Communications in Mathematics and Applications
Vol. 13, No. 3, pp. 1129–1141, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i3.1776

http://www.rgnpublications.com

Research Article

Some Identities of Dual Mersenne Numbers

Semra Kaya Nukran and İbrahim Gürgil*

Department of Mathematics, University of Usak, Usak, Turkey
***Corresponding author:** ibrahim.gurgil@usak.edu.tr

Received: December 28, 2021 **Accepted:** September 21, 2022

Abstract. The aim of this paper is to introduce the dual forms of the Mersenne, Jacobsthal and Jacobsthal-Lucas numbers which are called dual Mersenne, dual Jacobsthal and dual Jacobsthal-Lucas numbers. We give the widely known identities like, Binet to generalize these sequences, Catalan, and Cassini identities along with some useful properties of these dual sequences. We also show that identities of the dual forms of these sequences have a strong relation with their identities in their normal forms. We added the negative subscripts of dual Mersenne numbers. Finally, we show the relation of dual Mersenne numbers with dual Jacobsthal and dual Jacobsthal-Lucas numbers.

Keywords. Mersenne sequence, Dual Mersenne number, Dual Jacobsthal number, Dual Jacobsthal-Lucas number

Mathematics Subject Classification (2020). 11B39, 11B83, 11Y55

1. Introduction

In 1873, M. A. Clifford [3] introduced us the dual numbers which has many applications ranges from mechanic systems to screw systems. A dual unit is shown as ε which $\varepsilon^2 = 0$, $\varepsilon \neq 0$. Dual numbers are the extension of the real numbers in the form of

$$d = a + \varepsilon a^*.$$

The $\mathbb{D}$ set is defined as

$$\mathbb{D} = \{d = a + \varepsilon a^* \mid a, a* \in \mathbb{R}\}.$$

The algebra of dual set which is the two dimensional commutative associative over real numbers is a ring with addition and multiplication operations defined as:

$$(a+\varepsilon a^*)+(b+\varepsilon b^*)=(a+b)+\varepsilon(a^*+b^*)$$

and

$$(a+\varepsilon a^*)\cdot(b+\varepsilon b^*)=ab+\varepsilon(ab^*+a^*b).$$

Fibonacci and Lucas sequences have been widely studied and their dual forms have been introduced [7]. Other sequences are also the subject of several authors, like the Mersenne sequence. In the paper, [2], authors have worked on the properties of Mersenne sequence.

Some properties of Mersenne sequences can be found as:

$$M_{n+1}=2M_n+1, \tag{1.1}$$

where $M_0=0$ and $M_1=1$. With putting $n+1$ in the place of n, we get

$$M_{n+2}=2M_{n+1}+1. \tag{1.2}$$

Using these two equations, one can easily see that

$$M_{n+2}=3M_{n+1}-2M_n \tag{1.3}$$

is a new way to show the sequence of Mersenne numbers where $M_0=0$ and $M_1=1$.

If we look at the roots of the last equation, $r^2-3r+2=0$ are $r_1=2$ and $r_2=1$, we get the nth Mersenne number in the form of

$$M_n=2^n-1, \quad n\in\mathbb{N}^+ \tag{1.4}$$

is called Binet formula for Mersenne numbers [2].

Recall that the Jacobsthal numbers (recurrence of the second order form) are given by

$$J_{n+2}=J_{n+1}+2J_n, \tag{1.5}$$

where the initial conditions are $J_0=0$ and $J_1=1$.

Similar to Jacobsthal numbers, Jacobsthal-Lucas numbers (again in the recurrence of the second form) are given by

$$j_{n+2}=j_{n+1}+2j_n \tag{1.6}$$

with the initial conditions $j_0=2$ and $j_1=1$.

Binet formulas for these two sequences is known as

$$J_n=\frac{2^n-(-1)^n}{3} \tag{1.7}$$

and

$$j_n=2^n+(-1)^n. \tag{1.8}$$

For more information about these subjects are covered in [5,6].

We also know from [2] that Mersenne numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers are related in the forms of

$$M_k = \begin{cases} 3J_k, & k \text{ is even}, \\ 3J_k - 2, & k \text{ is odd}; \end{cases} \tag{1.9}$$

$$M_k = \begin{cases} j_k - 2, & k \text{ is even}, \\ j_k, & k \text{ is odd}. \end{cases} \tag{1.10}$$

Mersenne numbers are a popular topic for researchers in number theory and cryptosystems in computer science who are looking for prime numbers in the Mersenne sequence [1–3]. These numbers are called Mersenne primes.

