\kappa,n}\Phi_{\kappa,t+1} + \Phi_{\kappa,n-1}\Phi_{\kappa,t}.$$

This completes the proof of result (i).

(iii): First, using Lemma 2.1(b), we have

$$\xi_1^{2n} = \Psi_{\kappa,n}\xi_1^n - (-2)^n\,, \tag{2.17}$$

$$\xi_2^{2n} = \Psi_{\kappa,n}\xi_2^n - (-2)^n\,. \tag{2.18}$$

Multiplying an equation (2.17) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and (2.18) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we obtain

$$\frac{\xi_1^{2n+t}-\xi_2^{2n+t}}{\xi_1-\xi_2} = \Psi_{\kappa,n}\left(\frac{\xi_1^{n+t}-\xi_2^{n+t}}{\xi_1-\xi_2}\right) - (-2)^n\left(\frac{\xi_1^t-\xi_2^t}{\xi_1-\xi_2}\right),$$

i.e.

$$\Phi_{\kappa,2n+t} = \Psi_{\kappa,n}\Phi_{\kappa,n+t} - (-2)^n\Phi_{\kappa,t}\,.$$

Thus, the result (iii).

(v): By using Lemma 2.1(c), we have

$$\xi_1^{sn} = \xi_1^n\frac{\Phi_{\kappa,sn}}{\Phi_{\kappa,n}} - (-2)^n\frac{\Phi_{\kappa,(s-1)n}}{\Phi_{\kappa,n}}\,, \tag{2.19}$$

$$\xi_2^{sn} = \xi_2^n\frac{\Phi_{\kappa,sn}}{\Phi_{\kappa,n}} - (-2)^n\frac{\Phi_{\kappa,(s-1)n}}{\Phi_{\kappa,n}}\,. \tag{2.20}$$

Now, multiplying an equation (2.19) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and (2.20) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we attain

$$\frac{\xi_1^{sn+t}-\xi_2^{sn+t}}{\xi_1-\xi_2} = \frac{\Phi_{\kappa,sn}}{\Phi_{\kappa,n}}\left(\frac{\xi_1^{n+t}-\xi_2^{n+t}}{\xi_1-\xi_2}\right) - (-2)^n\left(\frac{\xi_1^t-\xi_2^t}{\xi_1-\xi_2}\right)\frac{\Phi_{\kappa,(s-1)n}}{\Phi_{\kappa,n}}\,.$$

Using the Binet formula of $\Phi_{\kappa,n}$, we get

$$\Phi_{\kappa,sn+t} = \frac{\Phi_{\kappa,sn}}{\Phi_{\kappa,n}}\Phi_{\kappa,n+t} - (-2)^n\frac{\Phi_{\kappa,(s-1)n}}{\Phi_{\kappa,n}}\Phi_{\kappa,t}\,,$$

i.e.

$$\Phi_{\kappa,sn+t}\Phi_{\kappa,n} = \Phi_{\kappa,sn}\Phi_{\kappa,n+t} - (-2)^n\Phi_{\kappa,(s-1)n}\Phi_{\kappa,t}\,.$$

This proves the result (v).

(vii): By making use of Lemma 2.1(d), we have

$$\xi_1^{sn}\Phi_{\kappa,rn} - \xi_1^{rn}\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,(r-s)n}\,, \tag{2.21}$$

$$\xi_2^{sn}\Phi_{\kappa,rn} - \xi_2^{rn}\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,(r-s)n}\,. \tag{2.22}$$

Now, multiplying an equation (2.21) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and (2.22) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we obtain

$$\left(\frac{\xi_1^{sn+t}-\xi_2^{sn+t}}{\xi_1-\xi_2}\right)\Phi_{\kappa,rn} - \left(\frac{\xi_1^{rn+t}-\xi_2^{rn+t}}{\xi_1-\xi_2}\right)\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,(r-s)n}\left(\frac{\xi_1^t-\xi_2^t}{\xi_1-\xi_2}\right),$$

i.e.

$$\Phi_{\kappa,sn+t}\Phi_{\kappa,rn} - \Phi_{\kappa,rn+t}\Phi_{\kappa,sn} = (-2)^{sn}\Phi_{\kappa,t}\Phi_{\kappa,(r-s)n}\,.$$

