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Research Article

Analytical Solution of Diffraction by a Composite Cylinders Above a Special Bottom Undulation in Uniform Water

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Abstract. We consider the co-axial cylindrical structure as a composite submerged solid cylinder above a special bottom undulation, i.e., a circular plate at the impermeable horizontal bottom. We consider the diffraction problem of the proposed structure in water of finite depth. This diffraction problem can be expressed as a wave energy converter. The variables of separation and eigenfunction expansion methods are utilized to determine the analytical solutions for the diffraction problem in their identified sub-domains. By using the methods of expansion of eigenfunction and the orthogonality of Bessel functions to the expression of the diffracted velocity potentials. We achieve a system of linear equations after suitably truncated to the obtain infinite series. This work may be helpful to the wave designer to design appropriate device so that one can extract maximum wave energy. The created wave energy may be used in many applications of conventional energy.

Keywords. Diffraction, Uniform depth, Virtual boundary, Exciting forces, Analytical solution

Mathematics Subject Classification (2020). 76B07, 76B15

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1. Introduction

The interaction of water waves due to the different geometrical shapes have been investigated by a number of researchers on basis of the linear water waves theory. Here, the current study is also affiliated to the water waves diffraction by composite cylinder and bottom-mounted cylinder.

We consider a semi-submerged composite cylindrical structure in presence of coaxial bottom-mounted cylindrical plate in a uniform water depth. This proposed device can be considered a one of the wave energy device. Appropriate positioning of this wave device help us to generate maximum energy. Such wave device assumes vital acceptation in the application of water waves. An exact prognosis of the exerted wave power by the proposed device is prime importance in order to frame most suitable energy device.

In this research field, there are few related works have been done by a number of researchers. Bhatta and Rahman [2] evaluated the wave forces corresponding to the translational and rotational motions of the cylinder in water having finite depth. Garrett [3] has determined and shown the various forces on basis of Galerkin's method for the solution of the problem numerically. Hassan and Bora [4] discussed the wave loads due to two coaxial cylinders demonstrated the wave forces for various parameters of the device. Rahman and Bhatta [6] investigated non linear wave forces exerted by cylinders on basis of theorem of Graf's addition of Bessel functions. Shen *et al*. [7] investigated wave loads of a rectangular structure under bottom sill. Siddorn and Taylor [8] discussed the encounter of waves of water with a series of cylinders and evaluated wave forces, hydrodynamic coefficients to the cylinders Wu *et al*. [9, 10] discussed the water wave problems for two solid cylinders and determined the respective potentials with the help of the expansion method of eigenfunction and discuss the effect of the caisson. The radiation and diffraction forces were shown for different ratios of radii of cylinders. Zhang *et al*. [11] discuss the water wave problems by two cylinders in uniform depth and determine the solution of the problems, and evaluated radiation force under some particular cases. As far as a composite coaxial floating cylinder above a bottom-mounted cylinder placed in uniform water depth is exercised, the authors did not find any analysis that has used an analytical solution based on separation of variable and eigenfunction expansion methods. Since the proposed device can be assume as one of the wave energy device. The derived solutions of the proposed problem of diffraction for the coaxial composite cylinder above a bottom mounted cylinder and it is assumed to help the designer and engineer for selecting physical variables of the proposed model, so that one can extract maximum energy.

2. Formulation of the Problem

Let us consider the motion is irrotational and the fluid incompressible then we formulate this diffraction problem with the help of the theory of linear water waves. The depth of the ocean assume to a height h. The system of Cartesian co-ordinate is identified along with the undisturbed free surface is $z = 0$ and z-axis measured positive in the upwards direction. The propagating wave is along the direction of the x-axis. Let us consider the radii of cylinders are R' and R, as shown in Figure 1. The drafts of the cylinders are denoted by h_1, h_2 and h_3.

Let the velocity potential to be $\Phi(r,\theta,z,t) = \Re[\phi(r,\theta,z)\, e^{-i\omega t}]$ here, ω is denoting the angular frequency, Re stands the real part, and the spatial part $\phi(r,\theta,z)$ obeys Laplace's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.1}$$

The potential ϕ is divided into the diffracted ϕ_d and the incident potential ϕ_i due to the consideration of diffraction problem only.