While the dual forms of Fibonacci and Lucas numbers has been identified [8], as the best of our knowledge, there is a missing part for the dual Mersenne numbers, dual Jacobshtal numbers, and dual Jacobsthal-Lucas numbers. The main purpose of this paper is to introduce the work on these sequences. The definition and some properties of the dual forms of Mersenne, Jacobsthal and Jacobsthal-Lucas numbers, are given in this paper.

2. Dual Mersenne Numbers

The sequence of Mersenne numbers and some of the properties have been mentioned above. We define the dual Mersenne numbers as:

$$\widetilde{M}_n = M_n + \varepsilon M_{n+1}. \tag{2.1}$$

Some of the members of dual Mersenne numbers are

$$\varepsilon, 1+3\varepsilon, 3+7\varepsilon, 7+15\varepsilon, 15+31\varepsilon, \dots$$

with the initial condition as $\widetilde{M}_0 = \varepsilon$.

Theorem 2.1. *Let $\widetilde{M}_n$ be a dual Mersenne number. Then, the followings are hold:*

(i) $\widetilde{M}_{n+1} = 2\widetilde{M}_n + 1 + \varepsilon$,

(ii) $\widetilde{M}_{n+2} = 3\widetilde{M}_{n+1} - 2\widetilde{M}_n$.

Proof. (i) From equation (2.1),

$$\begin{aligned} \widetilde{M}_{n+1} &= M_{n+1} + \varepsilon M_{n+2} \\ &= 2M_n + 1 + \varepsilon(2M_{n+1} + 1) \\ &= 2\widetilde{M}_n + 1 + \varepsilon. \end{aligned}$$

(ii) By using equations (1.1) and (2.1),

$$\widetilde{M}_{n+2} = M_{n+2} + \varepsilon M_{n+3}$$

$$= 3M_{n+1} - 2M_n + \varepsilon(3M_{n+2} - 2M_{n+1})$$

$$= 3\widetilde{M}_{n+1} - 2\widetilde{M}_n. \qquad \square$$

Lemma 2.2. *Let $\widetilde{M}_n$ and $\widetilde{M}_{n-r}$ be dual Mersenne numbers, then*

$$\widetilde{M}_n - \widetilde{M}_{n-r} = 2^{n-r}M_r(\widetilde{M}_1 - \widetilde{M}_0)$$

holds when $n \geq r$.

Proof.

$$\widetilde{M}_n - \widetilde{M}_{n-r} = M_n + \varepsilon M_{n+1} - (M_{n-r} + \varepsilon M_{n-r+1})$$

$$= M_n - M_{n-r} + \varepsilon(M_{n+1} - M_{n+1-r})$$

$$= 2^{n-r}M_r + \varepsilon 2^{n+1-r}M_r$$

$$= 2^{n-r}M_r(1 + 2\varepsilon)$$

$$= 2^{n-r}M_r(\widetilde{M}_1 - \widetilde{M}_0). \qquad \square$$

Theorem 2.3. *Let $\widetilde{M}_n$ be a dual Mersenne number than Binet formula for dual Mersenne numbers can be written as*

$$\widetilde{M}_n = \alpha(\alpha 2^n - 1),$$

where $\alpha = 1 + \varepsilon$.

Proof.

$$\widetilde{M}_n = M_n + \varepsilon M_{n+1}$$

$$= (2^n - 1) + \varepsilon(2^{n+1} - 1)$$

$$= (1 + \varepsilon)(2^n(1 + \varepsilon) - 1)$$

$$= \alpha(\alpha 2^n - 1). \qquad \square$$

Theorem 2.4. *Let $\widetilde{M}_n$ be a dual Mersenne number, then limit of ratio of consecutive quotients is*

$$\lim_{n\to\infty} \frac{\widetilde{M}_{n+1}}{\widetilde{M}_n} = r_1,$$

where $r_1 = \lim_{n\to\infty} \frac{M_{n+1}}{M_n}$ which is shown in [2].

