Communications in Mathematics and Applications
Vol. 13, No. 2, pp. 507–527, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i2.1802

http://www.rgnpublications.com

Research Article

Some Results on Fractional Differential Equation With Mixed Boundary Condition via *S*-Iteration

Haribhau L. Tidke*[1] and Gajanan S. Patil[2]

[1] *Department of Mathematics, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, Maharashtra, India*
[2] *Department of Mathematics, PSGVPM's ASC College, Shahada, Nandurbar, Maharashtra, India*
***Corresponding author:** tharibhau@gmail.com

Received: December 21, 2021 **Accepted:** June 13, 2022

Abstract. The present paper discuss the existence, uniqueness and other properties of solutions of nonlinear differential equation of fractional order involving the Caputo fractional derivative with mixed boundary condition. The analysis of obtained results is based on application of S-iteration method. Since the study of qualitative properties in general required differential and integral inequalities, but here S-iteration method itself has equally important contribution to study various properties such as dependence on boundary data, closeness of solutions and dependence on parameters and functions involved therein. The results obtained are illustrated through example.

Keywords. Existence and uniqueness, S-iterative method, Fractional derivative, Continuous dependence, Closeness, Parameters, Boundary value problem

Mathematics Subject Classification (2020). 34A08, 34A12, 26A33, 35B30, 34B15

1. Introduction

We consider the following boundary value problem involving the Caputo fractional derivative with mixed boundary condition of the type:

$$(D_*^{\alpha})y(t) = \mathcal{F}(t, y(t)), \quad t \in I = [0,b],\ 0 < \alpha < 1, \tag{1}$$

with the given boundary conditions

$$m_1 y(0) + m_2 y(b) = d, \tag{2}$$

where $\mathcal{F}: I \times X \to X$ is continuous function and m_1, m_2 are real constants with $m_1 + m_2 \neq 0$. The element $d \in X$ is given.

Several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [2–8, 10, 11, 13–18, 21–23]). The most of iterations devoted for both analytical and numerical approaches. The S-iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this paper.

Authors are motivated by the above mentioned results and influenced by [1, 24]. The main objective of this paper is to extend the some results of the paper [9] by the use of normal S-iteration method which establish the existence and uniqueness of solutions of the boundary value problem (1)-(2) and other qualitative properties of solutions.

2. Preliminaries

Before proceeding to the statement of our main results, we shall set-forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $I = [0, b]$ denotes an interval of the real line $\mathbb{R}$. We denote $B = C^1(I, X)$, as a Banach space of all continuous functions from I into X, endowed with the norm

$$\|y\|_B = \sup\{\|y(t)\| : y \in B\}, \quad t \in I.$$

Definition 1 ([20]). The Riemann-Liouville fractional integral (left-sided) of a function $h \in C^1[a, b]$ of order $\alpha \in \mathbf{R}_+ = (0, \infty)$ is defined by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler gamma function.

Definition 2 ([20]). Let $n-1 < \alpha \le n$, $n \in \mathbb{N}$. Then the expression

$$D_a^{\alpha} h(t) = \frac{d^n}{dt^n}[I_a^{n-\alpha} h(t)], \quad t \in [a, b]$$

is called the (left-sided) Riemann-Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 3 ([19]). Let $h \in C^n[a, b]$ and $n-1 < \alpha \le n$, $n \in \mathbb{N}$. Then the expression

$$(D_{*a}^{\alpha}) h(t) = I_a^{n-\alpha} h^{(n)}(t), \quad t \in [a, b]$$

is called the (left-sided) Caputo derivative of h of order α

Lemma 1 ([12]). *If the function $f = (f_1, \cdots, f_n) \in C^1[a, b]$, then the initial value problems*

$$(D_*^{\alpha_i}) y(t) = f_i(t, y_1, \cdots, y_n), \quad y_i^{(k)}(0) = c_k^i,\ i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, m_i,$$

where $m_i < \alpha_i \le m_i + 1$ is equivalent to Volterra integral equations:

$$y_i(t) = \sum_{k=0}^{m_i} c_k^i \frac{t^k}{k!} + I^{\alpha_i} f_i(t, y_1, \cdots, y_n), \quad 1 \le i \le n.$$

As a consequence of the above lemma and following results of [9, 19, 20, 25], it is easy to observe that if $y \in B$, then $y(t)$ satisfies the following integral equation

$$y(t) = \frac{d}{m_1+m_2} - \frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,y(s))ds, \quad t \in I \tag{3}$$

which is equivalent to (1)-(2).

We need the following pair of known results:

Theorem 1 ([21, p. 194]). *Let C be a nonempty closed convex subset of a Banach space X and $T: C \to C$ a contraction operator with contractivity factor $k \in [0,1)$ and fixed point x^*. Let α_n and β_n be two real sequences in $[0,1]$ such that $\alpha \le \alpha_n \le 1$ and $\beta \le \beta_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_n, v_n and w_n in C as follows:*

S-*iteration process* : $\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)u_n + \beta_n Tu_n, \quad n \in \mathbb{N} \end{cases}$

Picard iteration : $v_{n+1} = Tv_n$, $n \in \mathbb{N}$

Mann iteration process : $w_{n+1} = (1-\beta_n)w_n + \beta_n Tw_n$, $n \in \mathbb{N}$

Then, we have the following:

(a) $\|u_{n+1} - x^*\| \le k^n[1-(1-k)\alpha\beta]^n \|u_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

(b) $\|v_{n+1} - x^*\| \le k^n \|v_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

(c) $\|w_{n+1} - x^*\| \le [1-(1-k)\beta]^n \|w_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

Moreover, the S*-iteration process is faster than the Picard and Mann iteration processes.*

In particular, for $\alpha_n = 1$, $n \in \mathbb{N}$, the S-iteration process can be written as:

$$\begin{cases} y_0 \in C, \\ y_{n+1} = Tz_n, \\ z_n = (1-\beta_n)y_n + \beta_n Ty_n, \quad n \in \mathbb{N}. \end{cases} \tag{4}$$

Lemma 2 ([23, p.4]). *Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ one has satisfied the inequality*

$$\beta_{n+1} \le (1-\mu_n)\beta_n + \mu_n\gamma_n, \tag{5}$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \ge 0$, for all $n \in \mathbb{N}$. Then the following inequality holds

$$0 \le \limsup_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \gamma_n. \tag{6}$$

3. Existence and Uniqueness of Solutions via *S*-iteration

Now, we are able to state and prove the following main theorem which deals with the existence and uniqueness of solutions of the equation (1)-(2).