Since the fluid domain is fractionated into four subdomains, so let the diffraction potentials be ϕ_1, ϕ_2, ϕ_3 and ϕ_4 in the identified sub-domains I, II, III and IV, respectively as exhibit in Figure 1. The potential of incident wave having angular frequency ω and of unit amplitude, coming from the positive x-direction is written by (MacCamy and Fuchs [5])

$$\phi_i = -\frac{ig}{\omega}\frac{\cosh k(z+h)}{\cosh(kh)}\sum_{n=0}^{\infty}\mu_n J_n(kr)\cos n\theta, \tag{2.2}$$

where $i=\sqrt{-1}$, g are purely imaginary number and the gravitational acceleration, respectively, k denotes the wavenumber which to be obtained by using the dispersion equation $\omega^2 = gk\tanh(k\cdot h)$; $J_n(\cdot)$ be Bessel function of order n, whereas μ_n can be expressed as

$$\mu_n = \begin{cases} 2i^n & n>0, \\ 1 & n=0. \end{cases} \tag{2.3}$$

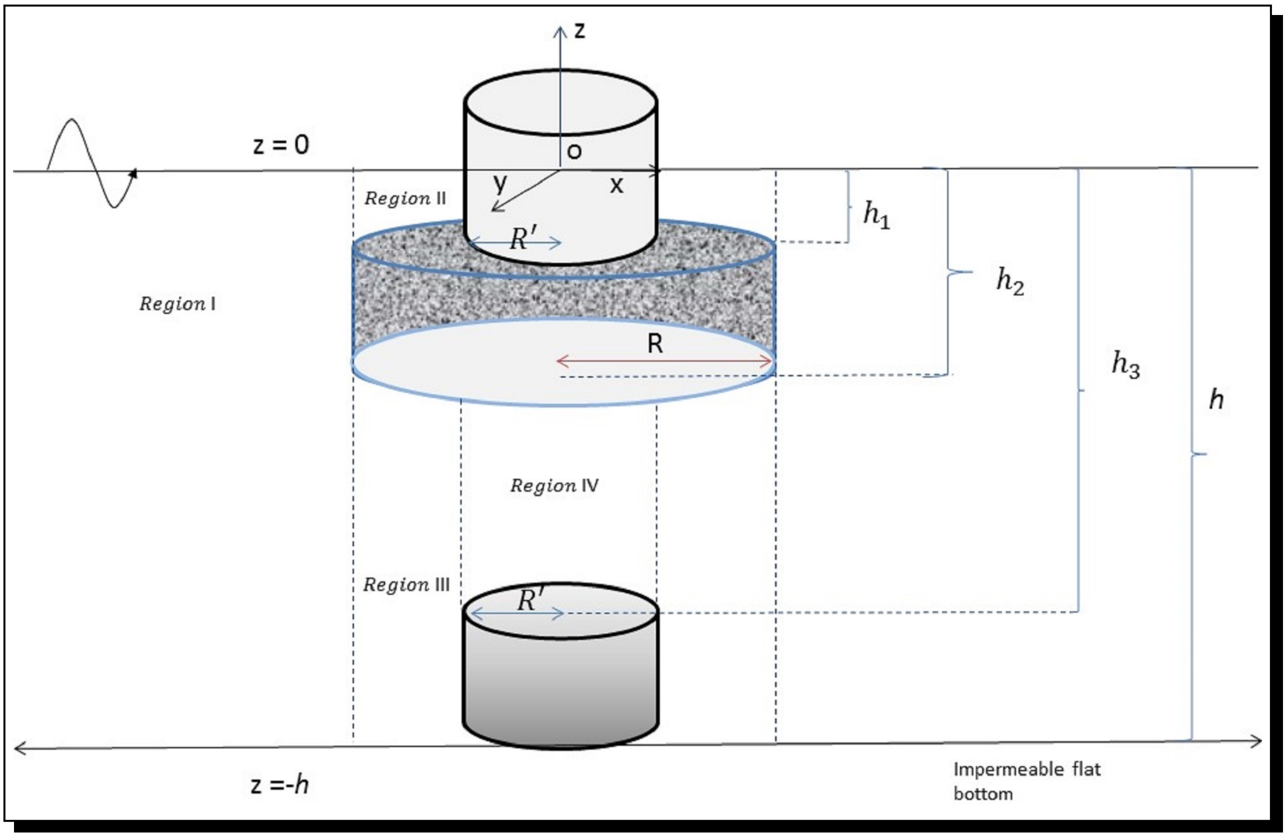

Figure 1. Diagram of the device

3. Boundary-value Problems

We split the whole fluid domain of the proposed device into four sub-domains as identified. Therefore, we can formulate the four boundary value problems as follows:

