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Research Article

A Common Fixed Point Theorem for Two Compatible Self-maps of a *S*-Metric Space

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Abstract. In this paper, we prove a common fixed point theorem for two compatible self-maps of a S-metric space.

Keywords. S-metric space, Fixed point, Contractive modulus, Associated sequence of a point relative to two self-maps

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

Fixed point theory is an important branch of non-linear analysis due to its application potential. Banach's contraction principle [4] is one of the most important result in non-linear analysis. This theorem has been generalized either by generalizing the underlying space or by viewing it as a common fixed point theorem along with other selfmaps.

Sedghi *et al*. [5] introduced D^*-metric spaces. In 2006, Mustafa and Sims [3] have initiated G-metric spaces as generalization of metric spaces. Later, Sedghi *et al*. [6] proposed S-metric spaces in 2012. These S-metric spaces evinced interest in many researchers. Several fixed point theorems are established on these spaces.

The notion of compatibility of self-maps is introduced as a generalization of commuting maps by Jungck [1,2]. Recently, common fixed theorems were established by using compatibility in [7].

In the present paper, we establish a necessary and sufficient condition for the existence of a common fixed point for two selfmaps of a S-metric space. Further we deduce two interesting consequences of our main theorem.

2. Preliminaries

We now recall some basic definitions which will be useful in our later discussion.

Definition 2.1 ([6]). Let X be a non empty set. By S-metric, we mean a function $S : X^3 \to [0,\infty)$ which satisfies the following conditions for each $x, y, z, w \in X$

(a) $S(x,y,z) \geq 0$;

(b) $S(x,x,y) = 0$ if and only if $x = y = z$;

(c) $S(x,y,z) \leq S(x,x,w) + S(y,y,w) + S(z,z,w)$.

In this case (X,S) is called a S-metric space.

Example 2.2. Let $X = \mathbb{R}$ and $S : \mathbb{R}^3 \to [0,\infty)$ be defined by

$$S(x,y,z) = |y+z-2x| + |y-z|, \quad \text{for } x,y,z \in \mathbb{R},$$

then (X,S) is a S-metric space.

Remark 2.3. It is shown ([6, Lemma 2.5]) in a S-metric space that

$$S(x,x,y) = S(y,y,x), \quad \text{for all } x,y \in X.$$

Definition 2.4 ([6]). Let (X,S) be a S-metric space. A sequence $\{y_n\}$ in X is said to be convergent, if there is a $y \in X$ such that $S(y_n, y_n, y) \to 0$, that is for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(y_n, y_n, y) < \epsilon$ and in this case we write $\lim\limits_{n\to\infty} y_n = y$.

Definition 2.5 ([6]). Let (X,S) be a S-metric space. A sequence $\{y_n\}$ in X is called a Cauchy sequence if to each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(y_n, y_n, y) < \epsilon$ for each $n, m \geq n_0$.

Definition 2.6 ([6]). Let (X,S) be a S-metric space. If there exists sequences $\{y_n\}$ and $\{x_n\}$ such that $\lim\limits_{n\to\infty} y_n = y$ and $\lim\limits_{n\to\infty} x_n = x$ then $\lim\limits_{n\to\infty} S(y_n, y_n, x_n) = S(y,y,x)$, then we say that $S(y,x,z)$ is continuous in y and x.

Definition 2.7 ([6]). If B and A are self-maps of a S-metric space (X,S) such that for every sequence $\{y_n\}$ in X with

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ay_n = u, \quad \text{for some } u \in X.$$

We have

$$\lim_{n\to\infty} S(BAy_n, BAy_n, ABx_n) = 0,$$

then B and A are said to be compatible.

Clearly, commuting self-maps of a S-metric space are compatible but not conversely.

Definition 2.8. A function $\chi : [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\chi(0) = 0$ and $\chi(r) < r$ for $r > 0$.

Definition 2.9. If B and A be self-maps of a non empty set X such that $B(X) \subseteq A(X)$, then for any $y_0 \in X$, if $\{y_n\}$ is a sequence in X such that $Ay_n = By_{n-1}$ for $n \geq 1$ then $\{y_n\}$ is called an associated sequence of y_0 relative to two self-maps B and A.

3. Main Theorem

Theorem 3.1. *Suppose A is continuous selfmap of a S-metric space (X,S), then A has fixed point in X if and only if there is a contractive modulus χ and a selfmap B of X such that*

(i) *A and B are compatible,*

(ii) *$S(Bx,Bx,By) \leq \chi(S(Ax,Ax,Ay))$ for all $x,y \in X$, and*

(iii) *there is a point $y_0 \in X$ and an associated sequence $\{y_n\}$ of y_0 relative to the selfmaps A and B such that the sequence $\{Ay_n\}$ converges to some point u of X. Further, Bu is the unique common fixed point of A and B.*

Proof. First assume that A has a fixed point say 'p', $p \in X$ then $Ap = p$.
Define $B : X \to X$ by $Bx = p$ for all $x \in X$.
Now for any $x \in X$, we have $BA(x) = B(Ax) = p$ and $(AB)x = ABx = Ap = p$ giving that $AB = BA$ showing that A and B are compatible, proving condition (i) of Theorem 3.1.
We have

$$S(Bx,Bx,By) = S(p,p,p) = 0 \leq \chi(S(Ax,Ax,Ay), \quad \text{for any } x,y \in X,$$

proving condition (ii) of Theorem 3.1.