Theorem 2. *Assume that there exists a function $p \in C(I,\mathbb{R}_+)$ such that*

$$\|\mathcal{F}(t,u_1)-\mathcal{F}(t,v_1)\| \le p(t)\|u_1-v_1\|. \tag{7}$$

Let $\{\xi_k\}_{k=0}^{\infty}$ be a real sequence in $[0,1]$ satisfying $\sum_{k=0}^{\infty}\xi_k=\infty$. If

$$\Theta=\left[\frac{|m_2|}{|m_1+m_2|}I^{\alpha}p(b)+I^{\alpha}p(t)\right]<1,$$

then the equation (1)-(2) has a unique solution $y\in B$ and normal S-iterative method (4) converges to $y\in B$ with the following estimate:

$$\|y_{k+1}-y\|_B \le \frac{\Theta^{k+1}}{e^{(1-\Theta)\sum_{i=0}^{k}\xi_i}}\|y_0-y\|_B. \tag{8}$$

Proof. Let $y(t)\in B$ and define the operator

$$(Ty)(t)=\frac{d}{m_1+m_2}-\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}\mathcal{F}(s,y(s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathcal{F}(s,y(s))ds, \quad t\in I. \tag{9}$$

Let $\{y_k\}_{k=0}^{\infty}$ be iterative sequence generated by normal S-iteration method (4) for the operator given in (9).

We will show that $y_k\to y$ as $k\to\infty$.

From (4), (9) and assumptions, we obtain

$$\begin{aligned}
\|y_{k+1}(t)-y(t)\| &= \|(Tz_k)(t)-(Ty)(t)\|\\
&=\Bigg\|\frac{d}{m_1+m_2}-\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}\mathcal{F}(s,z_k(s))ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathcal{F}(s,z_k(s))ds\\
&\quad-\frac{d}{m_1+m_2}+\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}\mathcal{F}(s,y(s))ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathcal{F}(s,y(s))ds\Bigg\|\\
&\le\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\mathcal{F}(s,y(s))\|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\mathcal{F}(s,y(s))\|ds\\
&\le\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}p(s)\|z_k(s)-y(s)\|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)\|z_k(s)-y(s)\|ds.
\end{aligned} \tag{10}$$

Now, by taking supremum in the inequality (10), we obtain

$$\|y_{k+1}-y\|_B\le\frac{|m_2|\|z_k-y\|_B}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}p(s)ds+\frac{\|z_k-y\|_B}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)ds$$

$$\leq \left[\frac{|m_2|}{|m_1+m_2|} I^\alpha p(b) + I^\alpha p(t)\right] \|z_k - y\|_B$$
$$\leq \Theta \|z_k - y\|_B. \tag{11}$$

Now, we estimate

$$\|z_k(t) - y(t)\| = [(1-\xi_k)\|y_k(t) - y(t)\| + \xi_k \|(Ty_k)(t) - (Ty)(t)\|]$$
$$\leq (1-\xi_k)\|y_k(t) - y(t)\| + \xi_k \Big\{ \frac{|m_2|}{|m_1+m_2|} \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} p(s) \|y_k(s) - y(s)\| ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \|y_k(s) - y(s)\| ds \Big\}. \tag{12}$$

Similarly, by taking supremum in the inequality (12) to get

$$\|z_k - y\|_B \leq \left[1 - \xi_k (1 - \frac{|m_2|}{|m_1+m_2|} I^\alpha p(b) + I^\alpha p(t))\right] \|y_k - y\|_B$$
$$= [1 - \xi_k (1-\Theta)] \|y_k - y\|_B. \tag{13}$$

Therefore, using (13) in (11), we have

$$\|y_{k+1} - y\|_B \leq \Theta [1 - \xi_k (1-\Theta)] \|y_k - y\|_B. \tag{14}$$

Thus, by induction, we get

$$\|y_{k+1} - y\|_B \leq \Theta^{k+1} \prod_{j=0}^{k} [1 - \xi_k (1-\Theta)] \|y_0 - y\|_B. \tag{15}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}$, the definition of Θ yields $\xi_k \leq 1$ and $\Theta < 1$

$$\Rightarrow \quad \xi_k \Theta < \xi_k$$
$$\Rightarrow \quad \xi_k (1-\Theta) < 1, \quad \text{for all } k \in \mathbb{N}. \tag{16}$$

From the classical analysis, we know that

$$1 - x \leq e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, \quad x \in [0,1].$$

Hence by utilizing this fact with (16) in (15), we obtain

$$\|y_{k+1} - y\|_B \leq \Theta^{k+1} e^{-(1-\Theta)\sum_{j=0}^{k} \xi_j} \|y_0 - y\|_B$$
$$= \frac{\Theta^{k+1}}{e^{(1-\Theta)\sum_{i=0}^{k} \xi_i}} \|y_0 - y\|_B. \tag{17}$$

This is (8). Since $\sum_{k=0}^{\infty} \xi_k = \infty$,

$$e^{-(1-\Theta)\sum_{j=0}^{k} \xi_j} \to 0 \quad \text{as} \quad k \to \infty. \tag{18}$$

Hence using this, the inequality (17) implies $\lim_{k\to\infty} \|y_{k+1} - y\|_B = 0$ and therefore, we have $y_k \to y$ as $k \to \infty$. □

Remark. It is an interesting to note that the inequality (17) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the equation (1)-(2) for $t \in I$.

4. Continuous Dependence via *S*-iteration

In this section, we shall deal with continuous dependence of solution of the problem (1) on the boundary data, functions involved therein and also on parameters.

4.1 Dependence on Boundary Data

Suppose $y(t)$ and $\bar{y}(t)$ are solutions of (1) with boundary data

$$m_1 y(0)+m_2 y(b)=d \tag{19}$$

and

$$m_1\bar{y}(0)+m_2\bar{y}(b)=\bar{d}, \tag{20}$$

where d, $\bar{d}$ are given elements in X.

Then looking at the steps as in the proof of Theorem 2, we define the operator for the equations (1)-(20)

$$\begin{aligned}(\bar{T}\bar{y})(t)&=\frac{\bar{d}}{m_1+m_2}-\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,\bar{y}(s))ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,\bar{y}(s))ds, \quad t\in I.\end{aligned} \tag{21}$$

We shall deal with the continuous dependence of solutions of equation (1) on boundary data.

