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Research Article

Common Fixed Point Theorems Using Subcompatible and Subsequentially Continuous Mappings on Partial Metric Spaces

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Abstract. The purpose of this paper is to generate two *common fixed point* (CFP) theorems using subcompatible and reciprocally continuous and subsequentially continuous and compatible on partial metric spaces. Further, we extend our results with suitable examples.

Keywords. Fixed point, Partial metric space, Subcompatible, Subsequentially continuous, Reciprocally continuous and compatible

Mathematics Subject Classification (2020). 54H25, 47H10

1. Introduction

The notion of *partial metric space* (shortly PMS) was introduced by Matthews [8] in 1994 and is an expansion of metric space. In PMS, the condition $d(\alpha,\alpha)$ need not to be zero and the condition $d(\alpha,\alpha)=0$ is replaced by the condition $d(\alpha,\alpha)\le d(\alpha,\beta)$. PMS play vital role in framing models in the theory of computation and to study the data flow networks. Many researchers proved different results in PMS e.g. [2], [3], [7] and [8].

In the recent past, many investigations have been conducted on the possibility of generalizing current metric fixed point theorems to partial metric spaces. Jungck [4] developed the concept of compatible mappings that are weaker than the weakly commuting mappings in 1986. Jungck and Rhoades [6] also expanded the compatibility criteria to include weakly compatible mappings. Jungck and Rhoades [5] have developed the concept of occasionally weakly compatible mappings that are weaker than weakly compatible mappings.

The notion of subcompatible and subsequentially continuous mappings was introduced by Bouhadjera and Godeti-Thobie[1] in 2009 and which is weaker than compatible and reciprocally continuous mappings. In this research article, we deal with two common fixed point theorems for four self-maps using subcompatible, subsequentially continuous, compatible and reciprocally continuous mappings.

2. Preliminaries

Definition 2.1. Suppose χ is a nonempty set and let $p : \chi \times \chi \to [0,\infty)$ satisfy

(PM1) $s = t$ if and only if $p(s,s) = p(t,t) = p(s,t)$,

(PM2) $p(s,s) \le p(s,t)$,

(PM3) $p(s,t) = p(t,s)$,

(PM4) $p(s,t) \le p(s,r) + p(r,t) - p(r,r)$,

for all s,t and $r \in \chi$. Then (χ,p) is said to be a partial metric space, and p is considered a metric on χ.

Definition 2.2. Suppose (χ,p) is a PMS, then a sequence $\{u_n\}$

(i) converges to $u \in \chi$ if and only if $p(u,u) = p(u,u_n)$ as $n \to \infty$;

(ii) is known to be Cauchy sequence if and only if $p(u_m,u_n)$ as $m,n \to \infty$ exists;

(iii) is known to be (χ,p) complete if every Cauchy sequence $\{u_n\}$ in it converges.

Remark 2.1. In partial metric space (χ,p), the following are true.

(a) If $p(s,t) = 0$ then $s = t$.

(b) If $s \ne t$ then $p(s,t) > 0$.

Definition 2.3 ([4])**.** The mappings E and F of a PMS (χ,p) defined as compatible if $\{EFu_n\} = \{FEu_n\}$ as $n \to \infty$ whenever sequence $\{u_n\}$ in χ such that $\{Eu_n\}$ and $\{Fu_n\}$ converges to δ as $n \to \infty$ for some $\delta \in \chi$.

Definition 2.4 ([9])**.** The mappings E and F of a PMS (χ,p) defined reciprocally continuous if $\{EFu_n\} = E\delta$ and $\{FEu_n\} = F\delta$ as $n \to \infty$ whenever $\{u_n\}$ in χ such that $\{Eu_n\}$, $\{Fu_n\}$ converges to δ as $n \to \infty$ for some $\delta \in \chi$.

Definition 2.5 ([1])**.** The mappings E and F of a PMS (χ,p) are defined as subcompatible if and only if there exists a sequence $\{u_n\}$ in χ such that $\{Eu_n\} = \{Fu_n\} = \delta$ as $n \to \infty$, $\delta \in \chi$ and which satisfy $\{EFu_n\} = \{FEu_n\}$ as $n \to \infty$.

