Communications in Mathematics and Applications
Vol. 13, No. 3, pp. 921–933, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i3.1789

http://www.rgnpublications.com

Research Article

A Study on Parametric S-metric Spaces

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Received: January 10, 2022 **Accepted:** June 21, 2022

Abstract. In this paper, some topological properties are discussed and some results on completeness, totally boundedness, compactness on parametric S-metric spaces are developed.

Keywords. Parametric S-metric, Parametric S-metric topology, Totally bounded set

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

In 1906, French mathematician M. Frechet [6] introduced metric space as a general notion of distance function for abstract spaces. After that several researchers have tried to generalize the concept of metric space by modifying or reducing some of the metric axioms. As a result we have come to know many generalized metric spaces ([2–4, 8, 9, 11]). Parametric S-metric is one of those generalized distance functions, brought to light by Taş and Özgür [13] in 2016. There is another one direction of investigation which includes the generalization of the celebrated Banach Contraction Principle [1]. It may be replacement of contractive condition or reconstruction in a new setting.

In 2014, Hussain *et al.* [7] introduced the notion of parametric metric by adjoining a parameter $t\,(>0)$ to the metric axioms and established the notion of convergence of a sequence, Cauchy sequence, etc. in that space. Many authors investigated fixed-point theorems for some contraction mappings on a complete parametric and triangular intuitionistic fuzzy metric space ([5, 7, 10, 12]). On the other hand, Taş and Özgür [13] generalized the notion of a parametric metric [7] and S-metric [11] by developing parametric S-metric. After defining convergence of a sequence and completeness in parametric S-metric spaces, authors proved fixed point theorems for expansive mappings on parametric S-metric spaces [13].

Some of the generalized metrics are failed to satisfy all the standard metric properties. That is why, in this paper, our motive is to study whether parametric S-metric satisfies all the standard metric properties or not. We establish some topological results including completeness, boundedness, and compactness, etc. also and justify the promising results by proper examples.

The organization of this article is as follows:

Section 2, consists of some preliminary results, related to our work. In Section 3, we have discussed open and closed balls and parametric S-metric topology in Section 4, there are some results related to the completeness, boundedness, and compactness of parametric S-metric spaces.

2. Preliminaries

Definition 2.1 ([7])**.** Let X be a non-empty set. A function $P: X\times X\times(0,\infty)\to\mathbb{R}_{\geq 0}$ is said to be a parametric metric if it satisfies:

(i) $P(a,b,t)=0$, $\forall\ t>0$ if and only if $a=b$;

(ii) $P(a,b,t)=P(b,a,t)$, $\forall\ a,b\in X$ and $\forall\ t>0$;

(iii) $P(a,b,t)\leq P(a,x,t)+P(b,x,t)$, $\forall\ a,b,x\in X$ and $\forall\ t>0$.

The pair (X,P) is called a parametric metric space.

Definition 2.2 ([11])**.** Let X be a nonempty set. Define a function $S: X\times X\times X\to\mathbb{R}_{\geq 0}$ which satisfies the following conditions:

(S1) $S(x,y,z)=0$ if and only if $x=y=z$;

(S2) $S(x,y,z)\leq S(x,x,w)+S(y,y,w)+S(z,z,w)$, $\forall\ x,y,z,w\in X$.

The function S is called an S-metric and the pair (X,S) is called an S-metric space.

Definition 2.3 ([13])**.** Let X be a non-empty set. A parametric S-metric, denoted by P_S is a function defined from $X\times X\times X\times(0,\infty)$ to $\mathbb{R}_{\geq 0}$ which satisfies

(i) $P_S(a,b,c,t)=0$, $\forall\ t>0$ if and only if $a=b=c$;

(ii) $P_S(a,b,c,t)\leq P_S(a,a,x,t)+P_S(b,b,x,t)+P_S(c,c,x,t)$, $\forall\ a,b,c,x\in X$ and $\forall\ t>0$.

The pair (X,P_S) is called a parametric S-metric space.

Some definitions and results in parametric S-metric space are given below.

Lemma 2.4 ([13])**.** *In a parametric S-metric space* (X,P_S), $P_S(x,x,y,t)=P_S(y,y,x,t)$, $\forall\ x,y\in X$ *and* $\forall\ t>0$.

Definition 2.5 ([13])**.** Let(X,P_S) be a parametric S-metric space and $\{x_n\}\subset X$ then

(a) $\{x_n\}$ is said to converge to x if for any $\epsilon>0$, $\exists\ N\in\mathbb{N}$ such that

$$P_S(x_n,x_n,x,t)<\epsilon,\quad \forall\ n>N \text{ and } \forall\ t>0.$$

(b) $\{x_n\}$ is said to be a Cauchy sequence if for any $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$P_S(x_n, x_n, x_m, t) < \epsilon, \quad \forall\ m, n > N \text{ and } \forall\ t > 0.$$

(c) (X, P_S) is called complete if every Cauchy sequence is convergent.