The total diffracted potential Φ is revealed as $\Phi = Re[\phi(r,\theta,z)e^{-i\omega t}]$, here ϕ is the spatial part with $0 \le \theta < 2\pi$ and satisfy the following governing equation:

$$\nabla^2\phi = 0;\ (0 < r < \infty, -h < z < 0, \text{or}, -h_1 < z < 0, \text{or}, -h < z < -h_2, -h_3 < z < -h_2). \tag{3.1}$$

3.1 The Boundary-Value Problem for Sub-domain I

For this particular sub-domain the diffraction potential Φ_1 is revealed in the form of $\Phi = Re[\phi_1(r,\theta,z)e^{-i\omega t}]$, heree ϕ_1 is the spatial part along with $0 \le \theta < 2\pi$, executes the following boundary-value problem:

$$\nabla^2\phi_1 = 0\,; \qquad (R < r < \infty,\ -h < z < 0\,), \tag{3.2}$$

$$\frac{\partial\phi_1}{\partial z} - \frac{\omega^2}{g}\phi_1 = 0\,; \qquad (\text{at } z = 0), \tag{3.3}$$

$$\frac{\partial\phi_1}{\partial z} = 0\,; \qquad (\text{at } z = -h), \tag{3.4}$$

$$\frac{\partial(\phi_1 + \phi_i)}{\partial r} = 0\,; \qquad (-h_2 < z < -h_1,\ \text{at } r = R). \tag{3.5}$$

The condition which is called far-field condition which is valid in the sub-domain I only as it is unbounded sub-domain and given by

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial\phi_1}{\partial r} - ik\phi_1\right) = 0. \tag{3.6}$$

3.2 The Boundary-Value Problem for Sub-domain II

For this sub-domain the diffracted velocity potential Φ_2 with $0 \le \theta \le 2\pi$ can be revealed as Φ=$Re[\phi_2(r,\theta,z)e^{-i\omega t}]$, here the spatial part ϕ_2 which excutes to the boundary-value problem:

$$\nabla^2\phi_2 = 0\,; \qquad (R' < r < R, -h_1 < z < 0), \tag{3.7}$$

$$\frac{\partial\phi_2}{\partial z} - \frac{\omega^2}{g}\phi_2 = 0\,; \qquad (\text{at } z = 0), \tag{3.8}$$

$$\frac{\partial\phi_2}{\partial z} = 0\,; \qquad (R' < r < R,\ \text{at } z = -h_1), \tag{3.9}$$

$$\frac{\partial(\phi_2 + \phi_i)}{\partial r} = 0\,; \qquad (-h_1 < z < 0, \text{at } r = R'). \tag{3.10}$$

3.3 The Boundary-Value Problem for Sub-domain III

For this sub-domain the diffracted velocity potential Φ_3 along with $0 \le \theta \le 2\pi$ is revealed as $\Phi = Re[\phi_3(r,\theta,z)e^{-i\omega t}]$, here ϕ_3 is the spatial part which executes the following boundary-value problem:

$$\nabla^2\phi_3 = 0\,; \qquad (R' < r < R, -h < z < -h_2), \tag{3.11}$$

$$\frac{\partial\phi_3}{\partial z} = 0\,; \qquad (R' < r < R,\ \text{at } z = -h), \tag{3.12}$$

$$\frac{\partial\phi_3}{\partial z} = 0\,; \qquad (\text{at } z = -h_2\, R' < r < R), \tag{3.13}$$

$$\frac{\partial(\phi_3+\phi_i)}{\partial r}=0\,; \qquad (-h<z<-h_3,\ \text{at } r=R'). \tag{3.14}$$