Proof.

$$\lim_{n\to\infty} \frac{\widetilde{M}_{n+1}}{\widetilde{M}_n} = \lim_{n\to\infty} \frac{M_{n+1} + \varepsilon M_{n+2}}{M_n + \varepsilon M_{n+1}}$$

$$= \lim_{n\to\infty} \frac{(M_{n+1} + \varepsilon M_{n+2})}{(M_n + \varepsilon M_{n+1})} \cdot \frac{(M_n - \varepsilon M_{n+1})}{(M_n - \varepsilon M_{n+1})}$$

$$= \lim_{n\to\infty} \frac{M_{n+1}\cdot M_n - \varepsilon(M_n\cdot M_{n+2} - {M_{n+1}}^2)}{{M_n}^2}$$
$$= \lim_{n\to\infty} \frac{M_{n+1}}{M_n} + \varepsilon \lim_{n\to\infty} \frac{M_n\cdot M_{n+2} - {M_{n+1}}^2}{{M_n}^2}$$
$$= r_1 - \varepsilon \lim_{n\to\infty} \frac{2^n}{(2^n-1)^2}$$
$$= r_1. \qquad \square$$

Corollary 2.5. *When $\widetilde{M}_n$ and $\widetilde{M}_{n+1}$ are two serial dual Mersenne numbers, then*

$$\lim_{n\to\infty} \frac{\widetilde{M}_n}{\widetilde{M}_{n+1}} = \frac{1}{r_1}$$

holds.

Theorem 2.6. *Let $\widetilde{M}_n$ and $\widetilde{M}_m$ be dual Mersenne numbers, then d'Ocagne's identity for dual Mersenne numbers can be identified as*

$$\widetilde{M}_m\widetilde{M}_{n+1} - \widetilde{M}_{m+1}\widetilde{M}_n = 2^n M_{m-n}\cdot\widetilde{M}_1.$$

Proof.

$$\widetilde{M}_m\widetilde{M}_{n+1} - \widetilde{M}_{m+1}\widetilde{M}_n$$
$$= (M_m + \varepsilon M_{m+1})(M_{n+1} + \varepsilon M_{n+2}) - (M_{m+1} + \varepsilon M_{m+2})(M_n + \varepsilon M_{n+1})$$
$$= M_m M_{n+1} - M_{m+1}M_n + \varepsilon(M_{m+1}M_{n+1} - M_{m+2}M_n + M_m M_{n+2} - M_{m+1}M_{n+1})$$
$$= 2^n M_{m-n} + \varepsilon(2^n M_{m+1-n} + 2^{n+1}M_{m-n-1})$$
$$= 2^n M_{m-n} + \varepsilon(2^n(3.M_{m-n}))$$
$$= 2^n M_{m-n}\cdot\widetilde{M}_1. \qquad \square$$

Recall that for Mersenne numbers M_n, M_{n-r} and M_{n+r}, Catalan's identity is

$$M_{n-r}M_{n+r} - {M_n}^2 = 2^{n+1} - 2^{n-r} - 2^{n+r}.$$

Theorem 2.7. *If $\widetilde{M}_{n-r}$, $\widetilde{M}_n$ and $\widetilde{M}_{n+r}$ are dual Mersenne numbers, then the following holds*

$$\widetilde{M}_{n-r}\widetilde{M}_{n+r} - \widetilde{M}_n^2 = 2^{n+1} - 2^{n-r} - 2^{n+r} - 3\varepsilon 2^{n-r}M_r^2.$$

Proof.

(i) $$\widetilde{M}_{n-r}\widetilde{M}_{n+r} = (M_{n-r} + \varepsilon M_{n-r+1})\cdot(M_{n+r} + \varepsilon M_{n+r+1})$$
$$= M_{n-r}M_{n+r} + \varepsilon(M_{n-r+1}M_{n+r} + M_{n-r}M_{n+r+1})$$

(ii) $$\widetilde{M}_n^2 = (M_n + \varepsilon M_{n+1})^2 = {M_n}^2 + 2\varepsilon M_n M_{n+1}$$

$$(\mathrm{i}) - (\mathrm{ii}) = M_{n-r}.M_{n+r} - {M_n}^2 + \varepsilon(M_{n-r+1}M_{n+r} + M_{n-r}M_{n+r+1} - 2M_n M_{n+1})$$

$$\text{(i)} - \text{(ii)} = 2^{n+1} - 2^{n-r} - 2^{n+r} + \varepsilon(M_{n-r+1}M_{n+r} + M_{n-r}M_{n+r+1} - 2M_nM_{n+1})$$

$$\text{(i)} - \text{(ii)} = A + \varepsilon B.$$

In B part of the equation add and subtract $M_{n-r+1}M_{n+r}$,

$$B = M_{n-r}M_{n+r+1} - M_{n-r+1}M_{n+r} + 2M_{n-r+1}M_{n+r} - 2M_nM_{n+1}$$

to use d'Ocagne's identity.