Thus, the result (vii). □

Theorem 2.3. *Let $n,r,s,t \in \mathbb{Z}^+$ with $t \geq 1$. Then*

(i) $\Phi_{\kappa,rn+t} = \sum_{i=0}^{n}\binom{n}{i}\Phi_{\kappa,r}^i\Phi_{\kappa,r-1}^{n-i}\Phi_{\kappa,i+t}\,,$

(ii) $\Psi_{\kappa,rn+t} = \sum_{i=0}^{n} \binom{n}{i} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} \Psi_{\kappa,i+t}$,

(iii) $\Phi_{\kappa,rn+t} \Psi_{\kappa,r}^{n} = \sum_{i=0}^{n} \binom{n}{i} (-2)^{r(n-i)} \Phi_{\kappa,2ri+t}$,

(iv) $\Psi_{\kappa,rn+t} \Psi_{\kappa,r}^{n} = \sum_{i=0}^{n} \binom{n}{i} (-2)^{r(n-i)} \Psi_{\kappa,2ri+t}$,

(v) $\Phi_{\kappa,t} \Phi_{\kappa,r-1}^{n} = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Phi_{\kappa,r}^{n-i} \Phi_{n+(r-1)i+t}$,

(vi) $\Psi_{\kappa,t} \Phi_{\kappa,r-1}^{n} = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Phi_{\kappa,r}^{n-i} \Psi_{n+(r-1)i+t}$.

Proof. From Lemma 2.1(a), we have

$$\xi_1^r = \Phi_{\kappa,r}\xi_1 + 2\Phi_{\kappa,r-1},$$
$$\xi_2^r = \Phi_{\kappa,r}\xi_2 + 2\Phi_{\kappa,r-1}.$$

Using the binomial theorem, we get

$$\xi_1^{rn} = \sum_{i=0}^{n} \binom{n}{i} 2^{(n-i)} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} \xi_1^{i}, \tag{2.23}$$

$$\xi_2^{rn} = \sum_{i=0}^{n} \binom{n}{i} 2^{(n-i)} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} \xi_2^{i}. \tag{2.24}$$

Now, by multiplying an equation (2.23) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and equation (2.24) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we attain

$$\frac{\xi_1^{rn+t} - \xi_2^{rn+t}}{\xi_1 - \xi_2} = \sum_{i=0}^{n} \binom{n}{i} 2^{(n-i)} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} \left(\frac{\xi_1^{i+t} - \xi_2^{i+t}}{\xi_1 - \xi_2} \right),$$

i.e.

$$\Phi_{\kappa,rn+t} = \sum_{i=0}^{n} \binom{n}{i} 2^{(n-i)} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} \Phi_{\kappa,i+t}.$$

This proves the result (i).

Again, by multiplying equation (2.23) by ξ_1^t and (2.24) by ξ_2^t and adding, we get

$$\xi_1^{rn+t} + \xi_2^{rn+t} = \sum_{i=0}^{n} \binom{n}{i} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} (\xi_1^{i+t} + \xi_2^{i+t}),$$

i.e.

$$\Psi_{\kappa,rn+t} = \sum_{i=0}^{n} \binom{n}{i} \Phi_{\kappa,r}^{i} \Phi_{\kappa,r-1}^{n-i} \Psi_{\kappa,i+t}.$$

Thus, the result (ii).

The proofs of (iii)-(vi) are analogous to (i) and (ii). Hence, we omit the proofs. □

Theorem 2.4. *Let* $n,r,s,t \in \mathbb{Z}^+$ *with* $t \geq 1$. *Then prove that*

(i) $\Phi_{\kappa,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(r+1)(n-i)} 2^{r(n-i)} \Psi_{\kappa,r}^{i} \Phi_{\kappa,ri+t}$,

(ii) $\Psi_{\kappa,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(r+1)(n-i)} 2^{r(n-i)} \Psi_{\kappa,r}^{i} \Psi_{\kappa,ri+t}$,