3.4 The Boundary-Value Problem for Sub-domain IV

As same way, for this sub-domain also, the diffracted potential Φ_4 is revealed as $\Phi = Re[\phi_4(r,\theta,z)e^{-i\omega t}]$, here also ϕ_4 with $0 \le \theta < 2\pi$ which executes to the boundary-value problem as:

$$\nabla^2\phi_4=0\,; \qquad (0<r<R',\ -h_3<z<-h_2), \tag{3.15}$$

$$\frac{\partial\phi_4}{\partial z}=0\,; \qquad (0<r<R',\ \text{at } z=-h_3), \tag{3.16}$$

$$\frac{\partial\phi_4}{\partial z}=0\,; \qquad (0<r<R',\ \text{at } z=-h_2). \tag{3.17}$$

We can find the solutions of the respective boundary-value problems in each specified sub-domain. In the above equations, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are representing the respective diffraction potentials in sub-domains I, II, III and IV, respectively.

4. Solution to the Boundary Value Problems

Diffracted Potentials

With the help of the method of variables of separation in each sub-domain, we can determine the diffracted potentials in respective sub-domains by utilizing variables of separation method. Since the obtained expressions are in the infinite series of orthogonal functions. As the continuity of pressure and fluid flow must be preserved, so we apply this continuity at the interface of physical and virtual boundary between the sub-domains, in order to use it, apply the matching conditions at the interface between the virtual and physical boundary at $r=R$ and $r=R'$, respectively. Hence, the expression of diffraction potentials in respective sub-domain from Wu *et al*. [10], can be given by

$$\phi_1=\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}A_{p,q}\frac{Z_p(\alpha_q r)}{Z_p(\alpha_q R)}\cos[\alpha_q(z+h)]\cos p\theta, \tag{4.1}$$

$$\phi_2=-\phi_i+\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\left[B_{p,q}\frac{R_p(\beta_q r)}{R_p(\beta_q R')}+C_{p,q}\frac{S_p(\beta_q r)}{S_p(\beta_q R')}\right]\cos[\beta_q(z+h_1)]\cos p\theta, \tag{4.2}$$

$$\phi_3=-\phi_i+\sum_{p=0}^{\infty}\left[D_{p,1}r^p+\sum_{q=2}^{\infty}D_{p,q}\frac{I_p(\gamma_q r)}{I_p(\gamma_q R')}+\sum_{q=2}^{\infty}E_{p,q}\frac{K_p(\gamma_q r)}{K_p(\gamma_q R')}\right]\cos[\gamma_q(z+h)]\cos p\theta, \tag{4.3}$$

$$\phi_4=-\phi_i+\sum_{p=0}^{\infty}\left[F_{p,1}r^p+\sum_{q=2}^{\infty}F_{p,q}\frac{I_p(\eta_q r)}{I_p(\eta_q R')}\cos[\eta_q(z+h_2)]\right]\cos p\theta, \tag{4.4}$$

where $A_{p,q}$, $B_{p,q}$, $C_{p,q}$, $D_{p,q}$, $E_{p,q}$ and $F_{p,q}$ are the undetermined coefficients and α_n, β_n and γ_n are determined from the following dispersion relations which are given as:

$$\begin{cases}\alpha_q=-ik,\ \omega^2=gk\tanh(kh), & q=1,\\ \omega^2=-g\alpha_q\tan(\alpha_q h), & q=2,3,\dots,\end{cases} \tag{4.5}$$

$$\begin{cases} \beta_q = -ik',\ \omega^2 = gk' \tanh(k'h_1), & q = 1, \\ \omega^2 = -g\beta_q \tan(\beta_q h_1), & q = 2,3,\dots, \end{cases} \tag{4.6}$$

$$\gamma_q = \frac{(q-1)\pi}{h-h_2}, \quad q = 1,2,3,\dots, \tag{4.7}$$

and

$$\eta_q = \frac{(q-1)\pi}{h_3-h_2}, \quad q = 1,2,3,\dots, \tag{4.8}$$

along with k, k' respectively, denoting the wave numbers in the sub-domains I and II, respectively.

