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Lower Order Eigenvalues of the Schrödinger Operator*

Research Article

Bingqing Ma
Department of Mathematics, Henan Normal University, Xinxiang 453007, Henan, People's Republic of China
bqma@henannu.edu.cn

Abstract. Making use of the method introduced by Brands in [2], we consider lower order eigenvalues of the Schrödinger operator in Euclidean domains. We extend an estimate on eigenvalues obtained by Ashbaugh and Benguria in [1].

Keywords. Membrane eigenvalue; Schrödinger operator; Rayleigh-Ritz inequality

MSC. 35P15; 58C40

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1. Introduction

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is called the *fixed membrane problem*. Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to +\infty$ denote the successive eigenvalues for (1.1), where each eigenvalue is repeated according to its multiplicity. In the case of $n = 2$, Payne-Pólya-Weinberger [5] proved

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \le 6. \tag{1.2}$$

Subsequently, in 1964, Brands [2] sharpen (1.2) to

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \le 5 + \frac{\lambda_1}{\lambda_2}. \tag{1.3}$$

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In 1993, for general dimensions $n \geq 2$, Ashbaugh and Benguria [1] proved (see the inequality (6.10) in [1])

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_2}. \tag{1.4}$$

Recently, the inequality (1.4) has been extended to some Riemannian manifolds, see [6, 3, 4] and the references therein.

In this note, we consider eigenvalue problem of the following Schrödinger operator

$$\begin{cases} (-\Delta + V)u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where V is a continuous bounded function on $\overline{\Omega}$. Using the method of Brands [2], we study the eigenvalue problem (1.5) for general dimensions $n \geq 2$ and extend the inequality (1.4) as follows:

Theorem. *Let λ_i be the i-th eigenvalue of the eigenvalue problem* (1.5)*. Then*

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + \frac{(M+1)(3\xi + 4M + 1)}{\xi + M}, \tag{1.6}$$

where $M = \sup_{\overline{\Omega}} |V|/\lambda_1$ and $\xi = \lambda_2/\lambda_1$.

Remark. If $V = 0$ in (1.6), from (1.6), it is easy to see that

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_2},$$

(1.4) follows. Hence, (1.6) extends the inequality (1.4).

2. Proof of Theorem

Let u_i be the orthonormal eigenvalue function with respect to L^2 inner product corresponding to λ_i, that is,

$$\int_\Omega u_i u_j = \delta_{ij}, \quad \text{for any } i, j.$$

We choose rectangular coordinates $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n$ of the Euclidean space $\mathbb{R}^n$ by taking as origin the center of gravity of Ω with mass-distribution u_1^2 such that

$$\int_\Omega \tilde{x}^i u_1^2 = 0, \quad \text{for } i = 1, 2, \cdots, n. \tag{2.1}$$

Defining an $n \times n$-matrix B as follows:

$$B := (b_{ij})$$

where $b_{ij} = \int_\Omega \widetilde{x}^i u_1 u_{j+1}$. Using the orthogonalization of Gram and Schmidt, we know that there exist an upper triangle matrix $R = (R_{ij})$ and an orthogonal matrix $Q = (q_{ij})$ such that $R = QB$, that is,

$$R_{ij} = \sum_{k=1}^n q_{ik} b_{kj} = \int_k \sum_{k=1}^n q_{ik} \widetilde{x}^k u_1 u_j = 0, \quad 2 \le j \le i \le n. \tag{2.2}$$

Setting $x^i = \sum_{j=1}^n q_{ij} \widetilde{x}^j$. From (2.1) and (2.2), we arrive at

$$\int_\Omega x_i u_1 u_j = 0, \quad \text{for } 1 \le j \le i \le n. \tag{2.3}$$

Let $\varphi_i = x_i u_1$. Then $\varphi_i = 0$ on $\partial\Omega$ and

$$\int_\Omega \varphi_i u_j = 0, \quad \text{for } 1 \le j \le i \le n.$$

One gets from Rayleigh-Ritz inequality that

$$\lambda_{i+1} \le \frac{\int_\Omega \varphi_i (-\Delta + V)\varphi_i}{\int_\Omega \varphi_i^2}. \tag{2.4}$$