[1] H. Bouhadjera and C. Godet-Thobie, Common fixed point theorems for pairs of subcompatible maps, (2009), https://arxiv.org/abs/0906.3159 [math.FA].

It is clear that two occasionally weakly compatible mappings are subcompatible but two subcompatible mappings are not occasionally weakly compatible mappings. For this, we give an example.

Example 2.1. Let $\chi = [0,\infty)$ be endue with PMS $p(x,y) = \max\{x,y\}$, $\forall\ x,y \in \chi$. Define E and F as follows

$$Ex = \begin{cases} x^2, & \text{if } 0 \le x < 1,\\ 3x-1, & \text{if } 1 \le x < \infty, \end{cases} \quad \text{and} \quad Fx = \begin{cases} 5x-4, & \text{if } 0 \le x < 1,\\ x+1, & \text{if } 1 \le x < \infty. \end{cases}$$

Consider a sequence $\{u_n\}$ by $u_n = 1 - \frac{1}{2n}$, for $n \ge 1$,

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n}\right)^2 = 1$$

and

$$\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} 5\left(1 - \frac{1}{2n}\right) - 4 = \lim_{n\to\infty} \left(1 - \frac{5}{2n}\right) = 1.$$

Now

$$\lim_{n\to\infty} EFu_n = \lim_{n\to\infty} E\left(1 - \frac{5}{2n}\right) = \lim_{n\to\infty} \left(1 - \frac{5}{2n}\right)^2 = 1$$

and

$$\lim_{n\to\infty} FEu_n = \lim_{n\to\infty} F\left(1 - \frac{1}{2n}\right)^2 = \lim_{n\to\infty} 5\left(1 - \frac{1}{2n}\right)^2 - 4 = 1.$$

Thus,

$$\lim_{n\to\infty} FEu_n = \lim_{n\to\infty} EFu_n$$

implies that E and F are subcompatible but E and F are not occasionally weakly compatible mappings as,

$$E(1) = 2 = F(1) \ \text{ and } \ EF(1) = E(2) = 5 \ne FE(1) = F(2) = 3.$$

Definition 2.6 ([9])**.** Two mappings E and F of a PMS (χ,p) are said to be reciprocally continuous if $\{EFu_n\} = E\delta$ and $\{FEu_n\} = F\delta$ as $n \to \infty$ whenever $\{u_n\}$ in χ such that $\{Eu_n\},\{Fu_n\}$ converges to δ as $n \to \infty$ for some $\delta \in \chi$.

Definition 2.7 ([1])**.** Two mappings E and F of a PMS (χ,p) are said to be subsequentially continuous if and only if there exists a sequence $\{u_n\}$ in χ such that $\{Eu_n\} = \{Fu_n\} = \delta$ as $n \to \infty$ for some $\delta \in \chi$ and satisfy $\{EFu_n\} = E\delta$ and $\{FEu_n\} = F\delta$ as $n \to \infty$.

It can be observed that if E and F are both continuous or reciprocally continuous then they are subsequentially continuous; however, the converse need not be true. For this, we give an example to show that subsequentially continuous mappings are not reciprocally continuous mappings.

Example 2.2. Let $\chi = [0,\infty)$ be endue with PMS $p(x,y) = \max\ x,y\}$, $\forall\ x,y \in \chi$ and E, F defined as

$$Ex = \begin{cases} x+3, & \text{if } x \in [0,6), \\ 2x, & \text{if } x \in [6,12], \end{cases} \quad \text{and} \quad Px = \begin{cases} -x+3, & \text{if } x \in [0,6], \\ 3x-3, & \text{if } x \in (6,12]. \end{cases}$$

Let the sequence $\{u_n\}$ be defined as $u_n = \frac{1}{n}$, $n = 1,2,3,\dots$. Then

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty}\left(3+\frac{1}{n}\right) = 3 = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty}\left(3-\frac{1}{n}\right).$$