Theorem 3. *Suppose the function $\mathcal{F}$ in equation* (1) *satisfies the condition* (7)*. Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\bar{y}_k\}_{k=0}^{\infty}$ generated normal* S*-iterative method associated with operators T in* (9) *and $\bar{T}$ in* (21)*, respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in* $[0,1]$ *satisfying $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$. If the sequence $\{\bar{y}_k\}_{k=0}^{\infty}$ converges to $\bar{y}$, then we have*

$$\|y-\bar{y}\|_B\le\frac{3\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)}{(1-\Theta)}. \tag{22}$$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\bar{y}_k\}_{k=0}^{\infty}$ generated normal S-iterative method associated with operators T in (9) and $\bar{T}$ in (21), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$. From iteration (4) and equations (9); (21) and assumptions, we obtain

$$\begin{aligned}\|y_{k+1}(t)-\bar{y}_{k+1}(t)\|&=\|(Tz_k)(t)-(\bar{T}\bar{z}_k)(t)\|\\ &=\Big\|\frac{d}{m_1+m_2}-\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,z_k(s))ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,z_k(s))ds\\ &\quad-\frac{\bar{d}}{m_1+m_2}+\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,\bar{z}_k(s))ds\\ &\quad-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,\bar{z}_k(s))ds\Big\|\\ &\le\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\mathcal{F}(s,\bar{z}_k(s))\|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\mathcal{F}(s,\bar{z}_k(s))\|ds\\
&\le \left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds.
\end{aligned}
\tag{23}
$$

Recalling the equations (11) and (13), the above inequality becomes

$$
\|y_{k+1}-\bar{y}_{k+1}\|_B \le \left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\Theta\|z_k-\bar{z}_k\|_B,
\tag{24}
$$

and similarly, it is seen that

$$
\|z_k-\bar{z}_k\|_B \le \xi_k\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+[1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B.
\tag{25}
$$

Therefore, using (25) in (24) and using hypothesis $\Theta<1$, and $\frac{1}{2}\le \xi_k$ for all $k\in\mathbb{N}$, the resulting inequality becomes

$$
\begin{aligned}
\|y_{k+1}-\bar{y}_{k+1}\|_B &\le \left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\|z_k-\bar{z}_k\|_B\\
&\le \left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\xi_k\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+[1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B\\
&\le 2\xi_k\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\xi_k\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+[1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B\\
&\le [1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B+\xi_k(1-\Theta)\frac{3\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)}{(1-\Theta)}.
\end{aligned}
\tag{26}
$$

We denote

$$
\begin{aligned}
&\beta_k=\|y_k-\bar{y}_k\|_B,\\
&\mu_k=\xi_k(1-\Theta)\in(0,1),\\
&\gamma_k=\frac{3\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)}{(1-\Theta)}\ge 0.
\end{aligned}
$$

The assumption $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$ implies $\sum_{n=0}^{\infty}\xi_k=\infty$. Now, it can be easily seen that (26) satisfies all the conditions of Lemma 2 and hence we have

$$
\begin{aligned}
&0\le \limsup_{k\to\infty}\beta_k \le \limsup_{k\to\infty}\gamma_k\\
\Rightarrow\quad &0\le \limsup_{k\to\infty}\|y_k-\bar{y}_k\|_B \le \limsup_{k\to\infty}\frac{3\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)}{(1-\Theta)}\\
\Rightarrow\quad &0\le \limsup_{k\to\infty}\|y_k-\bar{y}_k\|_B \le \frac{3\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)}{(1-\Theta)}.
\end{aligned}
\tag{27}
$$

Using the assumption $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \bar{y}_k = \bar{y}$, we get from (27) that

$$\|y-\bar{y}\|_B \le \frac{3\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)}{(1-\Theta)}, \tag{28}$$

which shows that the dependency of solutions of BVPs (1)-(2) and (1)-(20) on given boundary data. □

4.2 Closeness of Solution via *S*-iteration

Consider the problem (1)-(2) and the corresponding problem

$$(D_*^{\alpha})\bar{y}(t) = \bar{\mathcal{F}}(t,\bar{y}(t)), \quad t \in I,\ 0<\alpha<1, \tag{29}$$

with the given boundary condition

$$m_1\bar{y}(0)+m_2\bar{y}(b) = \bar{d}, \tag{30}$$

where $\bar{\mathcal{F}}$ is defined as $\mathcal{F}$ and $\bar{d}$ is given element in X.

Then looking at the steps as in the proof of Theorem 2, we define the operator for the equation (29)-(30)

$$\begin{aligned}(\bar{T}\bar{y})(t) &= \frac{\bar{d}}{m_1+m_2} - \frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\bar{\mathcal{F}}(s,\bar{y}(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{\mathcal{F}}(s,\bar{y}(s))ds, \quad t\in I.\end{aligned} \tag{31}$$

The next theorem deals with the closeness of solutions of the problems (1)-(2) and (29)-(30).

Theorem 4. *Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\bar{y}_k\}_{k=0}^{\infty}$ generated normal* S*-iterative method associated with operators T in* (9) *and $\bar{T}$ in* (31)*, respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in* $[0,1]$ *satisfying $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$. Assume that*

(i) *all conditions of Theorem* 2 *hold, and $y(t)$ and $\bar{y}(t)$ are solutions of* (1)-(2) *and* (29)-(30)*, respectively.*

(ii) *there exist non negative constant ϵ such that*

$$\|\mathcal{F}(t,u_1)-\bar{\mathcal{F}}(t,u_1)\| \le \epsilon, \quad \textit{for all } t\in I. \tag{32}$$