Now an associated sequence of $y_0 = p$ relative to the selfmaps A and B is given by $y_n = p$ for $n = 0,1,2,\ldots$ and since $\{Ay_n\}$ is a constant sequence converging to $p \in X$.
Proving condition (iii) of Theorem 3.1.
Conversely, assume that there is a selfmap B on X and a contractive modulus χ satisfying conditions (i), (ii) and (iii) of Theorem 3.1.
Now from condition (iii) of Theorem 3.1, we get an associated sequence $\{y_n\}$ of y_0 relative to the selfmaps A and B such that the sequence $Ay_n = By_{n-1}$ for $n = 1,2,3\cdots$ and $Ay_n \to u$ as $n \to \infty$ for some $u \in X$. Then $By_n \to u$ as $n \to \infty$.
Now we claim that B is continuous on X.
Let $\{z_n\}$ be a sequence in X such that $z_n \to z$ as $n \to \infty$, $z \in X$. As A is continuous, we have $Az_n \to Az$ as $n \to \infty$, combining this with inequality (ii) of the theorem, we obtain $S(Bz_n,Bz_n,Bz) \leq \chi(S(Az_n,Az_n,Az) \to 0$ as $n \to \infty$ from which it follows that $Bz_n \to Bz$ as $n \to \infty$, proving B is continuous.
Moreover, we have $BAy_n \to Bu$, $ABy_n \to Au$ as $n \to \infty$, since $Ay_n \to t$, $By_n \to t$ as $n \to \infty$ and by the compatibility of A and B, we have

$$\lim_{n\to\infty} S(ABy_n,ABy_n,BAy_n) = 0$$

giving $S(At,At,Bt) = 0$. Hence $At = Bt$.
In order to prove $ABt = BAt$, take $x_n = u$ for $n = 1,2,3\cdots$, so that $Ax_n \to Au$ and $Bx_n \to u$ as $n \to \infty$. Since $Au = Bu$, A and B are compatible together with continuity of A and B, we have

$$\lim_{n\to\infty} S(ABx_n,ABx_n,BAx_n) = 0$$

which implies that $S(ABu,ABu,BAu)=0$ and hence $ABu=BAu$.
Further, we have

$$AAu=ABu=BAu=BBu. \tag{3.1}$$

If $Bu\neq BBu$, then $S(Bu,Bu,BBu)>0$.
Hence

$$\chi(S(Bu,Bu,BBu))<S(Bu,Bu,BBu). \tag{3.2}$$

But from (ii) of Theorem 3.1 and (3.1), we get

$$S(Bu,Bu,BBu)\leq\chi(S(Au,Au,ABu))=\chi(S(Bu,Bu,BBu)),$$

contradicting (3.2).
Therefore $Bu=BBu$. Using this in (3.1) we get $BBu=Bu=ABu$, showing that Bu is a common fixed point of A and B.
Now, it remains to show the uniqueness of the fixed point.
If $\alpha,\beta\in X$ with $\alpha\neq\beta$ such that $\alpha=A\alpha=B\alpha$ and $\beta=A\beta=B\beta$.
Since $\alpha\neq\beta$ we have

$$S(\alpha,\alpha,\beta)\neq 0,$$

thus

$$\chi(S(\alpha,\alpha,\beta))<S(\alpha,\alpha,\beta) \tag{3.3}$$

But from condition (ii) of Theorem 3.1, we have

$$S(\alpha,\alpha,\beta)=S(B\alpha,B\alpha,B\beta)\leq\chi(S(A\alpha,A\alpha,A\beta))=\chi(S(\alpha,\alpha,\beta)),$$

which contradicts (3.3) and hence $\alpha=\beta$.
Completing proof of the Theorem 3.1. □

Corollary 3.2. *Let A be a continuous selfmap of a S-metric space (X,S), then A has fixed point in X if and only if there is a contractive modulus χ and a selfmap B of X such that*

(i) $AB=BA$,

(ii) $S(Bx,Bx,By)\leq\chi(S(Ax,Ax,Ay))$ *for all* $x,y\in X$, *and*

(iii) *there is a point $y_0\in X$ and an associated sequence $\{y_n\}$ of y_0 relative to the selfmaps A and B such that the sequence $\{Ay_n\}$ converges to some point u of X. Further, Bu is unique common fixed point of A and B.*

Proof. Commuting pair of selfmaps are always compatible and hence the proof of the corollary follows from Theorem 3.1. □

Corollary 3.3. *Let A and B are selfmaps of a S-metric space (X,S). Suppose A is continuous and if there is a contractive modulus χ and a positive integer k such that*

(i) $AB=BA$,

(ii) $S(B^mx,B^mx,B^my)\leq\chi(S(Ax,Ax,Ay))$ *for all* $x,y\in X$, *and*

(iii) *there is a point $y_0\in X$ and an associated sequence $\{y_n\}$ of y_0 relative to the selfmaps A and B^m such that the sequence $\{Ay_n\}$ converges to some point u of X. Further, Bu is unique common fixed point of A and B.*

Proof. From condition (i) of Corollary 3.3, we get $AB^m = B^m A$. Thus A and B^m are commuting and hence satisfying the hypothesis of Theorem 3.1 and therefore A, B^m have a unique common fixed point say c, then $B^m c = c = Ac$. Now $B^m Bc = B^{m+1}c = BB^m c = Bc$ and $ABc = BAc = Bc$.
This shows that Bc is a common fixed point of A and B^m.
The uniqueness of c implies $Bc = c$, since $Ac = c$, showing that c is a common fixed point of A and B.
We now prove uniqueness of common fixed point of A and B.
Let $\alpha,\beta \in X$ such that $\alpha = A\alpha = B\alpha$ and $\beta = A\beta = B\beta$, so that $B^m\alpha = \alpha$ and $B^m\beta = \beta$, showing α,β are common fixed points of A and B^m.
From which it follows $\alpha = \beta$, since the fixed point of A and B^m is unique. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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