Now

$$\lim_{n\to\infty} EPu_n = \lim_{n\to\infty} E\left(3-\frac{1}{n}\right) = \lim_{n\to\infty}\left(6-\frac{3}{n}\right) = 6 = E(3)$$

and

$$\lim_{n\to\infty} PEu_n = \lim_{n\to\infty} P\left(3+\frac{1}{n}\right) = \lim_{n\to\infty}\left(6+\frac{3}{n}\right) = 6 = P(3).$$

Thus, E and P are subsequentially continuous.
Consider other sequence $\{u_n\}$ be defined as $u_n = 3+\frac{1}{n}$, $n = 1,2,3,\dots$. Then

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty}\left(6+\frac{2}{n}\right) = 6 = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty}\left(9+\frac{3}{n}-3\right).$$

Also

$$\lim_{n\to\infty} PEu_n = \lim_{n\to\infty} P\left(6+\frac{2}{n}\right) = \lim_{n\to\infty}\left(15+\frac{6}{n}\right) = 15 = P(6)$$

and

$$\lim_{n\to\infty} EPu_n = \lim_{n\to\infty} E\left(6+\frac{3}{n}\right) = \lim_{n\to\infty}\left(12+\frac{6}{n}\right) = 12 = E(6).$$

Thus, E and F are not reciprocally continuous mappings.

Now we will move on to our main results, which generalize and extend the existing theorem proved on compatible mappings in [7].

3. Main Results

Theorem 3.1. *Suppose E, F, P and Q are self-mappings of a complete PMS (χ,p) into itself with $E(\chi) \subseteq Q(\chi)$ and $F(\chi) \subseteq P(\chi)$. If there exists $\hbar \in [0,1)$ such that*

$$p(Ex,Fy) \le \hbar\varphi(x,y), \tag{3.1}$$

for any $x,y \in \chi$, where,

$$\varphi(x,y) = \max\left\{p(Ex,Px), p(Fy,Qy), p(Px,Qy), \frac{1}{2}[p(Ex,Qy)+p(Fy,Px)]\right\}. \tag{3.2}$$

The couple $\{E,P\}$ and $\{F,Q\}$ are subcompatible and reciprocally continuous. Then the mappings E, F, P and Q have a unique common fixed point.

Proof. Let u_0 be any point in χ, using the condition $E(\chi) \subset Q(\chi)$ gives such that $Eu_0 = Qu_1$ for some $u_1 \in \chi$ and also from the condition $F(\chi) \subset P(\chi)$, for $u_1 \in \chi$ and $Fu_1 \in P(\chi)$, there exist $u_2 \in \chi$ such that $Fu_1 = Pu_2$. In general, $u_{2n+1} \in \chi$ is chosen such that $Eu_{2n} = Qu_{2n+1}$ and $u_{2n+2} \in \chi$ such that $Fu_{2n+1} = Pu_{2n+2}$, we obtain a sequence $\{u_n\}$ in χ such that

$$u_{2n} = Eu_{2n} = Qu_{2n+1},\ u_{2n+1} = Fu_{2n+1} = Pu_{2n+2}, \quad \text{where } n \geq 0. \tag{3.3}$$

Next we show that $\{u_n\}$ is a cauchy sequence.
By (3.1) and using (3.3), we observe

$$\begin{aligned} p(u_{2n+1}, u_{2n+2}) &= p(Qu_{2n+1}, Pu_{2n+2}) \\ &= p(Eu_{2n}, Fu_{2n+1}) \\ &\leq \hbar\varphi(u_{2n}, u_{2n+1}), \end{aligned} \tag{3.4}$$

where

$$\varphi(u_{2n}, u_{2n+1}) = \max\Big\{p(Eu_{2n}, Pu_{2n}), p(Fu_{2n+1}, Qu_{2n+1}), p(Pu_{2n}, Qu_{2n+1}), \frac{1}{2}[p(Eu_{2n}, Qu_{2n+1}) + p(Fu_{2n+1}, Pu_{2n})]\Big\}$$