If the sequence $\{\bar{y}_k\}_{k=0}^{\infty}$ converges to $\bar{y}$, then we have

$$\|y-\bar{y}\|_B \le \frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^{\alpha}}{\Gamma(\alpha+1)}\right]}{(1-\Theta)}. \tag{33}$$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\bar{y}_k\}_{k=0}^{\infty}$ generated normal *S*-iterative method associated with operators T in (9) and $\bar{T}$ in (31), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$. From iteration (4) and equations (9); (31) and hypotheses, we obtain

$$\begin{aligned}\|y_{k+1}(t)-\bar{y}_{k+1}(t)\| &= \|(Tz_k)(t)-(\bar{T}\bar{z}_k)(t)\| \\ &= \left\|\frac{d}{m_1+m_2} - \frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,z_k(s))ds\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,z_k(s))ds \\
&-\frac{\bar{d}}{m_1+m_2}+\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\bar{\mathcal{F}}(s,\bar{z}_k(s))ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{\mathcal{F}}(s,\bar{z}_k(s))ds\Big\| \\
\le&\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\bar{\mathcal{F}}(s,\bar{z}_k(s))\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\bar{\mathcal{F}}(s,\bar{z}_k(s))\|ds \\
\le&\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,\bar{z}_k(s))-\bar{\mathcal{F}}(s,\bar{z}_k(s))\|ds \\
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\mathcal{F}(s,\bar{z}_k(s))\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,\bar{z}_k(s))-\bar{\mathcal{F}}(s,\bar{z}_k(s))\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s))-\mathcal{F}(s,\bar{z}_k(s))\|ds \\
\le&\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\epsilon ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\epsilon ds \\
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
\le&\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}+\frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} \\
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
\le&\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\frac{|m_2|}{|m_1+m_2|}\frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}+\frac{\epsilon b^\alpha}{\Gamma(\alpha+1)} \\
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
=&\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)} \\
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|z_k(s)-\bar{z}_k(s)\|ds. \tag{34}$$

Recalling the derivations obtained in equations (12) and (13), the above inequality becomes

$$\|y_{k+1}-y\|_B \le \left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]+\Theta\|z_k-\bar{z}_k\|_B, \tag{35}$$

and similarly, it is seen that

$$\|z_k-\bar{z}_k\|_B \le \xi_k\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]+[1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B. \tag{36}$$

Therefore, using (36) in (35) and using hypothesis $\Theta<1$, and $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$, the resulting inequality becomes

$$\begin{aligned}
\|y_{k+1}-\bar{y}_{k+1}\|_B &\le \left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]+\|z_k-\bar{z}_k\|_B\\
&\le \left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]\\
&\quad +\xi_k\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]+[1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B\\
&\le 2\xi_k\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]\\
&\quad +\xi_k\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]+[1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B\\
&\le [1-\xi_k(1-\Theta)]\|y_k-\bar{y}_k\|_B+\xi_k(1-\Theta)\frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]}{(1-\Theta)}.
\end{aligned} \tag{37}$$

We denote

$$\beta_k=\|y_k-\bar{y}_k\|_B,$$
$$\mu_k=\xi_k(1-\Theta)\in(0,1),$$
$$\gamma_k=\frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]}{(1-\Theta)}\ge 0.$$

The assumption $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$ implies $\sum_{n=0}^{\infty}\xi_k=\infty$. Now, it can be easily observed that (37) satisfies all the conditions of Lemma 2 and hence we have

$$0\le\limsup_{k\to\infty}\beta_k\le\limsup_{k\to\infty}\gamma_k$$

$$\Rightarrow\quad 0\le\limsup_{k\to\infty}\|y_k-\bar{y}_k\|_B\le\limsup_{k\to\infty}\frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]}{(1-\Theta)}$$

$$\Rightarrow\quad 0\le\limsup_{k\to\infty}\|y_k-\bar{y}_k\|_B\le\frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]}{(1-\Theta)}. \tag{38}$$

Using the assumption $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \bar{y}_k = \bar{y}$, we get from (38) that

$$\|y-\bar{y}\|_B \le \frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right)+\epsilon\left(\frac{|m_2|}{|m_1+m_2|}+1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]}{(1-\Theta)}, \tag{39}$$

which shows that the dependency of solutions of BVPs (1)-(2) and (29)-(30) on the function involved on the right hand side of the given equation. □

Remark. The inequality (39) relates the solutions of the problems (1)-(2) and (29)-(30) in the sense that if $\mathcal{F}$ and $\bar{\mathcal{F}}$ are close as $\epsilon \to 0$, then not only the solutions of the problems (1)-(2) and (29)-(30) are close to each other (i.e. $\|y-\bar{y}\|_B \to 0$), but also depend continuously on the functions involved therein and boundary data.

4.3 Dependence on Parameters

We next consider the following problems

$$(D_*^\alpha)y(t) = \mathcal{F}(t, y(t), \mu_1), \quad t \in I,\ 0<\alpha<1, \tag{40}$$

with the given boundary condition

$$m_1 y(0) + m_2 y(b) = d \tag{41}$$

and

$$(D_*^\alpha)\bar{y}(t) = \mathcal{F}(t, \bar{y}(t), \mu_2), \quad t \in I,\ 0<\alpha<1, \tag{42}$$

with the given boundary condition

$$m_1 \bar{y}(0) + m_2 \bar{y}(b) = \bar{d} \tag{43}$$

where $\mathcal{F}: I \times X \times \mathbb{R} \to X$ is continuous function, $d, \bar{d}$ are given elements in X and constants μ_1, μ_2 are real parameters.

Let $y(t), \bar{y}(t) \in B$ and following steps from the proof of Theorem 2, define the operators for the equations (40) and (42), respectively

$$\begin{aligned}(Ty)(t) &= \frac{d}{m_1+m_2} - \frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s, y(s), \mu_1)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s, y(s), \mu_1)ds, \quad t \in I\end{aligned} \tag{44}$$

and

$$\begin{aligned}(\bar{T}\bar{y})(t) &= \frac{\bar{d}}{m_1+m_2} - \frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s, \bar{y}(s), \mu_2)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s, \bar{y}(s), \mu_2)ds, \quad t \in I.\end{aligned} \tag{45}$$

The following theorem discuss the continuous dependency of solutions on parameters.

Theorem 5. *Consider the sequences* $\{y_k\}_{k=0}^\infty$ *and* $\{\bar{y}_k\}_{k=0}^\infty$ *generated normal* S*-iterative method associated with operators* T *in* (44) *and* $\bar{T}$ *in* (45)*, respectively with the real sequence* $\{\xi_k\}_{k=0}^\infty$ *in* $[0,1]$ *satisfying* $\frac{1}{2} \le \xi_k$ *for all* $k \in \mathbb{N}$*. Assume that*

(i) $y(t)$ *and* $\bar{y}(t)$ *are solutions of* (40)-(41) *and* (42)-(43)*, respectively.*

(ii) *the function* $\mathcal{F}$ *satisfy the conditions:*

$$\|\mathcal{F}(t,u_1,\mu_1)-\mathcal{F}(t,v_1,\mu_1)\| \le \bar{p}(t)\|u_1-v_1\|$$

and

$$\|\mathcal{F}(t,u_1,\mu_1)-\mathcal{F}(t,u_1,\mu_2)\| \le r(t)|\mu_1-\mu_2|,$$

where $\bar{p}, r \in C(I,\mathbb{R}_+)$.