Proposition 2.6 ([13]). *In a parametric S-metric space*

(i) *limit of a sequence is unique.*

(ii) *every convergent sequence is Cauchy.*

3. Parametric S-metric Topology

In this section, some basic topological results in parametric S-metric spaces are developed.

Definition 3.1. Let (X, P_S) be a parametric S-metric space. For $x \in X$ and $r > 0$, we define open ball and closed ball center at x with radius r as

$$B(x, r) = \{y \in X : P_S(y, y, x, t) < r,\ \forall\ t > 0\},$$
$$B[x, r] = \{y \in X : P_S(y, y, x, t) \leq r,\ \forall\ t > 0\}.$$

Proposition 3.2. *In a parametric S-metric space (X, P_S), for each $x \in X$, $B(x, r) \subseteq B(x, s)$ if and only if $r \leq s$.*

Proof. Proof is straightforward. □

Theorem 3.3. *Let (X, P_S) be a parametric S-metric space and define*

$$\tau = \{G \subseteq X : \text{for each } x \in G, \exists\ r > 0 \text{ such that } B(x, r) \subseteq G\}.$$

Then τ is a topology on X.

Proof. Obviously $\phi, X \in \tau$ and τ is closed under arbitrary union. To check the closedness of τ under finite intersection, let us consider $G_1, G_2 \in \tau$. We need to show $G_1 \cap G_2 \in \tau$.
Take any $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$. So $\exists\ r_1, r_2 > 0$ such that $B(x, r_1) \subseteq G_1$ and $B(x, r_2) \subseteq G_2$.
Now if $r = \min\{r_1, r_2\}$, then $B(x, r) \subseteq B(x, r_1) \subseteq G_1$ and $B(x, r) \subseteq B(x, r_2) \subseteq G_2$.
Thus $B(x, r) \subseteq G_1 \cap G_2$. So, $G_1 \cap G_2 \in \tau$. □

Definition 3.4. Let (X, P_S) be a parametric S-metric space and $A \subseteq X$. Then

(i) A is said to be an open set if $A \in \tau$.

(ii) A is said to be a closed set if $X \setminus A \in \tau$.

(iii) $x \in X$ is said to be a limit point of A if for any $\epsilon > 0$, $(B(x, \epsilon) \setminus \{x\}) \cap A$ is nonempty.

(iv) Closure of A, denoted by $\bar{A}$, is the set containing A and all the limit points of A.

Remark 3.5. In parametric S-metric spaces, an open ball may not be an open set.
For, we consider $X = \{f \mid f : (0, \infty) \to \mathbb{R}$ be a function $\}$ and a parametric S-metric function defined as $P_S(f, g, h, t) = |f(t) - h(t)| + |g(t) - h(t)| + |f(t) - g(t)|$, $\forall\ f, g, h \in X$ and $\forall\ t > 0$.

Now consider the open ball $B(0,2) \subset X$ where

$$B(0,2) = \{f \in X \mid P_S(f,f,0,t) < 2,\ \forall\ t > 0\} = \{f \in X \mid |f(t)| < 1,\ \forall\ t > 0\}.$$

Let $f(t) = 1 - \frac{e^{-t}}{2}$, $\forall\ t > 0$. Then $|f(t)| < 1$, $\forall\ t > 0$ which implies $f \in B(0,2)$.
If possible suppose that $B(0,2)$ is an open set. Then as $f \in B(0,2)$, there exists a positive real number s such that $B(f,s) \subset B(0,2)$.
Now, we define a function g on $(0,\infty)$ by

$$g(t) = \begin{cases} f(t), & \text{if } 0 < t \le \ln\left(\dfrac{4}{s}\right), \\ 1 + \frac{s}{4} - e^{-t}, & \text{if } t > \ln\left(\dfrac{4}{s}\right). \end{cases}$$

Then

$$P_S(f,f,g,t) = 2|g(t) - f(t)| = \begin{cases} 0, & \text{if } 0 < t \le \ln\left(\dfrac{4}{s}\right), \\ 2\left|\frac{s}{4} - \frac{e^{-t}}{2}\right|, & \text{if } t > \ln\left(\dfrac{4}{s}\right). \end{cases}$$

Now

$$\begin{aligned} & \ln\left(\frac{4}{s}\right) < t < \infty \\ \Longrightarrow\quad & \frac{s}{8} < \frac{s}{4} - \frac{e^{-t}}{2} < \frac{s}{4} \\ \Longrightarrow\quad & \frac{s}{4} < 2\left|\frac{s}{4} - \frac{e^{-t}}{2}\right| < \frac{s}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} & 2|g(t) - f(t)| < s, \quad \forall\ t > 0 \\ \Longrightarrow\quad & P_S(f,f,g,t) < s, \quad \forall\ t > 0 \\ \Longrightarrow\quad & g \in B(f,s) \end{aligned}$$

Again for $\ln\left(\frac{4}{s}\right) < t < \infty$, we have

$$\begin{aligned} & e^{-t} < \frac{s}{4} \\ \Longrightarrow\quad & 1 + \frac{s}{4} - e^{-t} > 1 \\ \Longrightarrow\quad & g(t) > 1 \\ \Longrightarrow\quad & g \notin B(0,2) \end{aligned}$$

This contradicts that $B(f,s) \subset B(0,2)$. Hence $B(0,2)$ is not an open set.