Using (3.3), we observe

$$\varphi(u_{2n}, u_{2n+1}) = \max\Big\{p(Eu_{2n}, Fu_{2n-1}), p(Fu_{2n+1}, Eu_{2n}), p(Fu_{2n-1}, Eu_{2n}), \frac{1}{2}[p(Eu_{2n}, Eu_{2n}) + p(Fu_{2n+1}, Fu_{2n-1})]\Big\}. \tag{3.5}$$

From the definition (PM4), we observe

$$p(Fu_{2n-1}, Fu_{2n+1}) + p(Eu_{2n}, Eu_{2n}) \leq p(Fu_{2n-1}, Eu_{2n}) + p(Fu_{2n+1}, Eu_{2n}). \tag{3.6}$$

From (3.5) and (3.6), we observe

$$\varphi(u_{2n}, u_{2n+1}) = \max\{p(Eu_{2n}, Fu_{2n-1}), p(Fu_{2n+1}, Eu_{2n})\}. \tag{3.7}$$

But if $\varphi(u_{2n}, u_{2n+1}) = p(Fu_{2n+1}, Eu_{2n})$ then by (3.4), we observe

$$p(Fu_{2n+1}, Eu_{2n}) \leq \hbar p(Fu_{2n+1}, Eu_{2n}), \quad 0 \leq \hbar < 1, \tag{3.8}$$

this gives that $p(Eu_{2n+1}, Eu_{2n}) = 0$. Thus, $\varphi(u_{2n}, u_{2n+1}) = p(Fu_{2n-1}, Eu_{2n})$ and from (3.4), we get

$$p(Fu_{2n+1}, Eu_{2n}) \leq \hbar p(Fu_{2n-1}, Eu_{2n}), \tag{3.9}$$

which gives

$$p(u_{2n+2}, u_{2n+1}) \leq \hbar p(u_{2n+1}, u_{2n}), \quad \text{for all } \hbar \geq 0.$$

After simple calculation, noting $0 \leq \hbar < 1$, we conclude that $\{u_n\}$ as a Cauchy sequence. However (χ, p) being complete, this implies $\{u_n\}$ converges to some point $\delta \in \chi$. Consequently, the subsequences

$$\{Eu_{2n}\},\ \{Qu_{2n+1}\},\ \{Fu_{2n+1}\},\ \{Pu_{2n+2}\} \text{ also converges to } \delta \in \chi. \tag{3.10}$$

Since the pair (E,P) is subcompatible then there exists a sequence $\{u_n\}$ such that $\{Eu_n\}$, $\{Pu_n\}$ converges to α as $n \to \infty$ for some $\alpha \in \chi$ and satisfy

$$\{EPu_n\} = \{PEu_n\} \ \text{ as } \ n \to \infty \tag{3.11}$$

and also the pair (E,P) is reciprocally continuous then $\{EPu_n\} = E\alpha$ and

$$\{PEu_n\} = P\alpha \tag{3.12}$$

as $n \to \infty$.
Using (3.11) and (3.12), we have

$$E\alpha = P\alpha. \tag{3.13}$$

Similarly the couple (F,Q) is subcompatible and reciprocally continuous then there exists a sequence $\{v_n\}$ such that $\{Fv_n\} = \{Qv_n\} = \beta$ as $n \to \infty$ for some $\beta \in \chi$ and which satisfy

$$\{FQv_n\} = \{QFv_n\} \tag{3.14}$$

and

$$\{FQv_n\} = F\beta \ \text{ and } \ \{QFv_n\} = Q\beta. \tag{3.15}$$

Therefore, using (3.14) and (3.15), we have

$$F\beta = Q\beta. \tag{3.16}$$

From (3.13) and (3.16), we can observe that α is a coincidence point of the couple (E,P) and β is a coincidence point of the couple (F,Q).
Now we have to show that $\alpha = \beta$.
If possible suppose that $\alpha \neq \beta$. Then put $x = u_n, y = v_n$ in (3.1), we get

$$p(Eu_n, Fv_n) \leq \hbar\varphi(u_n, v_n) \tag{3.17}$$

where

$$\varphi(u_n, v_n) = \max\left\{p(Eu_n, Pu_n), p(Fv_n, Qv_n), p(Pu_n, Qv_n), \frac{1}{2}[p(Eu_n, Qv_n) + p(Fv_n, Pu_n)]\right\}.$$