For $m^* = n - r$ and $n^* = n + r$, d'Ocagne's identity:

$$M_{m^*}M_{n^*+1} - M_{m^*+1}M_{n^*} = 2^{n^*}M_{m^*-n^*}, \tag{2.2}$$

$$M_{n-r}M_{n+r+1} - M_{n-r+1}M_{n+r} = 2^{n+r}M_{-2r}. \tag{2.3}$$

Then B part becomes ($A = 2^{n+1} - 2^{n-r} - 2^{n+r}$)

$$B = 2^{n+r}M_{-2r} + 2(M_{n-r+1}M_{n+r} - M_nM_{n+1})$$

$$= 2^{n+r}M_{-2r} + 2((2^{n-r+1} - 1)(2^{n+r} - 1) - (2^n - 1)(2^{n+1} - 1)$$

$$= 2^{n-r} - 2^{n+r} - 2^{n-r+2} - 2^{n+r+1} + 6.2^n$$

$$= 3.2^n(2 - 2^r - 2^{-r})$$

$$= 3.2^n(M_{-r}M_r)$$

$$= 2^{n-r}M_r^2.$$

Then, we get

$$\widetilde{M}_{n-r}\widetilde{M}_{n+r} - \widetilde{M}_n^2 = 2^{n+1} - 2^{n-r} - 2^{n+r} - 3\,\varepsilon\, 2^{n-r}M_r^2. \qquad \square$$

Proposition 2.8 (Cassini's identity)**.** *For $r = 1$, in the Catalan's identity;*

$$\widetilde{M}_{n-1}\widetilde{M}_{n+1} - \widetilde{M}_n^2 = 2^{n+1} - 2^{n-1} - 2^{n+1} - 3\,\varepsilon\, 2^{n-1}M_1^2.$$

Since $M_1 = 1$,

$$\widetilde{M}_{n-1}\widetilde{M}_{n+1} - \widetilde{M}_n^2 = -2^{n-1}(1 + 3\varepsilon),$$

$$\widetilde{M}_{n-1}\widetilde{M}_{n+1} - \widetilde{M}_n^2 = -2^{n-1}(\widetilde{M}_1).$$

Definition 2.9. For the negative subscripts, dual Mersenne numbers are defined as

$$\widetilde{M}_{-n} = M_{-n} + \varepsilon M_{-(n+1)}.$$

Say $M_{-n} = G_n$, then we get

$$\widetilde{G}_n = \widetilde{M}_{-n} = G_n + \varepsilon G_{n+1}.$$

How to get the first member of dual Mersenne with negative subscripts numbers and first few members are shown as

$$\begin{aligned}\widetilde{M}_{-1} &= G_1 + \varepsilon G_2 \\ &= -\frac{M_1}{2} + \varepsilon - \frac{M_2}{4} \\ &= -\frac{(2M_1 + \varepsilon M_2)}{4} \\ &= -\frac{1}{2} - \frac{3}{4}\varepsilon.\end{aligned}$$

For the negative indices, dual Mersenne sequence has the members of

$$-\frac{1}{2} - \frac{3}{4}\varepsilon, -\frac{3}{4} - \frac{7}{8}\varepsilon, -\frac{7}{8} - \frac{15}{16}\varepsilon, -\frac{15}{16} - \frac{31}{32}\varepsilon, \dots.$$

Proposition 2.10. *Let $\widetilde{G}_n = \widetilde{M}_{-n}$ be a negative indexed dual Mersenne number and $\widetilde{M}_n$ is a dual Mersenne number, then*

$$\widetilde{G}_n = -\frac{1}{2^{n+1}}(\widetilde{M}_n + M_n)$$

holds.