Remark 3.6. Since open ball may not be an open set, hence the collection of open balls is not necessarily form a basis for the topology.

Theorem 3.7. *In a parametric S-metric space (X,P_S), every closed ball is a closed set.*

Proof. For any $x \in X$ and $r > 0$ consider the closed ball $B[x,r]$. To prove $B[x,r]$ is closed, it is enough to show that $X \setminus B[x,r] = A(\text{say})$ is open.

Choose $y \in A$. Then there exist a $t_0 > 0$ such that $P_S(y,y,x,t_0) > r$.
Let $P_S(y,y,x,t_0) = r_{t_0}$ (depends on t_0).
Take $s_{t_0} = \frac{r_{t_0}-r}{2} > 0$ and take $z \in B(y,s_{t_0})$. Then

$$\begin{aligned} & P_S(y,y,x,t_0) \le 2P_S(y,y,z,t_0) + P_S(z,z,x,t_0) \\ \implies\quad & P_S(z,z,x,t_0) \ge P_S(y,y,x,t_0) - 2P_S(y,y,z,t_0) \\ \implies\quad & P_S(z,z,x,t_0) > r_{t_0} - 2s_{t_0} = r \end{aligned}$$

Thus, there exist atleast one $t_0 > 0$ such that $P_S(z,z,x,t_0) > r$ which implies $z \notin B[x,r]$.
Therefore, $z \in B(y,s_{t_0})$ implies $z \in A$. Hence A is an open set, consequently $B[x,r]$ is a closed set. □

Theorem 3.8. *(X,P_S) is a Hausdorff space.*

Proof. Suppose (X,P_S) is not a Hausdorff space. Then for some $x \neq y \in X$, there does not exist any $r > 0$ such that $B\left(x,\frac{r}{3}\right) \cap B\left(y,\frac{r}{3}\right) = \phi$. So $\forall\ r > 0$ there exist $z_r \in B\left(x,\frac{r}{3}\right) \cap B\left(y,\frac{r}{3}\right)$ which implies $P_S(x,x,z_r,t) < \frac{r}{3}$ and $P_S(y,y,z_r,t) < \frac{r}{3}$, $\forall\ t > 0$. Then $\forall\ t > 0$, we have

$$\begin{aligned} P_S(y,y,x,t) &\le P_S(y,y,z_r,t) + P_S(y,y,z_r,t) + P_S(x,x,z_r,t) \\ &= 2P_S(y,y,z_r,t) + P_S(x,x,z_r,t) \\ &< 2\frac{r}{3} + \frac{r}{3} = r\,. \end{aligned}$$

Since $r > 0$ is chosen arbitrarily, we have $P_S(y,y,x,t) = 0$, $\forall\ t > 0$ which implies $x = y$. This is a contradiction to our assumption.
Hence (X,P_S) is a Hausdorff space. □

4. Completeness and Compactness in Parametric S-metric Space

In the previous section, we have shown that an open ball is necessarily not an open set, so (X,P_S) is not a metrizable topological space. For this, it is necessary to study the compactness including other characteristics of parametric S-metric spaces.

4.1 Bounded Set and Completeness

Definition 4.1. Let (X,P_S) be a parametric S-metric space and $F \subseteq X$. F is said to be bounded if $\exists\ K > 0$ such that $P_S(x,x,y,t) \le K$, $\forall\ x,y \in F$ and $\forall\ t > 0$.

Remark 4.2. In a parametric S-metric space, a finite set may not be bounded with respect to t.

Example 4.3. We consider the parametric S-metric space (X,P_S) where $X = \mathbb{R}$ and

$$P_S(a,b,c,t) = t[|a-b| + |b-c| + |c-a|], \quad \forall\ a,b,c \in X \ \text{ and } \ \forall\ t > 0.$$

We choose a subset A of X where $A = \{1,2,3\}$. It is clear that there does not exists any $K > 0$ such that $P_S(x,x,y,t) \le K$, $\forall\ x,y \in A$ and $\forall\ t > 0$ hold.

Definition 4.4. Diameter of $F \subseteq X$, denoted by $\delta(F)$ and defined by $\delta(F) = \sup_{x,y \in F} \sup_{t>0} P_S(x,x,y,t)$. Moreover, if $\delta(F) < \infty$, then F is bounded.