Note that

$$(-\Delta + V)\varphi_i = \lambda_1 x_i u_1 - 2u_{1,x_i},$$

where $u_{1,x_i} = \partial u_1 / \partial x_i$. It follows that

$$\begin{aligned}
\int_\Omega \varphi_i (-\Delta + V)\varphi_i &= \int_\Omega \varphi_i (\lambda_1 x_i u_1 - 2u_{1,x_i}) \\
&= \lambda_1 \int_\Omega \varphi_i^2 - 2\int_\Omega x_i u_1 u_{1,x_i} \\
&= \lambda_1 \int_\Omega \varphi_i^2 - \int_\Omega x_i (u_1^2)_{,x_i} \\
&= \lambda_1 \int_\Omega \varphi_i^2 + \int_\Omega u_1^2 \\
&= \lambda_1 \int_\Omega \varphi_i^2 + 1.
\end{aligned} \tag{2.5}$$

(2.5) combining with (2.4) yields

$$\lambda_{i+1} \le \lambda_1 + \left(\int_\Omega (x_i u_1)^2 \right)^{-1}. \tag{2.6}$$

By integration by parts, it holds that

$$\int_\Omega u_1^{\alpha+1} = -\int_\Omega x_i (u_1^{\alpha+1})_{,x_i} = -(\alpha+1) \int_\Omega (x_i u_1)(u_1^{\alpha-1} u_{1,x_i}).$$

For $\alpha > 1/2$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\left(\int_\Omega u_1^{\alpha+1}\right)^2 &= (\alpha+1)^2\left(\int_\Omega (x_iu_1)(u_1^{\alpha-1}u_{1,x_i})\right)^2\\ &\le (\alpha+1)^2\int_\Omega (x_iu_1)^2\int_\Omega (u_1^{\alpha-1}u_{1,x_i})^2\\ &= \frac{(\alpha+1)^2}{2\alpha-1}\int_\Omega (x_iu_1)^2\int_\Omega (u_1^{2\alpha-1})_{,x_i}u_{1,x_i}\\ &= \frac{-(\alpha+1)^2}{2\alpha-1}\int_\Omega (x_iu_1)^2\int_\Omega u_1^{2\alpha-1}u_{1,x_ix_i}.\end{aligned}$$

Thus

$$\left(\int_\Omega (x_iu_1)^2\right)^{-1} \le \frac{-(\alpha+1)^2}{2\alpha-1}\frac{\int_\Omega u_1^{2\alpha-1}u_{1,x_ix_i}}{\left(\int_\Omega u_1^{\alpha+1}\right)^2}. \tag{2.7}$$

Applying (2.7) to (2.6), one gets

$$\begin{aligned}\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1} &\le n + \frac{(\alpha+1)^2}{2\alpha-1}A(\alpha)\sup_{\overline{\Omega}}\left(1-\frac{V}{\lambda_1}\right)\\ &\le n + (M+1)\frac{(\alpha+1)^2}{2\alpha-1}A(\alpha),\end{aligned} \tag{2.8}$$

where $A(\alpha) = \int_\Omega u_1^{2\alpha}/\left(\int_\Omega u_1^{\alpha+1}\right)^2$.