Proof. From [4, equation (19)],

$$\begin{aligned}\widetilde{G}_n &= -\frac{M_n}{2^n} + \varepsilon - \frac{M_{n+1}}{2^{n+1}} \\ &= -\frac{(2M_n + \varepsilon M_{n+1})}{2^{n+1}} \\ &= -\frac{(M_n + \varepsilon M_{n+1})}{2^{n+1}} - \frac{M_n}{2^{n+1}} \\ &= \frac{\widetilde{M}_n}{2^{n+1}} - \frac{M_n}{2^{n+1}} \\ &= -\frac{1}{2^{n+1}}(\widetilde{M}_n + M_n).\end{aligned}$$

□

Theorem 2.11. *Let $\widetilde{G}_{n-r}$, $\widetilde{G}_n$ and $\widetilde{G}_{n+r}$ be three dual Mersenne numbers with negative subscripts, then*

$$\widetilde{G}_{n-r} \cdot \widetilde{G}_{n+r} - \widetilde{G}_n^2 = (1-\varepsilon)(G_{n-r} \cdot G_{n+r} - G_n^2) + \varepsilon\left(2 - \frac{1}{2^n}\right).$$

Proof.

$$\begin{aligned}\widetilde{G}_{n-r}.\widetilde{G}_{n+r} - \widetilde{G}_n^2 &= \left(-\frac{2M_{n-r} + \varepsilon M_{n-r+1}}{2^{n-r+1}}\right).\left(-\frac{2M_{n+r} + \varepsilon M_{n+r+1}}{2^{n+r+1}}\right) - \left(-\frac{2M_n + \varepsilon M_{n+1}}{2^{n+1}}\right)^2 \\ &= \frac{(4M_{n-r} \cdot M_{n+r} + 2\varepsilon M_{n-r+1} + 2\varepsilon M_{n+r+1} + \varepsilon^2 M_{n-r+1} M_{n+r+1})}{2^{2n+2}} \\ &\quad - \frac{4M_n^2 + 4\varepsilon M_n M_{n+1} + \varepsilon^2 M_{n+1}^2}{2^{2n+2}}\end{aligned}$$

$$=\frac{4(M_{n-r}M_{n+r}-M_n^2)}{2^{2n+2}}+2\varepsilon\frac{M_{n-r+1}+M_{n+r+1}-2M_nM_{n+1}}{2^{2n+2}}$$
$$=-\frac{2^{2r}-2^{r+1}+1}{2^{n+r}}+\varepsilon\frac{2^{n+1}+2^r+2^{-r}-3}{2^n}$$
$$=(G_{n-r}\cdot G_{n+r}-G_n^2)-\varepsilon(G_{n-r}\cdot G_{n+r}-G_n^2)+\varepsilon\left(2-\frac{1}{2^n}\right)$$
$$=(1-\varepsilon)(G_{n-r}\cdot G_{n+r}-G_n^2)+\varepsilon\left(2-\frac{1}{2^n}\right).$$ □

Proposition 2.12. *When we use as $r=1$ at Catalan's identity for dual Mersenne numbers with negative subscripts, we get the Cassini's identity as*

$$\widetilde{G}_{n-1}\cdot\widetilde{G}_{n+1}-\widetilde{G}_n^2=-\frac{1}{2^{n+1}}-2\varepsilon G_{n+2}.$$

3. Dual Jacobsthal and Dual Jacobsthal-Lucas Numbers

Binet formulas for Jacobsthal and Jacobsthal-Lucas numbers are given in (1.7) and (1.8), respectively.

3.1 Dual Jacobsthal Numbers

We define the dual Jacobsthal numbers as

$$\widetilde{J}_n=J_n+\varepsilon J_{n+1}. \tag{3.1}$$

From the equation above, one can easily see that the terms of the dual Jacobsthal sequence are

$$\widetilde{J}_0=\varepsilon,\ \widetilde{J}_1=1+\varepsilon,\ \widetilde{J}_2=1+3\varepsilon,\ \widetilde{J}_3=3+5\varepsilon,\dots.$$

Theorem 3.1 (Binet's formula)**.** *Let $\widetilde{J}_n$ be a dual Jacobsthal number, then*

$$\widetilde{J}_n=\widetilde{J}_1^2\cdot J_n+(-1)^n\cdot\widetilde{J}_0.$$

Proof.

