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Research Article

Unsteady Stokes Flow Past a Shear Free Sphere

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Abstract. A method of solution for the problem of an arbitrary unsteady Stokes flow in the presence of a shear free sphere is discussed. The corresponding Faxén [2] relations for a shear-free sphere are derived. Some previously known results are derived as limiting cases and are detailed in an example.

Keywords. Unsteady Stokes flow, Shear free sphere, Faxén relations

Mathematics Subject Classification (2020). 76D07, 35Q30

1. Introduction

Unsteady and steady Stokes flows in the presence of a spherical boundary have been studied extensively with different boundary conditions owing to their various scientific and engineering applications. Faxén's [2] laws for rigid boundary conditions are well known. Harper [3] derived a sphere theorem for an axisymmetric steady Stokes flow for a sphere with shear free boundary conditions. Rallison [4] gave Faxén [2] relations for a shear free particle in an arbitrary steady Stokes flow. The problem of an unsteady Stokes flow past a rigid spherical particle was discussed [1, 5] using a complete general solution, expressed in terms of two scalar functions A and B. In this paper, we discuss the problem of an arbitrary unsteady Stokes flow in the presence of a shear free sphere. Faxén's [2] laws are given and compared with previously known results. The results are illustrated by an example.

The equations of motion for the unsteady Stokes flow in a viscous, incompressible fluid in the absence of any external forces are given by

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} = -\nabla p + \mu\nabla^2\boldsymbol{V}, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{V} = 0, \tag{1.2}$$

where V is the fluid velocity, ρ is the density, μ is the coefficient of dynamic viscosity of the fluid. We rewrite equation (1.1) as

$$\mu\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\boldsymbol{V} = \nabla p, \tag{1.3}$$

where $\nu = \frac{\mu}{\rho}$ is the coefficient of kinematic viscosity. A general solution of unsteady Stokes equations (1.1) and (1.2) is given in [5] as follows:

$$\boldsymbol{V} = \mathrm{Curl}\,\mathrm{Curl}(\boldsymbol{r}A) + \mathrm{Curl}(\boldsymbol{r}B), \tag{1.4}$$

$$p = p_0 + \mu\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)A\right], \tag{1.5}$$

where p_0 is a constant and A, B are scalar functions that satisfy equations

$$\nabla^2\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)A = 0, \tag{1.6}$$

$$\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)B = 0. \tag{1.7}$$

A solution of (1.6) can be decomposed as follows:

$$A = A_1 + A_2, \tag{1.8}$$

where

$$\nabla^2 A_1 = 0, \tag{1.9}$$

$$\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)A_2 = 0. \tag{1.10}$$

The general solution of the equations (1.6) and (1.7) can therefore be written as $A = A_1 + A_2$, where

$$A_1 = \sum_{n=1}^{\infty}\left(\alpha_n r^n \frac{\beta_n}{r^{n+1}}\right)S_n(\theta,\varphi)e^{\lambda^2 \nu t}, \tag{1.11}$$

$$A_2 = \sum_{n=1}^{\infty}(\alpha'_n f_n(\lambda r) + \beta'_n g_n(\lambda r))S_n(\theta,\varphi)e^{\lambda^2 \nu t}, \tag{1.12}$$

$$B = \sum_{n=1}^{\infty}(\epsilon_n f_n(\lambda r) + \epsilon'_n g_n(\lambda r))T_n(\theta,\varphi)e^{\lambda^2 \nu t}, \tag{1.13}$$

$$S_n(\theta,\varphi) = \sum_{m=0}^{n} P_n^m(\cos\theta)(A_{nm}\cos m\varphi + B_{nm}\sin m\varphi), \tag{1.14}$$

$$T_n(\theta,\varphi) = \sum_{m=0}^{n} P_n^m(\cos\theta)(C_{nm}\cos m\varphi + D_{nm}\sin m\varphi), \tag{1.15}$$

where α_n, β_n, α'_n, β'_n, ϵ_n, ϵ'_n, A_{nm}, B_{nm}, C_{nm}, D_{nm} are constants and $Re(\lambda^2) \leq 0$. The functions $f_n(R) = \sqrt{\frac{\pi}{2R}}I_{n+\frac{1}{2}}(R)$, $g_n(R) = \sqrt{\frac{\pi}{2R}}K_{n+\frac{1}{2}}(R)$ are the modified Bessel functions of fractional order.

2. Shear Free Sphere in Unsteady Stokes Flow

In [5] the problem of an arbitrary unsteady Stokes flow in the presence of a rigid sphere was given. Let us now consider the unsteady Stokes flow in the presence of a shear-free sphere of radius a in a viscous, incompressible fluid with boundary conditions on $r = a$ given by

(i) normal velocity is zero, i.e., $q_r = 0$ on $r = a$;

(ii) tangential stress components $T_{r\theta}$ and $T_{r\varphi}$ are zero on $r = a$.