Letting $n \to \infty$ using $\{Eu_n\} = \{Pu_n\} = \alpha$ and $\{Fv_n\} = \{Qv_n\} = \beta$, we observe

$$\begin{aligned}\lim_{n\to\infty} \varphi(u_n, v_n) &= \max\left\{p(\alpha,\alpha), p(\beta,\beta), p(\alpha,\beta), \frac{1}{2}\left[p(\alpha,\beta) + p(\beta,\alpha)\right]\right\} \\ &= \max\{p(\alpha,\alpha), p(\beta,\beta), p(\alpha,\beta), p(\alpha,\beta)\} \\ &= p(\alpha,\beta).\end{aligned} \tag{3.18}$$

From (3.17) and (3.18) together on letting $n \to \infty$ gives

$$\lim_{n\to\infty} p(Eu_n, Fv_n) \le \hbar \lim_{n\to\infty} \varphi(u_n, v_n)$$

implies that $p(\alpha,\beta) \le \hbar p(\alpha,\beta)$, since $\hbar \in [0,1)$.
This gives that $\alpha = \beta$. Therefore from (3.16), we have

$$F\alpha = Q\alpha. \tag{3.19}$$

Now we prove that $E\alpha = \alpha$. If possible suppose that $E\alpha \ne \alpha$, then from the condition (3.1), on letting $x = \alpha$ and $y = v_n$, we have

$$p(E\alpha, Fv_n) \le \hbar\varphi(\alpha, v_n) \tag{3.20}$$

where

$$\varphi(\alpha, v_n) = \max\left\{p(E\alpha,P\alpha), p(Fv_n,Qv_n), p(P\alpha,Qv_n), \frac{1}{2}[p(E\alpha,Qv_n)+p(Fv_n,P\alpha)]\right\}.$$

Letting $n \to \infty$ using (3.13) and $\{Fv_n\} = \{Qv_n\} = \beta$, we get

$$\begin{aligned}\lim_{n\to\infty} \varphi(\alpha, v_n) &= \max\left\{p(E\alpha,E\alpha), p(\beta,\beta), p(E\alpha,\beta), \frac{1}{2}\left[p(E\alpha,\beta)+p(Fv_n,E\alpha)\right]\right\}\\ &= \max\{p(E\alpha,E\alpha), p(\beta,\beta), p(E\alpha,\beta), p(E\alpha,\beta)\}\\ &= p(E\alpha,\beta)\\ &= p(E\alpha,\alpha).\end{aligned} \tag{3.21}$$

From (3.19) and (3.20) together on letting $n \to \infty$ gives

$$\lim_{n\to\infty} \varphi(E\alpha,\beta) \le \hbar \lim_{n\to\infty} \varphi(\alpha, v_n)$$

implies that

$$p(E\alpha,\alpha) \le \hbar p(E\alpha,\alpha),$$

since $\hbar \in [0,1)$. This gives that $E\alpha = \alpha$.
Therefore

$$E\alpha = P\alpha = \alpha. \tag{3.22}$$

Now we prove that $F\alpha = \alpha$. If possible let $F\alpha \ne \alpha$ then using condition (3.1) with $x = u_n$ and $y = \alpha$, we have

$$p(Eu_n, F\alpha) \le \hbar p(u_n, \alpha) \tag{3.23}$$

where

$$\varphi(u_n, \alpha) = \max\left\{p(Eu_n,Pu_n), p(F\alpha,Q\alpha), p(Pu_n,Q\alpha), \frac{1}{2}[p(Eu_n,Q\alpha)+p(F\alpha,Pu_n)]\right\}.$$

