Communications in Mathematics and Applications
Vol. 13, No. 2, pp. 461–476, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i2.1757

http://www.rgnpublications.com

Research Article

Stability Results of Additive-Quadratic *n*-Dimensional Functional Equation: Fixed Point Approach

S. Karthikeyan*[1], John M. Rassias[2], S. Vijayakumar[1] and K. Sakthivel[1]

[1] *Department of Mathematics, R.M.K. Engineering College, Kavaraipettai 601206, Tamil Nadu, India*
[2] *Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, Athens 15342, Greece*
***Corresponding author:** karthik.sma204@yahoo.com

Received: December 6, 2021 **Accepted:** March 8, 2022

Abstract. In this paper, the authors discussed the Ulam-Hyers stability results of n-dimensional mixed type additive and quadratic functional equation:

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = \left(\frac{-n^2+7n-6}{2}\right)\sum_{i=1}^{n} f(x_i) + \left(\frac{-n^2+5n-2}{2}\right)\sum_{i=1}^{n} f(-x_i) + \left(\frac{n-4}{2}\right)\sum_{1\le i<j\le n}(f(x_i+x_j)+f(-x_i-x_j)),$$

where

$$x_{ij} = \begin{cases} -x_j & \text{if } i=j, \\ x_j & \text{if } i\neq j, \end{cases}$$

in Banach spaces using fixed point method.

Keywords. Additive functional equation, Quadratic functional equation, Generalized Ulam-Hyers stability, JMRassias stability

Mathematics Subject Classification (2020). 39B52, 39B82, 46B25

1. Introduction

The stability of functional equations had been first raised by S.M. Ulam [28]. In 1941, D.H. Hyers [10] gave a positive answer to the question of Ulam for Banach spaces. In 1950, T. Aoki [3] was the second author to treat this problem for additive mappings. Th.M. Rassias [23] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p)$, $p \in [0,1)$ to be unbounded. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called Hyers-Ulam-Rassias stability one can refer [1, 6, 11, 15].

In 1982, J.M. Rassias [21] followed the innovative approach of the Th.M. Rassias theorem [23] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^q$ for $p, q \in R$ with $p + q \neq 1$. A generalization of all the above results was obtained by P. Gavruta [9] in 1994 by replacing the unbounded Cauchy difference by a general control function $\phi(x, y)$ in the spirit of Rassias approach.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi *et al.* [27] by considering the summation of both the sum and the product of two p-norms in the spirit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 4, 5, 17, 18, 22, 27]) and reference cited there in.

The *additive functional equation* is

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

Since $f(x) = kx$ is the solution of the functional equation (1.1), every solution of the additive functional equation is called an additive mapping.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

is said to be *quadratic functional equation* because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.2). Quadratic functional equation was used to characterize inner product spaces [1, 16]. It is well known that a function f is a solution of (1.2) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see [16]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}[f(x+y) + f(x-y)]. \tag{1.3}$$

K.W. Jun and H.M. Kim [13] introduced the following generalized *quadratic and additive type functional equation*

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j) \tag{1.4}$$

in the class of function between real vector spaces. For $n = 3$, P.L. Kannappan proved that a function f satisfies the functional equation (1.4) if and only if there exists a symmetric bi-additive function A and additive function B such that $f(x) = B(x,x) + A(x)$ for all x (see [16]). The Hyers-Ulam stability for the equation (1.4) when $n = 3$ was proved by S.M. Jung [14].

The Hyers-Ulam-Rassias stability for the equation (1.4) when $n = 4$ was also investigated by I.S. Chang *et al.* [7].

The general solution and the generalized Hyers-Ulam stability for the *quadratic and additive type functional equation*

$$f(x+ay)+af(x-y) = f(x-ay)+af(x+y) \tag{1.5}$$

for any positive integer a with $a \neq -1,0,1$ was discussed by K.W. Jun and H.M. Kim [12]. Recently, A. Najati and M.B. Moghimi [20] investigated the generalized Hyers-Ulam-Rassias stability for the *quadratic and additive type functional equation* of the form

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.6}$$

In this paper, the authors established generalized Ulam-Hyers stability of n-dimensional mixed type additive and quadratic functional equation of the form

$$\begin{aligned}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) &= \left(\frac{-n^2+7n-6}{2}\right)\sum_{i=1}^{n} f(x_i) + \left(\frac{-n^2+5n-2}{2}\right)\sum_{i=1}^{n} f(-x_i) \\ &\quad + \left(\frac{n-4}{2}\right)\sum_{1\le i<j\le n}(f(x_i+x_j)+f(-x_i-x_j)),\end{aligned} \tag{1.7}$$

where

$$x_{ij} = \begin{cases} -x_j & \text{if } i = j, \\ x_j & \text{if } i \neq j, \end{cases}$$

using fixed point method.