In terms of A and B, the conditions (i) and (ii) are $A = 0$, $\frac{\partial^2 A}{\partial r^2} = 0$ and $\frac{\partial}{\partial r}\left(\frac{B}{r}\right) = 0$ on $r = a$. Further, $\boldsymbol{V} \to \boldsymbol{V_0}$ as $r \to \infty$, where $\boldsymbol{V_0}$ is the undisturbed flow given by

$$\boldsymbol{V_0} = \mathrm{Curl\,Curl}(\boldsymbol{r}A_0) + \mathrm{Curl}(\boldsymbol{r}B_0), \tag{2.1}$$

where

$$A_0 = \sum_{n=1}^{\infty} (\alpha_n r^n + \alpha'_n f_n(\lambda r)) S_n(\theta,\varphi) e^{\lambda^2 \nu t}, \tag{2.2}$$

$$B_0 = \sum_{n=1}^{\infty} \epsilon_n f_n(\lambda r) T_n(\theta,\varphi) e^{\lambda^2 \nu t}, \tag{2.3}$$

α_n, α'_n and ϵ_n being known constants. The disturbance caused due to the presence of the sphere of radius a modifies the flow so that the perturbed flow is represented by $\boldsymbol{V}$ and p as given in equations (1.4) and (1.5), respectively. The scalars A and B are assumed to be of the form given in equations (1.8), (1.11), (1.12) and (1.13). Then using the boundary conditions on $r = a$, we can determine the unknown constants as

$$\beta_n = a^{n+2}\lambda \left\{ \frac{\alpha_n(\lambda a g_n(\lambda a) + 2g_{n+1}(\lambda a)) a^n}{(4n+2-\lambda^2 a^2) g_n(\lambda a) - 2\lambda a g_{n+1}(\lambda a)} + \frac{2\alpha'_n [g_{n+1}(\lambda a) f_n(\lambda a) + f_{n+1}(\lambda a) g_n(\lambda a)]}{(4n+2-\lambda^2 a^2) g_n(\lambda a) - 2\lambda a g_{n+1}(\lambda a)} \right\}, \tag{2.4}$$

$$\beta'_n = -\left\{ \frac{(4n+2)\alpha_n a^n}{(4n+2-\lambda^2 a^2) g_n(\lambda a) - 2\lambda a g_{n+1}(\lambda a)} + \frac{\alpha'_n [(4n+2-\lambda^2 a^2) f_n(\lambda a) + 2\lambda a f_{n+1}(\lambda a)]}{(4n+2-\lambda^2 a^2) g_n(\lambda a) - 2\lambda a g_{n+1}(\lambda a)} \right\}, \tag{2.5}$$

$$\epsilon'_n = -\frac{[(n-1) f_n(\lambda a) + \lambda a f_{n+1}(\lambda a)]}{[(n-1) g_n(\lambda a) - \lambda a g_{n+1}(\lambda a)]} \epsilon_n. \tag{2.6}$$

The drag and torque on the sphere of radius a is therefore found to be

$$\begin{aligned} \boldsymbol{D} = 4\pi\mu\lambda^3 &\left\{ \frac{a^4(a\lambda g_1(\lambda a) + 2g_2(\lambda a))\alpha_1}{[2\lambda a g_2(\lambda a) - (6-\lambda^2 a^2) g_1(\lambda a)]} + \frac{2a^3(f_1(\lambda a) g_2(\lambda a) + f_2(\lambda a) g_1(\lambda a))\alpha'_1}{[2\lambda a g_2(\lambda a) - (6-\lambda^2 a^2) g_1(\lambda a)]} \right\} \\ &\cdot (A_{11}\hat{\boldsymbol{\imath}} + B_{11}\hat{\boldsymbol{\jmath}} + A_{10}\hat{\boldsymbol{k}}) e^{\lambda^2 \nu t}, \end{aligned} \tag{2.7}$$

$$\text{Torque} = \boldsymbol{T} = 0. \tag{2.8}$$

It can be easily show that

$$\begin{aligned} \boldsymbol{D} &= \frac{2\pi\mu\lambda^3 a^4 (a\lambda g_1(\lambda a) + 2g_2(\lambda a))}{[2\lambda a g_2(\lambda a) - (6-\lambda^2 a^2) g_1(\lambda a)]} [\boldsymbol{V_0}]_0 + \frac{\begin{pmatrix} 2\pi\mu[6a^3(f_1(\lambda a) g_2(\lambda a) + f_2(\lambda a) g_1(\lambda a)) \\ -a^4\lambda(a\lambda g_1(\lambda a) + 2g_2(\lambda a))] \end{pmatrix}}{[2\lambda a g_2(\lambda a) - (6-\lambda^2 a^2) g_1(\lambda a)]} [\nabla^2 \boldsymbol{V_0}]_0 \\ &= 4\pi\nabla a(\boldsymbol{B}_0 [\boldsymbol{V_0}]_0 + a^2 \boldsymbol{B}_2 [\nabla^2 \boldsymbol{V_0}]_0), \end{aligned} \tag{2.9}$$