Letting $n \to \infty$ and using (3.19) and $\{Eu_n\} = \{Pu_n\} = \alpha$, we get

$$\begin{aligned}\lim_{n\to\infty} \varphi(u_n,\alpha) &= \max\left\{p(\alpha,\alpha), p(F\alpha,F\alpha), p(\alpha,F\alpha), \frac{1}{2}[p(\alpha,F\alpha)+p(F\alpha,\alpha)]\right\} \\ &= \max\{p(\alpha,\alpha), p(F\alpha,F\alpha), p(\alpha,F\alpha), p(F\alpha,\alpha)\} \\ &= p(\alpha,F\alpha). \end{aligned} \tag{3.24}$$

From (3.23) and (3.24) together on letting $n \to \infty$ gives

$$\lim_{n\to\infty} p(Eu_n,F\alpha) \le \hbar \lim_{n\to\infty} p(u_n,\alpha)$$

implies that

$$p(\alpha,F\alpha) \le \hbar p(\alpha,F\alpha)$$

which is not possible since $\hbar \in [0,1[$. This gives $p(\alpha,F\alpha) = 0$ and implies that $F\alpha = \alpha$.
Therefore, $F\alpha = Q\alpha = \alpha$.
Hence, $E\alpha = P\alpha = F\alpha = Q\alpha = \alpha$. This shows that α is a CFP of E, F, P and Q.
To prove α is unique CFP, if possible suppose that there is another CFP β of E, F, P and Q. Then by using (3.1), on letting $x = \alpha, y = \beta$, we get

$$p(\alpha,\beta) = p(E\alpha,F\beta) \le \hbar\varphi(\alpha,\beta),$$

where

$$\begin{aligned}\varphi(\alpha,\beta) &= \max\left\{p(E\alpha,P\alpha), p(F\beta,Q\beta), p(P\alpha,Q\beta), \frac{1}{2}\left[p(E\alpha,Q\beta)+p(F\beta,P\alpha)\right]\right\} \\ &= \max\left\{p(\alpha,\alpha), p(\beta,\beta), p(\alpha,\beta), \frac{1}{2}[p(\alpha,\beta)+p(\beta,\alpha)]\right\} \\ &= p(\alpha,\beta).\end{aligned}$$

Thus $p(\alpha,\beta) \le \hbar p(\alpha,\beta)$, $0 \le \hbar < 1$ and provide that $\alpha = \beta$. So, α is becoming unique CFP of E, F, P and Q. □

Example 3.1. Let $\chi = [0,4]$ and $p : \chi \times \chi \to [0,\infty)$ be define by $p(x,y) = \max\{x,y\}$, $\forall\ x,y \in \chi$. Then (χ,p) is a complete PMS. $E,F,P,Q : \chi \to \chi$ are defined by

$$Ex = Fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,2), \\ 3x-4, & \text{if } x \in [2,4], \end{cases} \quad \text{and} \quad Px = Qx = \begin{cases} x-1, & \text{if } x \in [0,2), \\ x^2-2, & \text{if } x \in [2,4]. \end{cases}$$

We have $E(\chi) \subset Q(\chi)$ and $F(\chi) \subset P(\chi)$. Consider the sequence $\{u_n\}$ by $u_n = 2 + \frac{1}{n}$, for $n \ge 1$. Then

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty}\left(2+\frac{3}{n}\right) = 2 = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty}\left(2+\frac{1}{n}\right)^2 - 2.$$

Also,

$$\lim_{n\to\infty} EPu_n = \lim_{n\to\infty} E\left\{\left(2+\frac{1}{n}\right)^2 - 2\right\} = \lim_{n\to\infty}\left\{3\left(2+\frac{1}{n}\right)^2 - 6 - 4\right\} = 2 = E(2)$$

and

$$\lim_{n\to\infty} PEu_n = \lim_{n\to\infty} P\left(2+\frac{3}{n}\right) = \lim_{n\to\infty}\left(\left[2+\frac{3}{n}\right]^2-2\right) = 2 = P(2).$$

Therefore the pair (E,P) is subcompatible and reciprocally continuous.
Consider the other sequence $\{u_n\}$ by $u_n = 2-\frac{1}{n}$, for $n = 1,2,3,\ldots$. Then