If the sequence $\{\bar{y}_n\}_{n=0}^{\infty}$ *converges to* $\bar{y}$, *then we have*

$$\|y-\bar{y}\|_B \le \frac{3\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]}{(1-\bar{\Theta})}, \tag{46}$$

where $\bar{\Theta}=\Big[\frac{|m_2|}{|m_1+m_2|}I^\alpha \bar{p}(b)+I^\alpha \bar{p}(t)\Big]<1$.

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\bar{y}_k\}_{n=0}^{\infty}$ generated normal S-iterative method associated with operators T in (44) and $\bar{T}$ in (45), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$. From iteration (4) and equations (44); (45) and hypotheses, we obtain

$$\begin{aligned}
\|y_{k+1}(t)-\bar{y}_{k+1}(t)\| &= \|(Tz_k)(t)-(\bar{T}\bar{z}_k)(t)\| \\
&= \Big\|\frac{d}{m_1+m_2}-\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,z_k(s),\mu_1)ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,z_k(s),\mu_1)ds \\
&\quad -\frac{\bar{d}}{m_1+m_2}+\frac{m_2}{m_1+m_2}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\mathcal{F}(s,\bar{z}_k(s),\mu_2)ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}(s,\bar{z}_k(s),\mu_2)ds\Big\| \\
&\le \Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s),\mu_1)-\mathcal{F}(s,\bar{z}_k(s),\mu_2)\|ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s),\mu_1)-\mathcal{F}(s,\bar{z}_k(s),\mu_2)\|ds \\
&\le \Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,\bar{z}_k(s),\mu_1)-\mathcal{F}(s,\bar{z}_k(s),\mu_2)\|ds \\
&\quad +\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s),\mu_1)-\mathcal{F}(s,\bar{z}_k(s),\mu_1)\|ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,\bar{z}_k(s),\mu_1)-\mathcal{F}(s,\bar{z}_k(s),\mu_2)\|ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s,z_k(s),\mu_1)-\mathcal{F}(s,\bar{z}_k(s),\mu_1)\|ds \\
&\le \Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}r(s)|\mu_1-\mu_2|ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}r(s)|\mu_1-\mu_2|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\bar{p}(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{p}(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
\le &\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+\frac{|m_2|}{|m_1+m_2|}|\mu_1-\mu_2|I^\alpha r(b)+|\mu_1-\mu_2|I^\alpha r(t) \\
&+\frac{|m_2|}{|m_1+m_2|}\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\bar{p}(s)\|z_k(s)-\bar{z}_k(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{p}(s)\|z_k(s)-\bar{z}_k(s)\|ds. \qquad (47)
\end{aligned}
$$

Recalling the derivations obtained in equations (12) and (13), the above inequality becomes

$$
\|y_{k+1}-\bar{y}_{k+1}\|_B \le \Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]+\bar{\Theta}\|z_k-\bar{z}_k\|_B \qquad (48)
$$

and similarly, it is seen that

$$
\|z_k-\bar{z}_k\|_B \le \xi_k\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]+[1-\xi_k(1-\bar{\Theta})]\|y_k-\bar{y}_k\|_B. \qquad (49)
$$

Therefore, using (49) in (48) and using hypothesis $\bar{\Theta}<1$, and $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$, the resulting inequality becomes

$$
\begin{aligned}
\|y_{k+1}-\bar{y}_{k+1}\|_B &\le \Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]+\|z_k-\bar{z}_k\|_B \\
&\le \Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big] \\
&\quad +\xi_k\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]+[1-\xi_k(1-\bar{\Theta})]\|y_k-\bar{y}_k\|_B \\
&\le 2\xi_k\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big] \\
&\quad +\xi_k\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]+[1-\xi_k(1-\bar{\Theta})]\|y_k-\bar{y}_k\|_B \\
&\le [1-\xi_k(1-\bar{\Theta})]\|y_k-\bar{y}_k\|_B+\xi_k(1-\bar{\Theta})\frac{3\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]}{(1-\bar{\Theta})}. \qquad (50)
\end{aligned}
$$

We denote

$$
\begin{aligned}
\beta_k &= \|y_k-\bar{y}_k\|_B, \\
\mu_k &= \xi_k(1-\bar{\Theta})\in(0,1), \\
\gamma_k &= \frac{3\Big[\Big(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\Big)+|\mu_1-\mu_2|\Big(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\Big)\Big]}{(1-\bar{\Theta})}\ge 0.
\end{aligned}
$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$ implies $\sum_{n=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (50) satisfies all the conditions of Lemma 2 and hence we have

$$0 \le \limsup_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \gamma_k$$

$$\Rightarrow \quad 0 \le \limsup_{k\to\infty} \|y_k - \bar{y}_k\|_B \le \limsup_{k\to\infty} \frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right) + |\mu_1 - \mu_2|\left(\frac{|m_2|}{|m_1+m_2|} I^\alpha r(b) + I^\alpha r(t)\right)\right]}{(1-\bar{\Theta})}$$

$$\Rightarrow \quad 0 \le \limsup_{k\to\infty} \|y_k - \bar{y}_k\|_B \le \frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right) + |\mu_1 - \mu_2|\left(\frac{|m_2|}{|m_1+m_2|} I^\alpha r(b) + I^\alpha r(t)\right)\right]}{(1-\bar{\Theta})}. \tag{51}$$

Using the assumption $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \bar{y}_k = \bar{y}$, we get from (51) that

$$\|y - \bar{y}\|_B \le \frac{3\left[\left(\frac{\|d-\bar{d}\|}{|m_1+m_2|}\right) + |\mu_1 - \mu_2|\left(\frac{|m_2|}{|m_1+m_2|} I^\alpha r(b) + I^\alpha r(t)\right)\right]}{(1-\bar{\Theta})}, \tag{52}$$

which shows the dependence of solutions of the problem (1)-(2) is on parameters μ_1 and μ_2. □

Remark. The result dealing with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars and note that the boundary conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems.