Proposition 4.5. *For a subset A of a parametric S-metric space (X,P_S), $\delta(\bar{A}) = \delta(A)$.*

Proof. Since, $A \subseteq \bar{A}$, so

$$\delta(A) \leq \delta(\bar{A}). \tag{4.1}$$

Let $\epsilon > 0$ and $x, y \in \bar{A}$.
So there exist $x_1, y_1 \in A$ such that $x_1 \in B\left(x, \frac{\epsilon}{4}\right) \cap A$ and $y_1 \in B\left(y, \frac{\epsilon}{4}\right) \cap A$.
Then, $\forall\ t > 0$, $P_S(x,x,x_1,t) < \frac{\epsilon}{4}$ and $P_S(y,y,y_1,t) < \frac{\epsilon}{4}$. Thus,

$$\begin{aligned}
P_S(x,x,y,t) &\leq P_S(x,x,x_1,t) + P_S(x,x,x_1,t) + P_S(y,y,x_1,t) && \forall\ t>0\\
&\leq 2P_S(x,x,x_1,t) + P_S(y,y,y_1,t) + P_S(y,y,y_1,t) + P_S(x_1,x_1,y_1,t) && \forall\ t>0\\
&= 2P_S(x,x,x_1,t) + P_S(y,y,y_1,t) + P_S(x_1,x_1,y_1,t) && \forall\ t>0\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} + P_S(x_1,x_1,y_1,t) && \forall\ t>0\\
&\leq \epsilon + \delta(A) && \forall\ t>0.
\end{aligned}$$

Since x, y are two arbitrary elements of $\bar{A}$, so $\delta(\bar{A}) \leq \epsilon + \delta(A)$, which gives

$$\delta(A) \geq \delta(\bar{A}). \quad \text{(since } \epsilon > 0 \text{ is arbitrary)} \tag{4.2}$$

From (4.1) and (4.2), we get $\delta(A) = \delta(\bar{A})$. □

A formal definition of complete parametric S-metric space is given by Taş and Özgür [13]. Here we establish Cantor's intersection type theorem in parametric S-metric space.

Theorem 4.6 (Cantor's Intersection Type Theorem)**.** *A necessary and sufficient condition that the parametric S-metric space (X,P_S) be complete is that every nested sequence of non-empty closed subsets $\{F_i\}$ with $\delta(F_i) \to 0$ as $i \to \infty$ be such that $F = \bigcap_{i=1}^{\infty} F_i$ contains exactly one point.*

Proof. First suppose that X is complete. Consider a sequence of closed subsets $\{F_i\}$ such that $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $\delta(F_i) \to 0$ as $i \to \infty$.
Choose $a_n \in F_n$, $\forall\ n \in \mathbb{N}$. We verify that the sequence $\{a_n\}$ is a Cauchy sequence.
Now, $a_n \in F_n$ and $\forall\ p \in \mathbb{N}$, $a_{n+p} \in F_{n+p} \subset F_n$. So,

$$\begin{aligned}
& P_S(a_n, a_{n+p}, a_{n+p}, t) \leq \delta(F_n), \quad \forall\ n \in \mathbb{N} \text{ and } \forall\ t > 0\\
\Longrightarrow\quad & \lim_{n\to\infty} P_S(a_n, a_{n+p}, a_{n+p}, t) = 0, \quad \forall\ t > 0\\
\Longrightarrow\quad & \{a_n\} \text{ is a Cauchy sequence.}
\end{aligned}$$

Since X is complete, so $\{a_n\}$ converges to a limit $a \in X$.
Let k be an arbitrary positive integer and consider the set F_k. Then each $a_k, a_{k+1}, a_{k+2}, \cdots$ belongs to F_k. Since F_k closed, so $a \in F_k$. Now k being arbitrary positive integer, so we can conclude $a \in \bigcap_{i=1}^{\infty} F_i$.

Finally, we show that a is unique. For, let $\exists\ b(\neq a) \in \bigcap_{i=1}^{\infty} F_i$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned}
& a, b \in F_k \\
\implies\ & P_S(a,a,b,t) \leq \delta(F_k), \quad \forall\ t > 0 \\
\implies\ & P_S(a,a,b,t) = 0, \quad \forall\ t > 0 \ \text{(since } \delta(F_k) \to 0 \text{ as } k \to \infty) \\
\implies\ & a = b
\end{aligned}$$