(iii) $\Phi_{\kappa,n+t}\Phi_{\kappa,r}^{n} = \sum_{i=0}^{n} \binom{n}{i} (-2)^{n-i} \Phi_{\kappa,r-1}^{n-i} \Phi_{\kappa,ri+t}$,

(iv) $\Psi_{\kappa,n+t}\Phi_{\kappa,r}^{n} = \sum_{i=0}^{n} \binom{n}{i} (-2)^{n-i} \Phi_{\kappa,r-1}^{n-i} \Psi_{\kappa,ri+t}$,

(v) $\Phi_{\kappa,t}(-2)^{rn} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Psi_{\kappa,r}^{i} \Phi_{\kappa,(2n-i)r+t}$,

(vi) $\Psi_{\kappa,t}(-2)^{rn} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Psi_{\kappa,r}^{i} \Psi_{\kappa,(2n-i)r+t}$.

Proof. By using Lemma 2.1(b), we rewrite

$$\xi_1^{2r} = \xi_1^{r}\Psi_{\kappa,r} - (-2)^r,$$
$$\xi_2^{2r} = \xi_2^{r}\Psi_{\kappa,r} - (-2)^r.$$

By making use of the binomial theorem, we get

$$\xi_1^{2rn} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Psi_{\kappa,r}^{i} \xi_1^{ri} (-2)^{r(n-i)}, \tag{2.25}$$

$$\xi_2^{2rn} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Psi_{\kappa,r}^{i} \xi_2^{ri} (-2)^{r(n-i)}. \tag{2.26}$$

Now, multiplying an equation (2.25) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and equation (2.26) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we attain

$$\frac{\xi_1^{2rn+t} - \xi_2^{2rn+t}}{\xi_1 - \xi_2} = \sum_{i=0}^{n} \binom{n}{i} \Psi_{\kappa,r}^{i} (-1)^{(n-i)(r+1)} 2^{r(n-i)} \left(\frac{\xi_1^{ri+t} - \xi_2^{ri+t}}{\xi_1 - \xi_2} \right),$$

i.e.

$$\Phi_{\kappa,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} \Psi_{\kappa,r}^{i} (-1)^{(n-i)(r+1)} 2^{r(n-i)} \Phi_{\kappa,ri+t}.$$

Thus the result (i).

Again, by multiplying equation (2.25) by ξ_1^t and (2.26) by ξ_2^t and adding, we get

$$\xi_1^{2rn+t} + \xi_2^{2rn+t} = \sum_{i=0}^{n} \binom{n}{i} \Psi_{\kappa,r}^{i} (-1)^{(n-i)(r+1)} 2^{r(n-i)} (\xi_1^{ri+t} + \xi_2^{ri+t}),$$

$$\Psi_{\kappa,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} \Psi_{\kappa,r}^{i} (-1)^{(n-i)(r+1)} 2^{r(n-i)} \Psi_{\kappa,ri+t}.$$

Hence the result (ii).

The proofs of (iii)-(vi) are similar to (i) and (ii). Hence, we omit the proofs. □

Theorem 2.5. *Let $n,r,s,t,l \in \mathbb{Z}^+$ with $t \geq 1$. Then show that*

(i) $\Phi_{\kappa,trn+l}\Phi^n_{\kappa,r} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}2^{r(n-i)}\Phi^i_{\kappa,tr}\Phi^{(n-i)}_{\kappa,(t-1)r}\Phi_{\kappa,ri+l}$,

(ii) $\Psi_{\kappa,trn+l}\Phi^n_{\kappa,r} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}2^{r(n-i)}\Phi^i_{\kappa,tr}\Phi^{(n-i)}_{\kappa,(t-1)r}\Psi_{\kappa,ri+l}$,

(iii) $\Phi_{\kappa,rn+l}\Phi^n_{\kappa,tr} = \sum_{i=0}^{n}\binom{n}{i}(-2)^{r(n-i)}\Phi^i_{\kappa,r}\Phi^{(n-i)}_{\kappa,(t-1)r}\Phi_{\kappa,tri+l}$,

(iv) $\Psi_{\kappa,rn+l}\Phi^n_{\kappa,tr} = \sum_{i=0}^{n}\binom{n}{i}(-2)^{r(n-i)}\Phi^i_{\kappa,r}\Phi^{(n-i)}_{\kappa,(t-1)r}\Psi_{\kappa,tri+l}$,