The functions $Z_p(\cdot)$, $R_p(\cdot)$ and $S_p(\cdot)$ are expressed as:

$$Z_p(\alpha_q r) = H_p^{(1)}(i\alpha_1 r) = H_p^{(1)}(kr), \quad n = 1, \tag{4.9}$$

$$Z_p(\alpha_q r) = K_p(\alpha_q r), \quad n = 2,3,\dots, \tag{4.10}$$

$$R_p(\beta_q r) = H_p^{(1)}(k'r), \quad n = 1, \tag{4.11}$$

$$R_p(\beta_q r) = K_p(\beta_q r), \quad n = 2,3,\dots, \tag{4.12}$$

$$S_p(\beta_q r) = H_p^{(2)}(k'r), \quad n = 1, \tag{4.13}$$

$$T_p(\beta_q r) = I_p(\beta_q r), \quad n = 2,3,\dots. \tag{4.14}$$

Here, $H_p^{(1)}(\cdot)$ stands Hankel functions of order p, of first kind, $H_p^{(2)}(\cdot)$ denotes Hankel functions of order p, of second kind, and $I_p(\cdot)$ and $K_p(\cdot)$ are, the modified Bessel functions of order p.

All Matching Conditions

For the continuity of the flow, i.e., the continuity of the pressure and velocity at the interface between sub-domains must be ensured. So, we apply matching conditions along the interface of virtual and physical boundary as mentined in Figure 1. Along the interface of physical and the virtual boundary at $r = R$, between sub-domains I and II, and I and III, we have

$$\phi_1 = \phi_2, \quad (-h_1 \le z \le 0), \tag{4.15}$$

$$\phi_1 = \phi_3, \quad (-h \le z \le -h_2), \tag{4.16}$$

$$\frac{\partial \phi_1}{\partial r} = \begin{cases} \frac{\partial \phi_2}{\partial r}, & (-h_1 \le z \le 0), \\ -\frac{\partial \phi_i}{\partial r}, & (-h_2 \le z \le -h_1) \\ \frac{\partial \phi_3}{\partial r}, & (-h \le z \le -h_2). \end{cases} \tag{4.17}$$

Along the interface of the physical and the virtual boundary at $r = R'$, between sub-domains III and IV, we have

$$\phi_3 = \phi_4, \quad (-h_3 \le z \le -h_2), \tag{4.18}$$

$$\frac{\partial \phi_3}{\partial r} = \begin{cases} \frac{\partial \phi_4}{\partial r} & (-h_3 \le z \le -h_2), \\ -\frac{\partial \phi_i}{\partial r} & (-h \le z \le -h_3). \end{cases} \tag{4.19}$$

We use this matching conditions to obtain the unknown coefficients of the diffracted potential. Therefore, we put the expressions of diffracted potential in the matching conditions and subsequently, by using the orthogonality of cosine functions in the range $-h < z < 0$. Hence, we have a system of linear equations after truncating appropriately to a finite number ($N = 30$) to all obtained infinite series. The obtain linear system' of equations is of the form

$$Z_m X_m = Y_m, \tag{4.20}$$

where $X_m = [A_{i1}, A_{i2}, \dots, A_{iN}, B_{i1}, B_{i2}, \dots, B_{iN}, C_{i1}, C_{i2}, \dots, C_{iN}, D_{i1}, D_{i2}, \dots, D_{iN}, E_{i1}, E_{i2}, \dots, E_{iN}, F_{i1}, F_{i2}, \dots, F_{iN}]^T$, are the unknown coefficients, Z_m is square matrix which contains coefficients and this coefficients are evaluated on basis of [1], and Y_m is the column vectors. One can evaluate all unknown coefficients by using numerical tool to this system of linear equations, e.g., Matlab tool. The obtained solutions of this current problem for the proposed energy device is anticipated to assist the engineer in finding the appropriate parameters of the device so that maximum energy can be taken out.

5. Conclusions

On the basis of linear water and potential wave theory, we have formulated and solved the diffraction problem due to the encounter of water waves with a composite cylinder above a co-axial bottom mounted circular cylinder in uniform water depth. We divided the whole fluid domain into four sub-domains and formulated respective boundary value problem for each sub-domain. Hence, with the help of variables of separation and matched eigenfunction expansion, we have derived the respective diffracted velocity potentials of the boundary-value problem in each sub-domain. On application of matched eigenfunction expansion method due to the continuity of fluid velocity and pressure, we leads to the system of equations in linear form and then by using numerical technique to solve system of algebraic equations. The obtained solutions of this current problem for the proposed energy device is anticipated to assist the engineer in finding the appropriate parameters of the model in order to take maximum power.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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