Conversely, suppose that the condition of the theorem holds. To show that X is complete, we consider a Cauchy sequence $\{x_n\}$ in X. Let $H_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$, $\forall\ n \in \mathbb{N}$. For any $\epsilon > 0$, there exist a positive integer n_0 (say) such that

$$\begin{aligned}
& P_S(x_n, x_n, x_m, t) < \epsilon, \quad \forall\ n > m \geq n_0 \text{ and } \forall\ t > 0 \\
\implies\ & \delta(H_m) \leq \epsilon, \quad \forall\ m \geq n_0 \\
\implies\ & \delta(\bar{H}_m) \leq \epsilon, \quad \forall\ m \geq n_0 \\
\implies\ & \delta(\bar{H}_m) \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Clearly, $\forall\ n \in \mathbb{N}$, $H_{n+1} \subset H_n$ and thus $\bar{H}_{n+1} \subset \bar{H}_n$. Therefore $\{\bar{H}_n\}$ constitutes a closed, nested sequence of non-empty sets in X whose diameter tends to zero. By hypothesis, there exists a unique $x \in \bigcap_{n=1}^{\infty} \bar{H}_n$.

Now for each $n = 1, 2, \cdots$, $x_n \in H_n \subset \bar{H}_n$ implies

$$\begin{aligned}
& P_S(x_n, x_n, x, t) \leq \delta(\bar{H}_n), \quad \forall\ n \text{ and } \forall\ t > 0 \\
\implies\ & P_S(x_n, x_n, x, t) \to 0, \quad \text{as } n \to \infty,\ \forall\ t > 0 \\
\implies\ & x_n \to x \quad \text{as } n \to \infty
\end{aligned}$$

Therefore X is complete. □

4.2 Totally Bounded Set and Compactness

Definition 4.7. Let F be a subset of a parametric S-metric space (X, P_S) and ϵ be a positive number. A set $G \subset X$ is said to be an ϵ-net for F if for any $x \in F$, $\exists\ y \in G$ such that $P_S(x,x,y,t) < \epsilon$, $\forall\ t > 0$.

If G is finite and bounded with respect to t then F is said to be totally bounded.

Proposition 4.8. *Each totally bounded set F in a parametric S-metric space (X, P_S) is bounded.*

Proof. Let F be totally bounded and $\epsilon > 0$. Then there exist an ϵ-net G for F which is finite and bounded with respect to t. Let x_1, x_2 be two arbitrary elements in F. So there exists $y_1, y_2 \in G$ such that $\forall\ t > 0$, $P_s(x_1, x_1, y_1, t) < \frac{\epsilon}{4}$ and $P_s(x_2, x_2, y_2, t) < \frac{\epsilon}{4}$.

Now,

$$\begin{aligned}
P_S(x_1,x_1,x_2,t) &\leq P_S(x_1,x_1,y_1,t) + P_S(x_1,x_1,y_1,t) + P_S(x_2,x_2,y_1,t), && \forall\ t > 0 \\
&\leq 2P_S(x_1,x_1,y_1,t) + P_S(x_2,x_2,y_2,t) + P_S(x_2,x_2,y_2,t) + P_S(y_1,y_1,y_2,t), && \forall\ t > 0 \\
&= 2P_S(x_1,x_1,y_1,t) + 2P_S(x_2,x_2,y_2,t) + P_S(y_1,y_1,y_2,t), && \forall\ t > 0
\end{aligned}$$

$$< 2\cdot\frac{\epsilon}{4} + 2\cdot\frac{\epsilon}{4} + P_S(y_1,y_1,y_2,t), \qquad \forall\ t>0$$
$$< \epsilon + P_S(y_1,y_1,y_2,t), \qquad \forall\ t>0.$$

In particular, take $\epsilon = 1$. Again, since G is finite and bounded with respect to t, $\exists\ K>0$ such that $\sup_{t>0} P_S(y_1,y_1,y_2,t) < K$, for $y_1,y_2 \in G$. Then

$$\sup_{t>0} P_S(x_1,x_1,x_2,t) \le 1+K \qquad (\text{for } x_1,x_2 \in F)$$
$$\implies \sup_{x,y\in F}\sup_{t>0} P_S(x,x,y,t) \le 1+K$$

Hence F is bounded. □

Remark 4.9. The converse implication of Proposition 4.8 does not hold in general. To justify it we consider the following example.

Example 4.10. Let $X = l_2$ and the function $P_S : X\times X\times X\times(0,\infty)\to[0,\infty)$ be defined by

$$P_S(a,b,c,t) = g(t)\left\{\left(\sum_{i=1}^{\infty}|a_i-b_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|b_i-c_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|a_i-c_i|^2\right)^{\frac{1}{2}}\right\},$$

where $a=\{a_i\}$, $b=\{b_i\}$, $c=\{c_i\}\in l_2$ and $g:(0,\infty)\to(0,\infty)$ is a continuous function.

First, we show that(X,P_S) is a parametric S-metric space.

From definition, it is clear that $P_S(a,b,c,t)=0$, $\forall\ t>0 \iff a=b=c$.