where

$$\boldsymbol{B}_0 = \left(1 + \lambda a + \frac{\lambda^2 a^2}{2} + \frac{\lambda^3 a^3}{6}\right)\left(1 + \frac{\lambda a}{3}\right)^{-1},$$

$$\boldsymbol{B}_2 = \left(\frac{e^{\lambda a}}{a^2\lambda^2} - \left(\frac{\lambda a}{6} + \frac{1}{2} + \frac{1}{\lambda a} + \frac{1}{\lambda^2 a^2}\right)\right)\left(1 + \frac{\lambda a}{3}\right)^{-1},$$

where $\boldsymbol{V_0}$ is the velocity of the undisturbed flow and $[\ ]_0$ is the evaluation at the center of the sphere $r = 0$. We observe that when $\lambda \to 0$, the formula for torque and drag given (2.8) and (2.9), reduce to the Faxén's [2] laws for steady flows in shear-free case [4].

3. Shear-free Sphere in an Oscillatory Flow

Consider a shear-free sphere of radius a in an oscillatory flow of a viscous, incompressible fluid. This amounts to considering the velocity and pressure to be of the form $\boldsymbol{V_0} = \boldsymbol{U}e^{i\omega t}$ and $p = Pe^{i\omega t}$ respectively in the above analysis. Here we seek a solution satisfying the conditions (i) $\boldsymbol{V} = 0$ on $r = a$, (ii) $\boldsymbol{V} = \boldsymbol{U}e^{i\omega t}$ as $r \to \infty$ and $p \to Pe^{i\omega t}$ as $r \to \infty$. Here if $\boldsymbol{U} = U\hat{\imath}$, then

$$A_0 = \frac{U}{2} r \sin\theta \cos\varphi e^{i\omega t}, \tag{3.1}$$

$$B_0 = 0, \tag{3.2}$$

where U is a constant. The modified velocity and pressure owing to the presence of the sphere have the following representation

$$\boldsymbol{V} = \operatorname{Curl}\operatorname{Curl}(\boldsymbol{r}A) + \operatorname{Curl}(\boldsymbol{r}B),$$

$$p = p_0 + \mu\frac{\partial}{\partial r}(r(\nabla^2 - \lambda^2)A), \quad \lambda^2 = \frac{i\omega}{\nu},$$

where

$$\nabla^2(\nabla^2 - \lambda^2)A = 0, \quad (\nabla^2 - \lambda^2)B = 0.$$

In this example

$$A = \frac{U}{2}\left[r + \frac{a^4\lambda(\lambda a g_1(\lambda a) + 2g_2(\lambda a))}{r^2\left[(6 - \lambda^2 a^2)g_1(\lambda a) - 2\lambda a g_2(\lambda a)\right]} - \frac{6ag_1(\lambda r)}{[(6 - \lambda^2 a^2)g_1(\lambda a) - 2\lambda a g_2(\lambda a)]}\right]\sin\theta\cos\varphi e^{i\omega t}, \tag{3.3}$$

$$B = 0. \tag{3.4}$$

We rewrite A as

$$A = \frac{U}{2}\left[r - \left\{\frac{a^3}{r^2} + \left(\frac{\lambda a}{3} + 1 + \frac{2}{\lambda a} + \frac{2}{\lambda^2 a^2}\right) + \frac{2ae^{\lambda a - \lambda r}}{\lambda r}\left(1 + \frac{1}{\lambda r}\right)\right\}\left(1 + \frac{\lambda a}{3}\right)^{-1}\right]\sin\theta\cos\varphi e^{i\omega t}0. \tag{3.5}$$

In the limit $\lambda \to 0$, it reduces to

$$A = \frac{U}{2}(r - a)\sin\theta\cos\varphi. \tag{3.6}$$

We can identify the distribution of singularities from the expression for A given in equation (3.5). The image system consists of a potential dipole and a Stokeslet [5] due to a point force $\boldsymbol{F} = \frac{-4\pi\nabla U a e^{\lambda a} e^{i\omega t}}{1 + \frac{\lambda a}{3}}\hat{\imath}$ at the origin. The drag is given by the following expression

$$\begin{aligned}\boldsymbol{D} &= \frac{2\pi\lambda^3\nabla a^4 U(\lambda a g_1(\lambda a) + 2g_2(\lambda a))e^{i\omega t}}{[2\lambda a g_2(\lambda a) - (6 - \lambda^2 a^2)g_1(\lambda a)]}\hat{\imath} \\ &= 4\pi\nabla U a\left(1 + \lambda a + \frac{\lambda^2 a^2}{2} + \frac{\lambda^3 a^3}{6}\right)\left(1 + \frac{\lambda a}{3}\right)^{-1} e^{i\omega t}\hat{\imath}.\end{aligned} \tag{3.7}$$

The torque experienced by the shear-free sphere is zero. We can observe that the drag given in (3.7) reduces to the formula for the drag experienced by a shear-free sphere of radius 'a' in a steady, uniform flow in the limit $\lambda \to 0$ [4].

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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