$$\widetilde{J}_n=J_n+\varepsilon J_{n+1}$$
$$=\frac{2^n-(-1)^n}{3}+\varepsilon\frac{2^{n+1}-(-1)^{n+1}}{3}$$
$$=(1+\varepsilon)^2\cdot\frac{2^n-(-1)^n}{3}+(-1)^n\cdot\varepsilon$$
$$=\widetilde{J}_1^2\cdot J_n+(-1)^n.\widetilde{J}_0.$$ □

Theorem 3.2. *Let $\widetilde{J}_{n+2}$, $\widetilde{J}_{n+1}$ and $\widetilde{J}_n$, three serial dual Jacobsthal numbers, then*

$$\widetilde{J}_{n+2}=\widetilde{J}_{n+1}+2\widetilde{J}_n$$

holds.

Proof.

$$\widetilde{J}_{n+1} = J_{n+1} + \varepsilon J_{n+2}, \tag{3.2}$$

$$\widetilde{J}_{n} = J_{n} + \varepsilon J_{n+1}. \tag{3.3}$$

When we sum (3.2) and twice the (3.3),

$$\begin{aligned}\widetilde{J}_{n+1} + 2\widetilde{J}_{n} &= J_{n+1} + 2J_n + \varepsilon(J_{n+2} + 2J_{n+1}) \\ &= J_{n+2} + \varepsilon J_{n+3} \\ &= \widetilde{J}_{n+2}.\end{aligned}$$

□

Recall that Cassini-Like identities for Jacobsthal numbers are known as

$$J_{n+1} \cdot J_{n-1} - J_n^2 = (-1)^n \cdot 2^{n-1}.$$

Theorem 3.3 (Cassini-Like identities). *Let $\widetilde{J}_{n-1}$, $\widetilde{J}_n$ and J_{n+1} be dual Jacobsthal numbers, Cassini-Like identity for dual Jacobsthal numbers is*

$$\widetilde{J}_{n+1} \cdot \widetilde{J}_{n-1} - \widetilde{J}_n^2 = (-1)^n \cdot 2^{n-1} \cdot \widetilde{J}_1.$$

Proof.

$$\begin{aligned}\widetilde{J}_{n+1} \cdot \widetilde{J}_{n-1} - \widetilde{J}_n^2 &= (J_{n+1} + \varepsilon J_{n+2})(J_{n-1} + \varepsilon J_n) - (J_n + \varepsilon J_{n+1})^2 \\ &= J_{n+1} \cdot J_{n-1} - J_n^2 + \varepsilon(J_{n+2} \cdot J_{n-1} - J_{n+1} \cdot J_n) \\ &= (-1)^n \cdot 2^{n-1} + \varepsilon\left(\frac{2^{n+2} - (-1)^{n+2}}{3}\frac{2^{n-1} - (-1)^{n-1}}{3} - \frac{2^{n+1} - (-1)^{n+1}}{3}\frac{2^n - (-1)^n}{3}\right) \\ &= (-1)^n \cdot 2^{n-1} + \varepsilon\frac{1}{9} \cdot \left(2^n(-1)^n\frac{9}{2}\right) \\ &= 2^{n-1}(-1)^n(1+\varepsilon) \\ &= (-1)^n \cdot 2^{n-1} \cdot \widetilde{J}_1.\end{aligned}$$

□

Definition 3.4. For the dual Jacobsthal numbers with negative subscripts, we define

$$\widetilde{J}_{-n} = J_{-n} + \varepsilon J_{-(n+1)}.$$

Say $J_{-n} = H_n$, then we get

$$\widetilde{H}_n = \widetilde{J}_{-n} = H_n + \varepsilon H_{n+1}.$$

Negative subscripted dual Jacobsthal sequence are shown, with the help of [5], as

$$\begin{aligned}\widetilde{J}_{-1} = H_1 + \varepsilon H_2 &= \frac{J_1}{2} + \varepsilon - \frac{J_2}{4} \\ &= \frac{(2J_1 - \varepsilon J_2)}{4}\end{aligned}$$

$$= \frac{1}{2} - \frac{1}{4}\varepsilon.$$

The negative indexed dual Mersenne sequence has the members of

$$\frac{1}{2} - \frac{1}{4}\varepsilon, -\frac{1}{4} + \frac{3}{8}\varepsilon, \frac{3}{8} - \frac{5}{16}\varepsilon, -\frac{5}{16} + \frac{11}{32}\varepsilon, \dots.$$

Lemma 3.5. *Let $\widetilde{H}_n, \widetilde{H}_{n-1}$ be two dual Jacobsthal numbers with negative subscripts, then*

$$\widetilde{H}_n - \widetilde{H}_{n-1} = \frac{1}{2^{n+1}}(1 + \widetilde{J}_1)$$

holds.