Let $a=\{a_i\}$, $b=\{b_i\}$, $c=\{c_i\}$, $x=\{x_i\}\in l_2$. Then, $\forall\ t>0$, we have

$$P_S(a,b,c,t) = g(t)\left\{\left(\sum_{i=1}^{\infty}|a_i-b_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|b_i-c_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|a_i-c_i|^2\right)^{\frac{1}{2}}\right\},$$

$$P_S(a,a,x,t) = 2g(t)\left(\sum_{i=1}^{\infty}|a_i-x_i|^2\right)^{\frac{1}{2}},$$

$$P_S(b,b,x,t) = 2g(t)\left(\sum_{i=1}^{\infty}|b_i-x_i|^2\right)^{\frac{1}{2}},$$

$$P_S(c,c,x,t) = 2g(t)\left(\sum_{i=1}^{\infty}|c_i-x_i|^2\right)^{\frac{1}{2}}.$$

By Minkowski's inequality, $\forall\ a=\{a_i\}$, $b=\{b_i\}$, $c=\{c_i\}\in l_2$, we have

$$\left(\sum_{i=1}^{\infty}|a_i-b_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{\infty}|a_i-x_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|b_i-x_i|^2\right)^{\frac{1}{2}},$$

$$\left(\sum_{i=1}^{\infty}|b_i-c_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{\infty}|b_i-x_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|c_i-x_i|^2\right)^{\frac{1}{2}},$$

$$\left(\sum_{i=1}^{\infty}|a_i-c_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{\infty}|a_i-x_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}|c_i-x_i|^2\right)^{\frac{1}{2}}.$$

Adding the above three inequalities, we get

$$g(t)\left[\left(\sum_{i=1}^{\infty}|a_i-b_i|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty}|b_i-c_i|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty}|a_i-c_i|^2\right)^{\frac{1}{2}}\right]$$
$$\le 2g(t)\left\{\left(\sum_{i=1}^{\infty}|a_i-x_i|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty}|b_i-x_i|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty}|c_i-x_i|^2\right)^{\frac{1}{2}}\right\}, \quad \forall\ t>0$$
$$\Longrightarrow \quad P_S(a,b,c,t)\le P_S(a,a,x,t)+P_S(b,b,x,t)+P_S(c,c,x,t), \quad \forall\ t>0.$$

Hence (X,P_S) is a parametric S-metric space.

Here we take the continuous function g defined as the constant mapping $g(t)=1$, $\forall\ t\in(0,\infty)$.

Now, we consider the subset A of l_2 consisting the elements

$$x_1=(1,0,0,\cdots),\ x_2=(0,1,0,0,\cdots),\ x_3=(0,0,1,0,0,\cdots),\cdots.$$

Then, $\forall\ x\neq y\in A$ and $\forall\ t>0$ we have,

$$P_S(x,x,y,t)=2\sqrt{2}. \tag{4.3}$$

So A is bounded.

Now, we verify that A is not totally bounded.

Choose $0<\epsilon<\frac{1}{\sqrt{2}}$. If possible suppose A is totally bounded. So there exist an ϵ-net $G$for the set A which is finite and bounded with respect to t. Thus for x_i,x_j $(i\neq j)$ in A, there exists $y_i,y_j\in G$ such that $\forall\ t>0$, $P_s(x_i,x_i,y_i,t)<\epsilon$ and $P_s(x_j,x_j,y_j,t)<\epsilon$.

Now $x_i\neq x_j$ and their number is infinite and G contains only a finite number of elements. So some y_i,y_j must be equal. If $y_i=y_j$ $(i\neq j)$ then $\forall\ t>0$ we get,

$$\begin{aligned}P_S(x_i,x_i,x_j,t)&\le P_S(x_i,x_i,y_i,t)+P_S(x_i,x_i,y_i,t)+P_S(x_j,x_j,y_i,t)\\&=2P_S(x_i,x_i,y_i,t)+P_S(x_j,x_j,y_j,t) \qquad (\text{since } y_i=y_j)\\&<2\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{3}{\sqrt{2}}\end{aligned}$$

which contradicts the relation (4.3). Hence A is not totally bounded.

Definition 4.11. Let (X,P_S) be a parametric S-metric space and $A\subset X$. A is said to be compact if every sequence in A has a convergent subsequence which converges to some point in A.

Theorem 4.12. *Every compact subset A of a parametric S-metric space (X,P_S) is totally bounded.*

Proof. We assume that A is compact. Let $\epsilon>0$ be arbitrary and x_1 be an arbitrary element in X.

If $\forall\ x\in A$, $P_S(x_1,x_1,x,t)<\epsilon$, $\forall\ t>0$ then a finite ϵ-net, $B=\{x_1\}$ exists for A.

If not, $\exists\ x_2\in A$ such that $P_S(x_2,x_2,x_1,t_{12})\ge\epsilon$, for some $t_{12}>0$.