5. Example

We consider the following problem:

$$(D_*^\alpha)y(t) = \frac{3t}{5}\left[\frac{t - \sin(y(t))}{2}\right], \quad t \in [0,1],\ 0 < \alpha < 1, \tag{53}$$

with the given boundary condition

$$y(0) + y(1) = 1. \tag{54}$$

Comparing this equation with the equation (1), we get

$$\mathcal{F} \in C(I \times \mathbb{R}, \mathbb{R}) \text{ with } \mathcal{F}(t, y(t)) = \frac{3t}{5}\left[\frac{t - \sin(y(t))}{2}\right].$$

Now, we have

$$\begin{aligned} |\mathcal{F}(t,y(t)) - \mathcal{F}(t,\bar{y}(t))| &= \left|\frac{3t}{5}\left[\frac{t-\sin(y(t))}{2}\right] - \frac{3t}{5}\left[\frac{t-\sin(\bar{y}(t))}{2}\right]\right| \\ &\le \left|\frac{3t}{5}\right| \left|\frac{t-\sin(y(t))}{2} - \frac{t-\sin(\bar{y}(t))}{2}\right| \\ &\le \frac{3t}{10}|\sin(y(t)) - \sin(\bar{y}(t))|. \end{aligned} \tag{55}$$

Taking sup norm, we obtain

$$|\mathcal{F}(t,y(t)) - \mathcal{F}(t,\bar{y}(t))| \le \frac{3t}{10}|y - \bar{y}|, \tag{56}$$

where $p(t) = \frac{3t}{10}$.

5.1 Existence and Uniqueness of Solutions

Therefore, we the estimate

$$\begin{aligned}
\Theta &= \Big[\frac{|m_2|}{|m_1+m_2|} I^\alpha p(b) + I^\alpha p(t)\Big] \\
&= \Big[\frac{1}{2} I^\alpha p(b) + I^\alpha p(t)\Big] \\
&= \Big[\frac{1}{2} I^\alpha p(1) + I^\alpha \frac{3t}{10}\Big] \\
&= \frac{3}{10}\Big[\frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} s ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s ds\Big] \\
&= \frac{3}{10}\Big[\frac{1}{2}\frac{1^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\Big] \\
&\le \frac{3}{10}\Big[\frac{1}{2}\frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+2)}\Big] \qquad (t \le 1) \\
&= \frac{3}{10}\Big[\frac{1}{2}+1\Big]\frac{1}{\Gamma(\alpha+2)} \\
&= \frac{3\times 3}{10\times 2}\frac{1}{\Gamma(\alpha+2)} \\
&= \frac{9}{20}\frac{1}{\Gamma(\alpha+2)}.
\end{aligned} \tag{57}$$

Therefore, the condition $\Theta < 1$ is satisfied only if $\frac{9}{20}\frac{1}{\Gamma(\alpha+2)} < 1$.

In particular, we choose $\alpha = \frac{1}{2}$, then we have

$$\begin{aligned}
\frac{9}{20}\frac{1}{\Gamma(\alpha+2)} &= \frac{9}{20}\frac{1}{\Gamma\left(\frac{1}{2}+2\right)} \\
&= \frac{9}{20}\frac{1}{\Gamma\left(\frac{5}{2}\right)} \\
&= \frac{9}{20}\frac{1}{\frac{3\sqrt{\pi}}{4}} \\
&= \frac{3}{5}\frac{1}{\sqrt{\pi}} \\
&\simeq 0.3385 \\
&< 1.
\end{aligned}$$

We define the operator $T : B \to B$ for the given problem by

$$\begin{aligned}
(Ty)(t) = \frac{1}{2} - \frac{1}{2}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 (1-s)^{\frac{1}{2}-1}\mathcal{F}(s, y(s)) ds \\
+ \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1}\mathcal{F}(s, y(s)) ds
\end{aligned}$$

$$= \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{\pi}}\int_0^1 (1-s)^{-\frac{1}{2}}\frac{3s}{5}\Big[\frac{s-\sin(y(s))}{2}\Big]ds$$
$$+\frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-\frac{1}{2}}\frac{3s}{5}\Big[\frac{s-\sin(y(s))}{2}\Big]ds, \quad t\in I. \tag{58}$$

Since all conditions of Theorem 2 are satisfied and so by its conclusion, the sequence $\{y_n\}$ associated with the normal S-iterative method (4) for the operator T in (58) converges to a unique solution $y \in B$.

5.2 Error Estimate

Further, we also have for any $y_0 \in B$

$$|y_{k+1} - y|_B \le \frac{\Theta^{k+1}}{e^{(1-\Theta)\sum_{i=0}^{k}\xi_i}}|y_0 - y|_B$$
$$\le \frac{\left[\frac{3}{5}\frac{1}{\sqrt{\pi}}\right]^{k+1}}{e^{\left[1-\frac{3}{5}\frac{1}{\sqrt{\pi}}\right]\sum_{i=0}^{k}\xi_i}}|y_0 - y|, \tag{59}$$

where we have chosen $\xi_i = \frac{1}{1+i} \in [0,1]$. The estimate obtained in (59) is called a bound for the error (due to truncation of computation at the $k-$th iteration).

5.3 Continuous Dependence

One can check easily that the continuous dependence of solutions of equations (1) on boundary data. Indeed, for $y(0)+y(1) = d = 1$, $\bar{y}(0)+\bar{y}(1) = \bar{d} = \frac{1}{2}$, we have

$$|y-\bar{y}|_B \le \frac{3\left(\frac{|d-\bar{d}|}{|m_1+m_2|}\right)}{(1-\Theta)}$$
$$\le \frac{3\left(\frac{1-\frac{1}{2}}{2}\right)}{\left(1-\frac{3}{5}\frac{1}{\sqrt{\pi}}\right)}$$
$$\le \frac{3}{4\left(1-\frac{3}{5}\frac{1}{\sqrt{\pi}}\right)}$$
$$\simeq 1.1338. \tag{60}$$

5.4 Closeness of Solutions

Next, we consider the perturbed equation:

$$(D_*^{\frac{1}{2}})\bar{y}(t) = \frac{3t}{5}\Big[\frac{t-\sin(\bar{y}(t))}{2}\Big] - t + \frac{1}{7}, \quad t \in [0,1], \tag{61}$$

with the given boundary condition

$$\bar{y}(0)+\bar{y}(1) = \bar{d} = \frac{1}{2}. \tag{62}$$

Similarly, comparing it with the equation (29), we have

$$\bar{\mathcal{F}}(t,\bar{y}(t)) = \frac{3t}{5}\Big[\frac{t-\sin(\bar{y}(t))}{2}\Big] - t + \frac{1}{7}.$$

One can easily define the mapping $\bar{T}: B \to B$ by

$$(\bar{T}\bar{y})(t) = \frac{1}{4} - \frac{1}{2}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 (1-s)^{-\frac{1}{2}}\left\{\frac{3s}{5}\left[\frac{s-\sin(\bar{y}(s))}{2}\right] - s + \frac{1}{7}\right\}ds$$
$$+ \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{-\frac{1}{2}}\left\{\frac{3s}{5}\left[\frac{s-\sin(\bar{y}(s))}{2}\right] - s + \frac{1}{7}\right\}ds, \quad t \in I. \tag{63}$$

In perturbed equation, all conditions of Theorem 2 are also satisfied and so by its conclusion, the sequence $\{\bar{y}_n\}$ associated with the normal S-iterative method (4) for the operator $\bar{T}$ in (63) converges to a unique solution $\bar{y} \in B$.