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty}\left(1-\frac{1}{2n}\right) = 1 = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right).$$

Now

$$\lim_{n\to\infty} PEu_n = \lim_{n\to\infty} P\left(1-\frac{1}{2n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{2n}-1\right) = 0 = P(1)$$

and

$$\lim_{n\to\infty} EPu_n = \lim_{n\to\infty} E\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2n}\right) = \frac{1}{2} = E(1).$$

Thus $\lim_{n\to\infty} EPu_n = E(1)$ and $\lim_{n\to\infty} PEu_n = P(1)$ but $\lim_{n\to\infty} PEu_n \neq \lim_{n\to\infty} EPu_n$, which shows that the pair (E,P) is reciprocally continuous but not subcompatible. The contractive condition (3.1) holds for the value of $\hbar \in [0,1)$. we observe that 2 is the unique CFP of maps E, F, P and Q.

Now we generate another theorem on PMS using subsequentially continuous and compatible mappings.

Theorem 3.2. *Suppose (χ,p) is a complete PMS and E, F, P and Q are self-mappings on χ, with $E(\chi)\subseteq Q(\chi)$ and $F(\chi)\subseteq P(\chi)$. If there exists a $\hbar\in[0,1)$ such that*

$$p(Ex,Fy) \le \hbar\varphi(x,y), \tag{3.25}$$

for any $x,y\in\chi$, where

$$\varphi(x,y) = \max\left\{p(Ex,Px), p(Fy,Qy), p(Px,Qy), \frac{1}{2}[p(Ex,Qy)+p(Fy,Px)]\right\}. \tag{3.26}$$

The couple $\{E,P\}$ and $\{F,Q\}$ are subsequentially continuous and compatible. Then the mappings E, F, P and Q have one and only one common fixed point.

Proof. Since the couple (E,P) is subsequentially continuous and compatible, there exists a sequence $\{u_n\}$ in χ such that

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \alpha, \quad \text{for some } \alpha\in\chi \tag{3.27}$$

and

$$\lim_{n\to\infty} EPu_n = E\alpha \text{ and } \lim_{n\to\infty} PEu_n = P\alpha \text{ and } \lim_{n\to\infty} PEu_n = \lim_{n\to\infty} EPu_n. \tag{3.28}$$

This implies that

$$E\alpha = P\alpha. \tag{3.29}$$

Similarly the pair (F,Q) is subsequentially continuous and compatible, there exists a sequence $\{v_n\}$ in χ such that

$$\lim_{n\to\infty} Fv_n = \lim_{n\to\infty} Qv_n = \beta, \quad \text{for some } \beta \in \chi, \tag{3.30}$$

$$\lim_{n\to\infty} FTv_n = F\beta \text{ and } \lim_{n\to\infty} QBv_n = Q\beta \text{ and } \lim_{n\to\infty} FQv_n = \lim_{n\to\infty} QFv_n. \tag{3.31}$$

This gives that

$$F\beta = Q\beta. \tag{3.32}$$

Thus from (3.29) and (3.32), $E\alpha = P\alpha$ and $F\beta = Q\beta$. This shows that the pair (E,P) has coincidence point α whereas the pair (F,Q) has coincidence point β.
The rest of the proof of this theorem can be done easily as Theorem 3.1. □

We justify the above theorem with the following example:

Example 3.2. Let $\chi = [0,5]$ and $p : \chi \times \chi \to [0,\infty)$ is define by $p(x,y) = \max\{x,y\}$ for all $x,y \in \chi$. Then (χ,p) is a complete PMS. $E,F,P,Q : \chi \to \chi$ are defined by

$$Ex = Fx = \begin{cases} \frac{x}{5}, & \text{if } x \in [0,1], \\ \frac{x+4}{5}, & \text{if } x \in (1,5], \end{cases} \quad \text{and} \quad Px = Qx = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,1], \\ \frac{x+3}{4}, & \text{if } x \in (01,5]. \end{cases}$$