(v) $(-2)^{rn}\Phi_{\kappa,l}\Phi^n_{\kappa,(t-1)r} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\Phi^i_{\kappa,tr}\Phi^{(n-i)}_{\kappa,r}\Phi_{\kappa,ri+l}$,

(vi) $(-2)^{rn}\Psi_{\kappa,l}\Phi^n_{\kappa,(t-1)r} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\Phi^i_{\kappa,tr}\Phi^{(n-i)}_{\kappa,r}\Psi_{\kappa,ri+l}$.

Proof. By making use of Lemma 2.1(c), we have

$$\xi_1^{tr} = \xi_1^r\frac{\Phi_{\kappa,tr}}{\Phi_{\kappa,r}} - (-2)^r\frac{\Phi_{\kappa,(t-1)r}}{\Phi_{\kappa,r}},$$
$$\xi_2^{tr} = \xi_2^r\frac{\Phi_{\kappa,tr}}{\Phi_{\kappa,r}} - (-2)^r\frac{\Phi_{\kappa,(t-1)r}}{\Phi_{\kappa,r}}.$$

Now, using the binomial theorem, we get

$$\xi_1^{trn} = \frac{1}{\Phi^n_{\kappa,r}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}2^{r(n-i)}\Phi^i_{\kappa,tr}\xi_1^{ri}\Phi^{(n-i)}_{\kappa,(t-1)r}, \tag{2.27}$$
$$\xi_2^{trn} = \frac{1}{\Phi^n_{\kappa,r}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}2^{r(n-i)}\Phi^i_{\kappa,tr}\xi_2^{ri}\Phi^{(n-i)}_{\kappa,(t-1)r}. \tag{2.28}$$

By multiplying an equation (2.27) by $\frac{\xi_1^l}{\xi_1-\xi_2}$ and equation (2.28) by $\frac{\xi_2^l}{\xi_1-\xi_2}$ and subtracting, we obtain

$$\frac{\xi_1^{trn+l}-\xi_2^{trn+l}}{\xi_1-\xi_2} = \frac{1}{\Phi^n_{\kappa,r}}\sum_{i=0}^{n}\binom{n}{i}\Phi^i_{\kappa,tr}\Phi^{n-i}_{\kappa,(t-1)r}(-1)^{(n-i)(r+1)}2^{r(n-i)}\left(\frac{\xi_1^{ri+l}-\xi_2^{ri+l}}{\xi_1-\xi_2}\right),$$

i.e.

$$\Phi_{\kappa,trn+l}\Phi^n_{\kappa,rn} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}2^{r(n-i)}\Phi^i_{\kappa,r}\Phi^{n-i}_{\kappa,(t-1)r}\Phi_{\kappa,ri+l}.$$

Hence the proof of (i).

Furthermore, multiplying equation (2.27) by ξ_1^l and (2.28) by ξ_2^l and adding, we obtain

$$\xi_1^{trn+l}+\xi_2^{trn+l} = \frac{1}{\Phi^n_{\kappa,r}}\sum_{i=0}^{n}\binom{n}{i}\Phi^i_{\kappa,tr}\Phi^{n-i}_{\kappa,(t-1)r}(-1)^{(n-i)(r+1)}2^{r(n-i)}(\xi_1^{ri+l}+\xi_2^{ri+l}),$$

i.e.

$$\Psi_{\kappa,trn+l}\Phi^n_{\kappa,r} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}2^{r(n-i)}\Phi^i_{\kappa,tr}\Phi^{n-i}_{\kappa,(t-1)r}\Psi_{\kappa,ri+l}.$$

Thus the result (ii).
The proofs of (iii)-(vi) are analogous to (i) and (i). Hence, we omit the proofs. □

Theorem 2.6. *Let $n,r,s,t \in \mathbb{Z}^+$ with $t \geq 1$. Then*

(i) $(-2)^{smn}\Phi_{\kappa,t}\Phi^n_{\kappa,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\Phi^i_{\kappa,rm}\Phi^{(n-i)}_{\kappa,sm}\Phi_{\kappa,smi+rm(n-i)+t}$,