In that case $\forall\ x\in A$, either $P_S(x_1,x_1,x,t)<\epsilon$ or $P_S(x_2,x_2,x,t)<\epsilon$, $\forall\ t>0$. Then a finite ϵ-net, $B=\{x_1,x_2\}$ exists for A.

However, if this is not true, then $\exists x_3 \in A$ such that $P_S(x_3,x_3,x_1,t_{13}) \geq \epsilon$ and $P_S(x_3,x_3,x_2,t_{23}) \geq \epsilon$ for some $t_{12},t_{23} > 0$.
In that case $\forall\ x \in A$, either $P_S(x_1,x_1,x,t) < \epsilon$ or $P_S(x_2,x_2,x,t) < \epsilon$ or $P_S(x_3,x_3,x,t) < \epsilon$, $\forall\ t > 0$. Then a finite ϵ-net, $B = \{x_1,x_2,x_3\}$ exists for A.
Continuing in this way we obtain points $x_1,x_2,x_3,\cdots$ where $x_1 \in X$ and $x_i \in A$, $i \geq 2$, such that for $i = 1,2,3\cdots$,

$$\begin{aligned}
&P_S(x_i,x_i,x_{i+1},t_{i,i+1}) \geq \epsilon\\
&P_S(x_i,x_i,x_{i+2},t_{i,i+2}) \geq \epsilon\\
&P_S(x_i,x_i,x_{i+3},t_{i,i+3}) \geq \epsilon\\
&\qquad\qquad\vdots
\end{aligned}$$

There are two cases may arise:

Case I: The procedure may stops after finite number of steps, say k. Then, we obtain points $x_1,x_2,x_3,\cdots,x_k$ such that for every $x \in A$ at least one of the inequalities $P_S(x_i,x_i,x,t) < \epsilon$ for $i = 1,2,\cdots,k$; $\forall\ t > 0$ holds.
Let $B = \{x_1,x_2,\cdots,x_k\}$ and $K = \max\limits_{1 \leq i,r \leq k} P_S(x_i,x_i,x_{i+r},t_{i,i+r})$.
So $\epsilon \leq K < \infty$. Then $\sup\limits_{t>0} P_S(x,x,y,t) \leq K$, $\forall\ x,y \in B$.
Therefore, B is an ϵ-net for A which is finite and bounded with respect to t. Hence A is totally bounded.

Case II: The procedure continues infinitely.
Then, we obtain a infinite sequence $\{x_n\}$, $x \in X$ and $x_n \in A$, $n \geq 2$. For $i = 1,2,3,\cdots$ we have

$$\begin{aligned}
&P_S(x_i,x_i,x_{i+1},t_{i,i+1}) \geq \epsilon\\
&P_S(x_i,x_i,x_{i+2},t_{i,i+2}) \geq \epsilon\\
&P_S(x_i,x_i,x_{i+3},t_{i,i+3}) \geq \epsilon\\
&\qquad\qquad\vdots
\end{aligned}$$

The above relations implies that neither the sequence $\{x_n\}$ nor its any subsequence converges to some point of A. This contradicts the compactness of A.
Thus, *Case* II is not possible. □

Theorem 4.13. *Every compact subset A of a parametric S-metric space is closed and bounded.*

Proof. If possible, suppose A is not closed. Then there exists a sequence of points $\{x_n\} \subset A$ converges to a point $x \notin A$.
Since A is compact, $\{x_n\}$ has a subsequence which converge to a point in A. But the subsequence must converges to x which does not belong to A, which contradicts the compactness of A. Hence A is closed.
Again in parametric S-metric spaces, every compact subset is totally bounded (Theorem 4.12) and every totally bounded set is bounded (Proposition 4.8), therefore A is bounded. □

Remark 4.14. The converse of Theorem 4.13 does not hold in general. To justify it consider the following example:

Example 4.15. Consider the parametric S-metric space (X,P_S) and $A \subset X$, defined in Example 4.10.

Here we take $g(t) = \frac{1}{1+t}$, $\forall\ t \in (0,\infty)$.

For $x, y \in A$, we have

$$P_S(x,x,y,t) = \begin{cases} \frac{2\sqrt{2}}{1+t}, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Since $\sup\limits_{t\in(0,\infty)} \{\frac{1}{1+t}\} = 1$, we get $P_S(x,x,y,t) \leq 2\sqrt{2}$, $\forall\ x,y \in A$, $\forall\ t > 0$.

Thus A is bounded. Since A has no limit point, so A is closed.

But elements of A are itself a sequence which has no convergent subsequence, so A is not compact.

Theorem 4.16. *Every compact parametric S-metric space is complete.*

Proof. Let (X,P_S) be a compact parametric S-metric space and $\{x_n\}$ be a Cauchy sequence in (X,P_S). Choose $t_0 \in (0,\infty)$.