We have $E(\chi) \subset Q(\chi)$ and $F(\chi) \subset P(\chi)$.
Consider a sequence $\{u_n\}$ by $u_n = \frac{1}{n}$, where $n = 1,2,3,\ldots$, then

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} \left(\frac{1}{5n}\right) = 0$$

and

$$\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} \left(\frac{1}{4n}\right) = 0.$$

Next,

$$\lim_{n\to\infty} EFu_n = \lim_{n\to\infty} E\left(\frac{1}{4n}\right) = \lim_{n\to\infty} \left(\frac{1}{20n}\right) = 0 = E(0)$$

and

$$\lim_{n\to\infty} FEu_n = \lim_{n\to\infty} F\left(\frac{1}{5n}\right) = \lim_{n\to\infty} \left(\frac{1}{20n}\right) = 0 = F(0).$$

Therefore, E and F are subsequentially continuous and are compatible.

Consider another sequence $\{u_n\}$ by $u_n = 1 + \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Then

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty}\lim_{n\to\infty}\left(1+\frac{1}{5n}\right) = 1$$

and

$$\lim_{n\to\infty} Fu_n = \lim_{n\to\infty}\left(1+\frac{1}{4n}\right) = 1.$$

Also

$$\lim_{n\to\infty} EFu_n = \lim_{n\to\infty} E\left(1+\frac{1}{4n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{20n}\right) = 1 \neq E(1)$$

and

$$\lim_{n\to\infty} FEu_n = \lim_{n\to\infty} F\left(1+\frac{1}{5n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{20n}\right) = 1 \neq F(1).$$

Thus, E and F are not reciprocally continuous.

Also, the contractive condition (3.1) holds for the value of $\hbar \in [0,1)$. We observe that 0 is the unique common fixed point of maps E, F, P and Q.

4. Conclusion

In this research article, we generate two results. In the first result, two pairs are assumed to be subcompatible and reciprocally continuous and in the second result, two pairs are subsequentially continuous and compatible mappings. Further, these results are justified with appropriate examples. Thus, we assert our results generalized and extend the results proved in [7].

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. A. Al-Thagafi and N. Shahzad, Generalized I-nonexpansive selfmaps and invariant approximations, *Acta Mathematica Sinica* (English series) **24** (2008), 867 – 876, DOI: 10.1007/s10114-007-5598-x.

[2] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology and its Applications* **157**(18) (2010), 2778 – 2785, DOI: 10.1016/j.topol.2010.08.017.

[3] L. Ćirić, M. Abbas, R. Saadati and N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, *Applied Mathematics and Computation* **217**(12) (2011), 5784 – 5789, DOI: 10.1016/j.amc.2010.12.060.

[4] G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences* **9** (1986), Article ID 531318, 9 pages, DOI: 10.1155/S0161171286000935.

[5] G. Jungck and B. E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory* **7**(2) 2006), 287 – 296, URL: https://www.math.ubbcluj.ro/~nodeacj/download.php?f=062rhoades.pdf.

[6] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, *Indian Journal of Pure and applied Mathematics* **29**(3) (1998), 227 – 238.

[7] E. Karapınar and U. Yüksel, Some common fixed point theorems in partial metric spaces, *Journal of Applied Mathematics* **2011** (2011), Article ID 263621, 16 pages, DOI: 10.1155/2011/263621.

[8] S. G. Matthews, Partial metric topology, *Annals of the New York Academy of Sciences* **728**(1) (1994), 183 – 197, DOI: 10.1111/j.1749-6632.1994.tb44144.x.

[9] R. P. Pant, A common fixed point theorem under a new condition, *Indian Journal of Pure and applied Mathematics* **30**(2) (1999), 147 – 152.

[10] S. Ravi and V. Srinivas, Fixed point theorem using occasionally weakly compatible mappings in metric space, *Journal of Computer and Mathematical Sciences* **10**(3) (2019), 461 – 466.

[11] V. Srinivas and R. Sriramula, Generation of a common fixed point theorem using A-compatible and B-compatible mappings of type (E), *Global Journal of Pure and Applied Mathematics* **13**(6) (2017), 1735 – 1744.