Proof.

$$\begin{aligned} \widetilde{H}_n - \widetilde{H}_{n-1} &= (H_n + \varepsilon H_{n+1}) - (H_{n-1} + \varepsilon H_n) \\ &= (H_n - H_{n-1}) + \varepsilon(H_{n+1} - H_n) \\ &= \frac{1}{2^n} + \varepsilon \frac{1}{2^{n+1}} \\ &= \frac{1}{2^{n+1}}(1 + \widetilde{J}_1). \end{aligned}$$

□

3.2 Dual Jacobsthal-Lucas numbers

We define the dual Jacobsthal-Lucas numbers as

$$\widetilde{J}_n = j_n + \varepsilon j_{n+1}.$$

The terms of the dual Jacobsthal-Lucas numbers are as

$$2 + \varepsilon, 1 + 5\varepsilon, 5 + 7\varepsilon, 7 + 17\varepsilon, 17 + 31\varepsilon, \dots$$

with using the same initial conditions for Jacobsthal-Lucas numbers ([2]).

Theorem 3.6 (Binet's Formula)**.** *Let $\widetilde{J}_n$ be a Jacobsthal-Lucas number, Binet's formula for this sequence can be written as*

$$\widetilde{J}_n = 2^n(1 + 2\varepsilon) + (-1)^n(1 - \varepsilon).$$

Proof.

$$\begin{aligned} \widetilde{J}_n &= j_n + \varepsilon j_{n+1} \\ &= 2^n + (-1)^n + \varepsilon(2^{n+1} + (-1)^{n+1}) \\ &= 2^n(1 + 2\varepsilon) + (-1)^n(1 - \varepsilon). \end{aligned}$$

□

Theorem 3.7. *When $\widetilde{J}_{n+2}, \widetilde{J}_{n+1}$, and $\widetilde{J}_n$ are serial dual Jacobsthal-Lucas numbers, then*

$$\widetilde{J}_{n+2} = \widetilde{J}_{n+1} + 2\widetilde{J}_n$$

holds.

Proof.

$$\begin{aligned}\widetilde{J}_{n+2} &= j_{n+2} + \varepsilon j_{n+3} \\ &= j_{n+1} + 2j_n + \varepsilon(j_{n+2} + 2j_{n+1}) \\ &= \widetilde{J}_{n+1} + 2\widetilde{J}_n.\end{aligned}$$

□

Definition 3.8. The negative subscripts for Jacobsthal-Lucas numbers are defined as

$$\widetilde{J}_{-n} = j_{-n} + \varepsilon j_{-(n+1)}.$$

Say $j_{-n} = I_n$, then we get

$$\widetilde{I}_n = \widetilde{J}_{-n} = I_n + \varepsilon I_{n+1}.$$

Some of the members of this negative indexed Jacobsthal-Lucas sequence are

$$-\frac{1}{2} + \varepsilon\frac{5}{4}, \frac{5}{4} - \varepsilon\frac{7}{8}, -\frac{7}{8} + \varepsilon\frac{17}{16}, \ldots.$$

Lemma 3.9. *Let $\widetilde{I}_n$ and $\widetilde{I}_{n-1}$ be two dual Jacobsthal-Lucas numbers with negative subscripts, then*

$$\widetilde{I}_n + \widetilde{I}_{n-1} = \frac{3}{2^{n+1}}\widetilde{J}_0.$$

Proof.

$$\begin{aligned}\widetilde{I}_n + \widetilde{I}_{n-1} &= (I_n + \varepsilon I_{n+1}) + (I_{n-1} + \varepsilon I_n) \\ &= (I_n + I_{n-1}) + \varepsilon(I_{n+1} + I_n) \\ &= \frac{3}{2^n} + \varepsilon\frac{3}{2^{n+1}} \\ &= \frac{3}{2^{n+1}}\widetilde{J}_0.\end{aligned}$$