Now, we have the following estimate:

$$\begin{aligned}
|\mathcal{F}(t,y(t)) - \bar{\mathcal{F}}(t,y(t))| &= \left|\frac{3t}{5}\left[\frac{t-\sin(y(t))}{2}\right] - \frac{3t}{5}\left[\frac{t-\sin(y(t))}{2}\right] + t - \frac{1}{7}\right| \\
&= \left|t - \frac{1}{7}\right| \\
&\le |t| + \frac{1}{7} \\
&\le 1 + \frac{1}{7} \qquad (t \le 1) \\
&= \frac{8}{7} = \epsilon.
\end{aligned} \tag{64}$$

Consider the sequences $\{y_n\}_{n=0}^{\infty}$ with $y_n \to y$ as $n \to \infty$ and $\{\bar{y}_n\}_{n=0}^{\infty}$ with $\bar{y}_n \to \bar{y}$ as $n \to \infty$ generated normal S-iterative method associated with operators T in (58) and $\bar{T}$ in (63), respectively with the real sequence $\{\xi_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \le \xi_n$ for all $n \in \mathbb{N}$. Then we have from Theorem 3 that for $b = 1$, $d = 1$, $\bar{d} = \frac{1}{2}$, $\epsilon = \frac{8}{7}$

$$\begin{aligned}
|x - \bar{x}|_B &\le \frac{3\left[\left(\frac{|d-\bar{d}|}{|m_1+m_2|}\right) + \epsilon\left(\frac{|m_2|}{|m_1+m_2|} + 1\right)\frac{b^\alpha}{\Gamma(\alpha+1)}\right]}{(1-\Theta)} \\
&\le \frac{3\left[\frac{1}{4} + \frac{8}{7}\left(\frac{1}{2}+1\right)\frac{1}{\Gamma(\frac{1}{2}+1)}\right]}{\left(1 - \frac{3}{5}\frac{1}{\sqrt{\pi}}\right)} \\
&\le \frac{3\left[\frac{1}{4} + \frac{12}{7}\frac{1}{\Gamma(\frac{3}{2})}\right]}{\left(1 - \frac{3}{5}\frac{1}{\sqrt{\pi}}\right)} \\
&\le \frac{3\left[\frac{1}{4} + \frac{12}{7}\frac{1}{\frac{1}{2}\sqrt{\pi}}\right]}{\left(1 - \frac{3}{5}\frac{1}{\sqrt{\pi}}\right)} \\
&\le \frac{3\left[\frac{1}{4} + \frac{24}{7}\frac{1}{\sqrt{\pi}}\right]}{\left(1 - \frac{3}{5}\frac{1}{\sqrt{\pi}}\right)}
\end{aligned}$$

$$\simeq \frac{6.5531}{0.6615}$$
$$\simeq 9.9064. \tag{65}$$

This shows that the closeness and dependency of solutions on functions involved therein.

5.5 Dependence on Parameters

Finally, we shall prove the dependency of solutions on real parameters.

We consider the following integral equations involving real parameters $\mu_1,\ \mu_2$:

$$D_*^{\frac{1}{2}} y(t) = \frac{3t}{5}\left[\frac{t-\sin(y(t))}{2}+\mu_1\right], \quad t \in [0,1], \tag{66}$$

with the given boundary condition

$$y(0)+y(1) = d = 1 \tag{67}$$

and

$$D_*^{\frac{1}{2}} \bar{y}(t) = \frac{3t}{5}\left[\frac{t-\sin(\bar{y}(t))}{2}+\mu_2\right], \quad t \in [0,1], \tag{68}$$

with the given boundary condition

$$\bar{y}(0)+\bar{y}(1) = \bar{d} = \frac{1}{2}. \tag{69}$$

Following above discussion, one can observe that $p(t) = \bar{p}(t) = r(t) = \frac{3t}{5}$ and therefore, we have $\Theta = \bar{\Theta}$. Hence by making similar arguments and from Theorem 5, one can have ($a = 0$, $b = 1$, $p(t) = \bar{p}(t) = r(t) = \frac{3t}{5}$)

$$\begin{aligned}
|y-\bar{y}|_B &\le \frac{3\left[\left(\frac{|d-\bar{d}|}{|m_1+m_2|}\right)+|\mu_1-\mu_2|\left(\frac{|m_2|}{|m_1+m_2|}I^\alpha r(b)+I^\alpha r(t)\right)\right]}{(1-\bar{\Theta})}\\
&\le \frac{3\left[\left(\frac{|1-\frac{1}{2}|}{2}\right)+|\mu_1-\mu_2|\left(\frac{1}{2}I^\alpha r(1)+I^\alpha r(t)\right)\right]}{(1-\Theta)}\\
&\le \frac{3\left[\frac{1}{4}+|\mu_1-\mu_2|\left(\frac{1}{2}I^{\frac{1}{2}}r(1)+I^{\frac{1}{2}}r(t)\right)\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)}\\
&\le \frac{3\left[\frac{1}{4}+|\mu_1-\mu_2|\frac{9}{20}\frac{1}{\Gamma(\alpha+2)}\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)}.
\end{aligned} \tag{70}$$

In particular, if we choose $\mu_1 = 1$, $\mu_2 = \frac{1}{2}$, then we have from (70) that

$$\begin{aligned}
\|y-\bar{y}\|_B &\le \frac{3\left[\frac{1}{4}+|\mu_1-\mu_2|\frac{9}{20}\frac{1}{\Gamma(\alpha+2)}\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)}\\
&\le \frac{3\left[\frac{1}{4}+\left|1-\frac{1}{2}\right|\frac{9}{20}\frac{1}{\Gamma(\frac{1}{2}+2)}\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{3\left[\frac{1}{4}+\frac{9}{40}\frac{1}{\Gamma\left(\frac{5}{2}\right)}\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)} \\
&\le \frac{3\left[\frac{1}{4}+\frac{9}{40}\frac{1}{\frac{3\sqrt{\pi}}{4}}\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)} \\
&\le \frac{3\left[\frac{1}{4}+\frac{3}{10\sqrt{\pi}}\right]}{\left(1-\frac{3}{5\sqrt{\pi}}\right)} \\
&\le \frac{1.2578}{0.6615} \\
&\simeq 1.9014. \qquad (71)
\end{aligned}$$

This proves that the dependency of solutions on both boundary data and real parameters.