So for a given $\epsilon > 0$, $\exists$ a positive integer $n_0(t_0)$ such that

$$P_S(x_n,x_n,x_{n_0},t_0) < \frac{\epsilon}{8}, \quad \forall\ n \geq n_0(t_0). \tag{4.4}$$

Since X is compact, thus $\exists$ a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_{k_n} = x$ in X.

So $\exists$ a positive integer $m(t_0)$ such that

$$P_S(x_{k_m},x_{k_m},x,t_0) < \frac{\epsilon}{8}, \quad \forall\ m(t_0) \geq n_0(t_0). \tag{4.5}$$

Since $k_m(t_0) > m(t_0) \geq n_0(t_0)$, so from (4.4), we get

$$P_S(x_{k_m},x_{k_m},x_{n_0},t_0) < \frac{\epsilon}{8}. \tag{4.6}$$

Again, we have

$$\begin{aligned} P_S(x_n,x_n,x,t_0) &\leq P_S(x_n,x_n,x_{n_0},t_0) + P_S(x_n,x_n,x_{n_0},t_0) + P_S(x,x,x_{n_0},t_0) \\ &\leq 2P_S(x_n,x_n,x_{n_0},t_0) + P_S(x,x,x_{k_m},t_0) + P_S(x,x,x_{k_m},t_0) + P_S(x_{n_0},x_{n_0},x_{k_m},t_0) \\ &= 2P_S(x_n,x_n,x_{n_0},t_0) + 2P_S(x,x,x_{k_m},t_0) + P_S(x_{n_0},x_{k_m},x_{k_m},t_0). \end{aligned}$$

Finally, we obtain

$$P_S(x_n,x_n,x,t_0) < 2\cdot\frac{\epsilon}{8} + 2\cdot\frac{\epsilon}{8} + \frac{\epsilon}{8} < \epsilon \qquad (n \geq n_0(t_0))$$

$$\implies \quad \lim_{n\to\infty} P_S(x_n,x_n,x,t_0) = 0$$

Since $t_0 \in (0,\infty)$ is arbitrary, thus $\lim\limits_{n\to\infty} P_S(x_n,x_n,x,t) = 0$, $\forall\ t \in (0,\infty)$ which implies $\{x_n\}$ converges to $x \in X$.

Hence (X,P_S) is complete. □

Remark 4.17. The converse of Theorem 4.16 does not hold. To justify consider the following example:

Example 4.18. We consider the parametric S-metric space of Example 4.3.

Now, we show that $(\mathbb{R}, P_S)$ is complete. For, let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}, P_S)$. Therefore

$$\begin{aligned}
&\lim_{n,m\to\infty} P_S(x_n, x_n, x_m, t) = 0 \qquad (\forall\ t \in (0,\infty)) \\
\implies &\lim_{n,m\to\infty} g(t)\{|x_n - x_n| + |x_n - x_m| + |x_n - x_m|\} = 0 \qquad (\forall\ t \in (0,\infty)) \\
\implies &\lim_{n,m\to\infty} |x_n - x_m| = 0 \\
\implies &\{x_n\} \text{ is a Cauchy sequence in } \mathbb{R} \text{ with respect to usual metric.}
\end{aligned}$$

Since $\mathbb{R}$ is complete with respect to usual metric, so $\exists\ x \in \mathbb{R}$ such that $\lim_{n\to\infty} |x_n - x| = 0$. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} P_S(x, x, x_n, t) = \lim_{n\to\infty} 2g(t)|x_n - x| \qquad (\forall\ t \in (0,\infty)) \\
\implies &\lim_{n\to\infty} P_S(x, x, x_n, t) = 0 \qquad (\forall\ t \in (0,\infty))
\end{aligned}$$

Thus, the Cauchy sequence $\{x_n\}$ in $(\mathbb{R}, P_S)$ converges to $x \in \mathbb{R}$. Hence $(\mathbb{R}, P_S)$ is complete.

But $(\mathbb{R}, P_S)$ is not compact.

For, we consider the sequence $\{x_n\}$ in $\mathbb{R}$ by $x_n = n$, $\forall\ n \in \mathbb{N}$. It is easy to verify that neither $\{x_n\}$ nor its any subsequence converges in $(\mathbb{R}, P_S)$.

5. Conclusion

We see that parametric S-metric fails to satisfy some of the standard metric properties. In parametric S-metric spaces, the open ball is not necessarily an open set. The induced topology τ is Hausdorff but the collection of open balls may not form a basis for τ. We also study the completeness, compactness, and other characteristics of parametric S-metric which will be helpful for further development of results on parametric S-metric spaces.

Acknowledgement

Abhishikta Das is thankful to University Grant Commission (UGC), New Delhi, India for awarding her senior research fellowship. The authors are also thankful to the Department of Mathematics, Siksha-Bhavana, Visva-Bharati.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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