2. Stability Results: Fixed Point Method

This section deals with the generalized Ulam-Hyers stability of the functional equation (1.7) in Banach spaces.

Through out this section, let M be a normed space and N be a Banach space, respectively. Define mappings $Df, Df_o, Df_e : M \to N$ by

$$\begin{aligned}Df(x_1,x_2,\cdots,x_n) &= \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - \left(\frac{-n^2+7n-6}{2}\right)\sum_{i=1}^{n} f(x_i) \\ &\quad - \left(\frac{-n^2+5n-2}{2}\right)\sum_{i=1}^{n} f(-x_i) - \left(\frac{n-4}{2}\right)\sum_{1\le i<j\le n}(f(x_i+x_j)-f(-x_i-x_j)), \\ &\qquad \text{for all } x_1,x_2,\cdots,x_n \in M,\end{aligned}$$

$$Df_o(x_1,x_2,\cdots,x_n) = \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - (n-2)\sum_{j=1}^{n} f(x_j), \text{ for all } x_1,x_2,\cdots,x_n \in M, \text{ and}$$

$$\begin{aligned}Df_e(x_1,x_2,\cdots,x_n) &= \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2+6n-4)\sum_{i=1}^{n} f(x_i) - (n-4)\sum_{1\le i<j\le n} f(x_i+x_j) \\ &\qquad \text{for all } x_1,x_2,\cdots,x_n \in M.\end{aligned}$$

Now we will recall the fundamental results in fixed point theory.

Theorem 2.1 (Banach's Contraction Principle). *Let (X,d) be a complete metric space and consider a mapping $T:X\to X$ which is strictly contractive mapping, i.e.*

(A1) $d(Tx,Ty)\le Ld(x,y)$ *for some (Lipschitz constant) $L<1$. Then,*

- (i) *The mapping T has one and only fixed point $x^*=T(x^*)$;*
- (ii) *The fixed point for each given element x^* is globally attractive, i.e.*

(A2) $\lim_{n\to\infty} T^n x = x^*$, *for any starting point $x\in X$;*

- (iii) *One has the following estimation inequalities:*

(A3) $d(T^n x,x^*)\le \frac{1}{1-L}d(T^n x,T^{n+1}x)$, *for all $n\ge 0$, for all $x\in X$;*

(A4) $d(x,x^*)\le \frac{1}{1-L}d(x,x^*)$, *for all $x\in X$.*

Theorem 2.2 ([8], The Alternative of Fixed Point). *Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T:X\to X$ with Lipschitz constant L. Then, for each given element $x\in X$, either*

(B1) $d(T^n x,T^{n+1}x)=\infty$, *for all $n\ge 0$, or*

(B2) *there exists a natural number n_0 such that:*

- (i) $d(T^n x,T^{n+1}x)<\infty$ *for all $n\ge n_0$;*
- (ii) *the sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
- (iii) *y^* is the unique fixed point of T in the set $Y=\{y\in X: d(T^{n_0}x,y)<\infty\}$;*
- (iv) $d(y^*,y)\le \frac{1}{1-L}d(y,Ty)$ *for all $y\in Y$.*

2.1 Additive Stability Results

This subsection deals with the Ulam-Hyers stability results of additive functional equation using fixed point method in Banach spaces.

Theorem 2.3. *Let $f_o:M\to N$ be a mapping for which there exist a function $\Xi:M^n\to[0,\infty)$ with the condition*

$$\lim_{k\to\infty}\frac{1}{\pi_i^k}\Xi(\pi_i^k x_1,\pi_i^k x_2,\cdots,\pi_i^k x_n)=0, \tag{2.1}$$

where $\pi_i=n-2$ if $i=0$ and $\pi_i=\frac{1}{n-2}$ if $i=1$ such that the functional inequality with

$$\|Df_o(x_1,x_2,\cdots,x_n)\|\le \Xi(x_1,x_2,\cdots,x_n), \tag{2.2}$$

for all $x_1,x_2,\cdots,x_n\in M$. If there exists $L=L(i)<1$ such that the function

$$x\to\Gamma(x)=\frac{1}{n}\Xi\Big(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\Big), \tag{2.3}$$

has the property

$$\Gamma(x)\le L\frac{1}{\pi_i}\Gamma(\pi_i x), \tag{2.4}$$

for all $x\in M$, then there exists a unique additive mapping $\Delta:M\to N$ satisfying the functional equation (1.7) and

$$\|f_o(x)-\Delta(x)\|\le \frac{L^{1-i}}{1-L}\Gamma(x), \tag{2.5}$$

for all $x\in U$.