□

Theorem 3.10. *Let $\widetilde{M}_n$, $\widetilde{J}_n$ and $\widetilde{J}_n$ are the numbers of dual Mersenne, Jacobsthal and Jacobsthal-Lucas, then*

(i) $$\widetilde{M}_n = \begin{cases} 3\widetilde{J}_n - 2\varepsilon, & n \text{ is even}, \\ 3\widetilde{J}_n - 2, & n \text{ is odd} \end{cases}$$

and

(ii) $$\widetilde{M}_n = \begin{cases} \widetilde{J}_n - 2, & n \text{ is even}, \\ \widetilde{J}_n - 2\varepsilon, & n \text{ is odd} \end{cases}$$

holds.

Proof. For first part of the condition (i), assume $n = 2k$ then

$$\widetilde{M}_{2k} = M_{2k} + \varepsilon M_{2k+1}$$

$$= 3J_{2k} + \varepsilon(3J_{2k+1} - 2)$$
$$= 3(J_{2k} + \varepsilon J_{2k+1}) - 2\varepsilon$$
$$= 3\widetilde{J}_{2k} - 2\varepsilon.$$

Second part of (i) can be shown with the same way with the help of the equation (1.9).

For second part of the condition (ii), assume $n = 2k+1$ then,

$$\begin{aligned}\widetilde{M}_{2k+1} &= M_{2k+1} + \varepsilon M_{2k+2} \\ &= j_{2k+1} + \varepsilon(j_{2k+2} - 2) \\ &= j_{2k+1} + \varepsilon J_{2k+2} - 2\varepsilon \\ &= \widetilde{j}_{2k+1} - 2\varepsilon.\end{aligned}$$

First part of the equation can be derived with similar way with the help of the eq. (1.10). □

4. Conclusion

We defined a new version of Dual numbers using Mersenne sequence's numbers. With similar approaches, we introduced the Dual formats of the Jacobsthal and Jacobsthal-Lucas numbers. We have given interesting results and important inequalities by using these sequences. Binet formula, Cassine equality etc. which are widely known for Fibonacci numbers are given for these new sequences in Dual numbers. For future reference, we are thinking of working on their matrix forms and their properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. Aggarwal, A. Joux, A. Prakash and M. Santha, *A New Public-Key Cryptosystem via Mersenne Numbers*, in: H. Shacham and A. Boldyreva (eds.), *Advances in Cryptology – CRYPTO 2018* (CRYPTO 2018), *Lecture Notes in Computer Science*, Vol. **10993**, Springer, Cham. (2018), DOI: 10.1007/978-3-319-96878-0_16.

[2] P. Catarino, C. Helena and P. Vasco, On the Mersenne sequence, *Annales Mathematicae et Informaticae* **46** (2016), 37 – 53, URL: http://publikacio.uni-eszterhazy.hu/3253/1/AMI_46_from37to53.pdf.

[3] M. A. Clifford, Preliminary sketch of bi-quaternions, *Proceedings of the London Mathematical Society* **4** (1873), 381 – 395, DOI: 10.1112/plms/s1-4.1.381.

[4] A. Daşdemir, Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers with negative subscripts, *Acta Mathematica Universitatis Comenianae* **LXXXVIII** (2019), 145 – 156, URL: http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/906/645.

[5] A. Daşdemir and G. Bilgici, Gaussian Mersenne numbers and generalized Mersenne quaternions, *Notes on Number Theory and Discrete Mathematics* **25** (2019), 87 – 96, DOI: 10.7546/nntdm.2019.25.3.87-96.

[6] T. Goy, On new identities for Mersenne numbers, *Applied Mathematics E-Notes* **18** (2018), 100 – 105, URL: https://www.emis.de/journals/AMEN/2018/AMEN-170424.pdf.

[7] İ. A. Güven and S. K. Nurkan, A new approach to Fibonacchi, Lucas numbers and dual vectors, *Advances in Applied Clifford Algebras* **25**(1) (2015), 577 – 590, DOI: 10.1007/s00006-014-0516-7.

[8] S. K. Nurkan and İ. A. Güven, Dual Fibonacci quaternions, *Advances in Applied Clifford Algebras* **25**(1) (2015), 403 – 414, DOI: 10.1007/s00006-014-0488-7.