In the following, we will find an upper bound of

$$\frac{(\alpha+1)^2}{2\alpha-1}A(\alpha). \tag{2.9}$$

Define

$$\phi = u_1^\alpha - u_1\int_\Omega u_1^{\alpha+1}, \text{for } \alpha > 1.$$

Then we have

$$\int_\Omega \phi u_1 = 0.$$

This means that

$$\lambda_2 \le \frac{\int_\Omega \phi(-\Delta+V)\phi}{\int_\Omega \phi^2}. \tag{2.10}$$

Note that

$$\begin{aligned}\alpha\int_\Omega u_1^{\alpha-1}|\nabla u_1|^2 &= \int_\Omega u_1^\alpha(-\Delta)u_1\\ &= \int_\Omega (\lambda_1-V)u_1^{\alpha+1},\\ (2\alpha-1)\int_\Omega u_1^{2\alpha-2}|\nabla u_1|^2 &= \int_\Omega u_1^{2\alpha-1}(-\Delta)u_1\\ &= \int_\Omega (\lambda_1-V)u_1^{2\alpha},\end{aligned}$$

and

$$(-\Delta+V)\phi=-\alpha(\alpha-1)u_1^{\alpha-2}|\nabla u_1|^2+(\alpha\lambda_1-\alpha V+V)u_1^{\alpha}-\lambda_1 u_1\int_\Omega u_1^{\alpha+1}.$$

Hence, we have

$$\begin{aligned}\int_\Omega \phi(-\Delta+V)\phi &= \frac{\alpha^2}{2\alpha-1}\lambda_1\int_\Omega u_1^{2\alpha}-\frac{(\alpha-1)^2}{2\alpha-1}\int_\Omega Vu_1^{2\alpha}-\lambda_1\left(\int_\Omega u_1^{\alpha+1}\right)^2\\ &\le\left(\frac{\alpha^2}{2\alpha-1}+\frac{(\alpha-1)^2}{2\alpha-1}M\right)\lambda_1\int_\Omega u_1^{2\alpha}-\lambda_1\left(\int_\Omega u_1^{\alpha+1}\right)^2.\end{aligned}\tag{2.11}$$

From (2.10) and (2.11), we arrive at

$$\frac{\lambda_2}{\lambda_1}\le\frac{\left(\frac{\alpha^2}{2\alpha-1}+\frac{(\alpha-1)^2}{2\alpha-1}M\right)A(\alpha)-1}{A(\alpha)-1}.\tag{2.12}$$

Again, by using the Cauchy-Schwarz inequality, one gets

$$\left(\int_\Omega u_1^{\alpha+1}\right)^2=\left(\int_\Omega u_1^{\alpha}u_1\right)^2\le\int_\Omega u_1^{2\alpha}\int_\Omega u_1^2=\int_\Omega u_1^{2\alpha}.$$

This means that $A(\alpha)>1$ for $\alpha>1$. If α is restricted to the condition

$$\xi-\left(\frac{\alpha^2}{2\alpha-1}+\frac{(\alpha-1)^2}{2\alpha-1}M\right)>0,$$

that is

$$1<\alpha<\frac{(M+\xi)+\sqrt{(M+\xi)(\xi-1)}}{M+1}.\tag{2.13}$$

Then (2.12) is equivalent to

$$A(\alpha)\le\frac{\xi-1}{\xi-\left(\frac{\alpha^2}{2\alpha-1}+\frac{(\alpha-1)^2}{2\alpha-1}M\right)},\tag{2.14}$$

where $\xi=\lambda_2/\lambda_1$. Inserting (2.14) into (2.8) yields

$$\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1}\le n+(M+1)(\xi-1)f(\alpha),\tag{2.15}$$

where

$$f(\alpha)=\frac{(\alpha+1)^2}{(2\alpha-1)\xi-[\alpha^2+(\alpha-1)^2M]}.$$

The minimum of $f(\alpha)$ as a function of α in the range (2.13) is

$$\frac{(M+1)(3\xi+4M+1)}{(\xi+M)(\xi-1)}$$

and this is attained at

$$\alpha = \frac{2\xi + 2M}{\xi + 2M + 1}.$$

Hence, (2.15) yields

$$\begin{aligned}\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} &\leq n + (M+1)(\xi - 1) f\left(\frac{2\xi + 2M}{\xi + 2M + 1}\right) \\ &= n + \frac{(M+1)(3\xi + 4M + 1)}{\xi + M}.\end{aligned}$$

This concludes the proof of theorem.

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