Proof. Consider the set

$$\Omega = \{p/p : M \to N, p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = d_\Gamma(p,q) = \inf\{K \in (0,\infty) : \|p(x) - q(x)\| \le K\Gamma(x), x \in M\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(x) = \frac{1}{\pi_i} p(\pi_i x),$$

for all $x \in M$. Now $p, q \in \Omega$, we have

$$d(p,q) \le K$$

$$\Rightarrow \quad \|p(x) - q(x)\| \le K\Gamma(x), \qquad x \in M$$

$$\Rightarrow \quad \left\|\frac{1}{\pi_i} p(\pi_i x) - \frac{1}{\pi_i} q(\pi_i x)\right\| \le \frac{1}{\pi_i} K\Gamma(\pi_i x), \quad x \in M$$

$$\Rightarrow \quad \left\|\frac{1}{\pi_i} p(\pi_i x) - \frac{1}{\pi_i} q(\pi_i x)\right\| \le LK\Gamma(x), \qquad x \in M$$

$$\Rightarrow \quad \|Tp(x) - Tq(x)\| \le LK\Gamma(x), \qquad x \in M$$

$$\Rightarrow \quad d_\Gamma(p,q) \le LK$$

This implies $d(Tp, Tq) \le Ld(p,q)$, for all $p, q \in \Omega$, i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, \cdots, x)$ in (1.7), we arrive

$$\left\|f_o(x) - \frac{f_o((n-2)x)}{n-2}\right\| \le \frac{1}{n(n-2)} \Xi(x, x, \cdots, x), \tag{2.6}$$

for all $x \in M$.

Using (2.3) and (2.4) for the case $i = 0$, it reduces to

$$\left\|f_o(x) - \frac{f_o((n-2)x)}{n-2}\right\| \le L\Gamma(x),$$

for all $x \in U$, i.e.,

$$d_\Xi(f, Tf) \le L \quad \Rightarrow \quad d(f, Tf) \le L \le L^1 < \infty.$$

Again replacing $x = \frac{x}{n-2}$ in (2.6), we get

$$\left\|(n-2) f_o\left(\frac{x}{n-2}\right) - f_o(x)\right\| \le \frac{1}{n} \Xi\left(\frac{x}{n-2}, \frac{x}{n-2}, \cdots, \frac{x}{n-2}\right), \tag{2.7}$$

for all $x \in M$. Using (2.3) and (2.4) for the case $i = 1$ it reduces to

$$\left\|f_o(x) - (n-2) f_o\left(\frac{x}{n-2}\right)\right\| \le \Gamma(x),$$

for all $x \in M$, i.e.,

$$d_\Xi(f, Tf_o) \le 1 \quad \Rightarrow \quad d(f, Tf_o) \le 1 \le L^0 < \infty.$$

In both cases, we arrive

$$d(f, Tf_o) \le L^{1-i}.$$

Therefore (A1) holds.

By (A2), it follows that there exists a fixed point Δ of T in Ω such that

$$\Delta(x) = \lim_{k\to\infty} \frac{1}{\pi_i^k}(f_o(\pi_i^k x)), \tag{2.8}$$

for all $x \in M$.

To prove $\Delta : M \to N$ is additive. Replacing $(x_1, x_2, \cdots, x_n)$ by $(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)$ in (2.2) and dividing by π_i^k, it follows from (2.1) that

$$\begin{aligned}\|\Delta(x_1, x_2, \cdots, x_n)\| &= \lim_{k\to\infty} \frac{\|D\ f_o(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)\|}{\pi_i^k} \\ &\le \lim_{k\to\infty} \frac{\Xi(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)}{\pi_i^k} = 0,\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in M$. i.e., Δ satisfies the functional equation (1.7).

By (A3), Δ is the unique fixed point of T in the set $\Delta = \{\Delta \in \Omega : d(f, Q) < \infty\}$, Δ is the unique function such that

$$\|f_o(x) - \Delta(x)\| \le K\Gamma(x),$$

for all $x \in M$ and $K > 0$. Finally, by (A4), we obtain

$$d(f_o, \Delta) \le \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f_o, \Delta) \le \frac{L^{1-i}}{1-L}$$

which yields

$$\|f_o(x) - \Delta(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x).$$

This completes the proof of the theorem. □

The following corollaries express the instant significance of the Theorem 2.3 concerning the Ulam-Hyers, Hyers-Ulam-Rassias, Ulam-Gavruta-Rassias and Rassias stability results of the functional equation (1.7).