Acknowledgment

The author are very grateful to the referees for their comments and remarks.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R.P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Georgian Mathematical Journal* **16** (2009), 401 – 411, DOI: 10.1515/gmj.2009.401.

[2] R.P. Agarwal, D. O'Regan and D. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *Journal of Nonlinear and Convex Analysis* **8** (2007), 61 – 79, URL: http://www.yokohamapublishers.jp/online2/jncav8.html.

[3] Y. Atalan and V. Karakaya, Iterative solution of functional Volterra-Fredholm integral equation with deviating argument, *Journal of Nonlinear and Convex Analysis* **18**(4) (2017), 675 – 684, URL: http://www.yokohamapublishers.jp/online2/jncav18-4.html.

[4] Y. Atalan and V. Karakaya, Stability of nonlinear Volterra-Fredholm integro differential equation: A fixed point approach, *Creative Mathematics and Informatics* **26**(3) (2017), 247 – 254, DOI: 10.37193/CMI.2017.03.01.

[5] Y. Atalan and V. Karakaya, An example of data dependence result for the class of almost contraction mappings, *Sahand Communications in Mathematical Analysis* **17**(1) (2020), 139 – 155, DOI: 10.22130/scma.2018.88562.464.

[6] Y. Atalan, F. Gürsoy and A.R. Khan, Convergence of S-iterative method to a solution of Fredholm integral equation and data dependency, *Facta Universitatis, Ser. Math. Inform.* **36**(4) (2021), 685 – 694, URL: http://casopisi.junis.ni.ac.rs/index.php/FUMathInf/article/view/4799/pdf.

[7] V. Berinde and M. Berinde, The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings, *Carpathian Journal of Mathematics* **21**(1-2) (2005), 13 – 20, URL: https://www.jstor.org/stable/43998882.

[8] V. Berinde, Existence and approximation of solutions of some first order iterative differential equations, *Miskolc Mathematical Notes* **11**(1) (2010), 13 – 26, DOI: 10.18514/MMN.2010.256.

[9] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surveys in Mathematics and Its Applications* **3** (2008), 1 – 12, URL: https://www.utgjiu.ro/math/sma/v03/p01.pdf.

[10] C.E. Chidume, Iterative approximation of fixed points of Lipschitz pseudocontractivemaps, *Proceedings of the American Mathematical Society* **129**(8) (2001), 2245 – 2251, DOI: 10.1090/S0002-9939-01-06078-6.

[11] R. Chugh, V. Kumar and S. Kumar, Strong convergence of a new three step iterative scheme in Banach spaces, *American Journal of Computational Mathematics* **2** (2012), 345 – 357, DOI: 10.4236/ajcm.2012.24048.

[12] V. Daftardar-Gejji and H. Jafari, Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives, *Journal of Mathematical Analysis and Applications* **328**(2) (2007), 1026 – 1033, DOI: 10.1016/j.jmaa.2006.06.007.

[13] F. Gürsoy and V. Karakaya, Some convergence and stability results for two new Kirk type hybrid fixed point iterative algorithms, *Journal of Function Spaces* **2014** (2014), 1 – 8, DOI: 10.1155/2014/684191.

[14] E. Hacioglu, F. Gürsoy, S. Maldar, Y. Atalan, and G.V. Milovanovic, Iterative approximation of fixed points and applications to two-point second-order boundary value problems and to machine learning, *Applied Numerical Mathematics* **167** (2021), 143 – 172, DOI: 10.1016/j.apnum.2021.04.020.

[15] N. Hussain, A. Rafiq, B. Damjanović and R. Lazović, On rate of convergence of various iterative schemes, *Fixed Point Theory and Applications* **1** (2011), 1 – 6, DOI: 10.1186/1687-1812-2011-45.

[16] S. Ishikawa, Fixed points by a new iteration method, *Proceedings of the American Mathematical Society* **44**(1) (1974), 147 – 150.

[17] S.M. Kang, A. Rafiq and Y.C. Kwun, Strong convergence for hybrid S-iteration scheme, *Journal of Applied Mathematics* **2013** (2013), Article ID 705814, 1 – 4,DOI: 10.1155/2013/705814.

[18] S.H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory and Applications* **1** (69)2013, 1 – 10, DOI: 10.1186/1687-1812-2013-69.

[19] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, **204** (2006), pages 1 – 523, URL: https://www.sciencedirect.com/science/book/9780444518323.

[20] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, Mathematics in Science and Engineering Series, Vol. **198**, pages 1 – 340, Academic Press, New York (1999), URL: https://www.sciencedirect.com/bookseries/mathematics-in-science-and-engineering/vol/198/suppl/C.

[21] D.R. Sahu, Applications of the S-iteration process to constrained minimization problems and split feasibility problems, *Fixed Point Theory* **12**(1) (2011), 187 – 204, UIRL: http://www.math.ubbcluj.ro/~nodeacj/vol__12(2011)_no_1.php.

[22] D.R. Sahu and A. Petruşel, Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces, *Nonlinear Analysis: Theory, Methods & Applications* **74**(17) (2011), 6012 – 6023, DOI: 10.1016/j.na.2011.05.078.

[23] S.M. Şoltuz and, Data dependence for Ishikawa iteration, *Lecturas Matemáticas* **25**(2) (2004), 149 – 155, URL: http://scm.org.co/archivos/revista/Articulos/755.pdf.

[24] J.Y. Sun, Z. Zeng and J. Song, Existence and uniqueness for the boundary value problems of nonlinear fractional differential equation, *Applied Mathematics* **8**(3) (2017), 312 – 323, DOI: 10.4236/am.2017.83026.

[25] S.Q. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electronic Journal of Differential Equations* **2006**(36) (2006), 1 – 12, URL: https://ejde.math.txstate.edu/Volumes/2006/36/zhang.pdf.