(ii) $(-2)^{smn}\Psi_{\kappa,t}\Phi^n_{\kappa,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\Phi^i_{\kappa,rm}\Phi^{(n-i)}_{\kappa,sm}\Psi_{\kappa,smi+rm(n-i)+t}$,

(iii) $\Phi^n_{\kappa,rm}\Phi_{\kappa,smn+t} = \sum_{i=0}^{n}\binom{n}{i}(-2)^{sm(n-i)}\Phi^i_{\kappa,sm}\Phi^{(n-i)}_{\kappa,(r-s)m}\Phi_{\kappa,rmi+t}$,

(iv) $\Phi^n_{\kappa,rm}\Psi_{\kappa,smn+t} = \sum_{i=0}^{n}\binom{n}{i}(-2)^{sm(n-i)}\Phi^i_{\kappa,sm}\Phi^{(n-i)}_{\kappa,(r-s)m}\Psi_{\kappa,rmi+t}$,

(v) $\Phi^n_{\kappa,sm}\Phi_{\kappa,rmn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(sm+1)(n-i)}(2)^{sm(n-i)}\Phi^i_{\kappa,rm}\Phi^{(n-i)}_{\kappa,(r-s)m}\Phi_{\kappa,smi+t}$,

(vi) $\Phi^n_{\kappa,sm}\Psi_{\kappa,rmn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(sm+1)(n-i)}(2)^{sm(n-i)}\Phi^i_{\kappa,rm}\Phi^{(n-i)}_{\kappa,(r-s)m}\Psi_{\kappa,smi+t}$.

Proof. Using Lemma 2.1(d), we have

$$\xi_1^{sm}\Phi_{\kappa,rm} - \xi_1^{rm}\Phi_{\kappa,sm} = (-2)^{sm}\Phi_{\kappa,(r-s)m},$$
$$\xi_2^{sm}\Phi_{\kappa,rm} - \xi_2^{rm}\Phi_{\kappa,sm} = (-2)^{sm}\Phi_{\kappa,(r-s)m}.$$

Thanks to the binomial theorem. By employing it, we get

$$(-2)^{smn}\Phi^n_{\kappa,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\Phi^i_{\kappa,rm}\Phi^{n-i}_{\kappa,sm}\xi_1^{smi+rm(n-i)}, \tag{2.29}$$

$$(-2)^{smn}\Phi^n_{\kappa,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\Phi^i_{\kappa,rm}\Phi^{n-i}_{\kappa,sm}\xi_2^{smi+rm(n-i)}. \tag{2.30}$$

Thus the result (i). Now, multiplying an equation (2.29) by $\frac{\xi_1^t}{\xi_1-\xi_2}$ and equation (2.30) by $\frac{\xi_2^t}{\xi_1-\xi_2}$ and subtracting, we attain

$$(-2)^{smn}\Phi^n_{\kappa,(r-s)m}\left(\frac{\xi_1^t-\xi_2^t}{\xi_1-\xi_2}\right) = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\Phi^i_{\kappa,rm}\Phi^{n-i}_{\kappa,sm}\left(\frac{\xi_1^{smi+rm(n-i)+t}-\xi_2^{smi+rm(n-i)+t}}{\xi_1-\xi_2}\right),$$

i.e.

$$(-2)^{smn}\Phi^n_{\kappa,(r-s)m}\Phi_{\kappa,t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\Phi^i_{\kappa,rm}\Phi^{n-i}_{\kappa,sm}\Phi_{\kappa,smi+rm(n-i)+t}.$$

Furthermore, by multiplying equation (2.29) by ξ_1^t and (2.30) by ξ_2^t and adding, we obtain

$$(-2)^{smn}\Phi^n_{\kappa,(r-s)m}\Psi_{\kappa,t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\Phi^i_{\kappa,rm}\Phi^{n-i}_{\kappa,sm}\Psi_{\kappa,smi+rm(n-i)+t}.$$

Thus the proof of the result (ii).
The proofs of (iii)-(vi) are similar to (i) and (ii). Hence, we omit the proofs. □

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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