Corollary 2.4. *Let $f_o : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|Df_o(x_1, x_2, \cdots, x_n)\| \le \tau, \tag{2.9}$$

for all $x_1, x_2, \cdots, x_n \in M$, then there exists a unique additive function $\Delta : M \to N$ such that

$$\|f_o(x) - \Delta(x)\| \le \frac{\tau}{n|n-3|}, \tag{2.10}$$

for all $x \in M$.

Proof. Setting $\Xi(x_1, x_2, \cdots, x_n) = \tau$, for all $x_1, x_2, \cdots, x_n \in M$. Now,

$$\begin{aligned}\frac{\Xi(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)}{\pi_i^k} &= \frac{\tau}{\pi_i^k} = \tau\pi_i^{-k} \\ &\to 0 \ \text{ as } \ k \to \infty.\end{aligned}$$

Thus, (2.1) is holds.

But we have $\Gamma(x)=\frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right)$ has the property $\Gamma(x)\le L\cdot\frac{1}{\pi_i}\Gamma(\pi_i x)$, for all $x\in M$. Hence

$$\Gamma(x)=\frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right)=\frac{\tau}{n}.$$

Now,

$$\frac{1}{\pi_i}\Gamma(\pi_i x)=\frac{\tau}{n\pi_i}=\pi_i^{-1}\Gamma(x),$$

for all $x\in M$. Now from (2.5), we prove the following cases:

Case 1: $L=(n-2)^{-1}$ for $s=0$ if $i=0$,

$$\|f_o(x)-\Delta(x)\|\le\frac{\tau}{n}\left(\frac{((n-2)^{-1})^{1-0}}{1-(n-2)^{(-1)}}\right)\le\frac{\tau}{n(n-3)}.$$

Case 2: $L=(n-2)^{1}$ for $s=0$ if $i=0$,

$$\|f_o(x)-\Delta(x)\|\le\frac{\tau}{n}\left(\frac{((n-2)^{1})^{1-1}}{1-(n-2)^{1}}\right)\le\frac{\tau}{-n(n-3)}.$$

Hence the proof is complete. □

Corollary 2.5. *Let $f_o:M\to N$ be a mapping and there exist real numbers τ and s such that*

$$\|Df_o(x_1,x_2,\cdots,x_n)\|\le\tau\sum_{i=1}^{n}\|x_i\|^s,\quad s\neq 1, \tag{2.11}$$

for all $x_1,x_2,\cdots,x_n\in M$, then there exists a unique additive function $\Delta:M\to N$ such that

$$\|f_o(x)-\Delta(x)\|\le\tau\frac{(\|x\|^s)}{|(n-2)-(n-2)^s|}, \tag{2.12}$$

for all $x\in M$.

Proof. Setting $\Xi(x_1,x_2,\cdots,x_n)=\tau\sum_{i=1}^{n}\|x_i\|^s$, for all $x_1,x_2,\cdots,x_n\in M$. Now,

$$\frac{\Xi(\pi_i^k x_1,\pi_i^k x_2,\cdots,\pi_i^k x_n)}{\pi_i^k}=\frac{\tau}{\pi_i^k}\sum_{i=1}^{n}\|\pi_i^k x_i\|^s=\tau\pi_i^{k(s-1)}\sum_{i=1}^{n}\|x_i\|^s \to 0 \ \text{as}\ k\to\infty.$$

Thus, (2.1) is holds.

But we have $\Gamma(x)=\frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right)$ has the property $\Gamma(x)\le L\cdot\frac{1}{\pi_i}\Gamma(\pi_i x)$, for all $x\in M$. Hence

$$\Gamma(x)=\frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right)=\frac{\tau}{n}\frac{n}{(n-2)^s}\|x\|^s.$$

Now,

$$\frac{1}{\pi_i}\Gamma(\pi_i x)=\frac{\tau}{\pi_i}\frac{1}{(n-2)^s}(\|\pi_i x\|^s)=\pi_i^{s-1}\Gamma(x).$$

Now from (2.5), we prove the following cases:

Case 3: $L=(n-2)^{s-1}$ for $s>1$ if $i=0$,

$$\|f_o(x)-\Delta(x)\|\le\frac{\tau}{(n-2)^s}\left(\frac{((n-2)^{(s-1)})^{1-0}}{1-(n-2)^{(s-1)}}\right)\|x\|^s\le\frac{\tau}{(n-2)-(n-2)^s}\|x\|^s.$$

Case 4: $L = (n-2)^{1-s}$ for $s < 1$ if $i = 1$,

$$\|f_o(x) - \Delta(x)\| \le \frac{\tau}{(n-2)^s}\left(\frac{((n-2)^{(1-s)})^{1-1}}{1-(n-2)^{(1-s)}}\right)\|x\|^s \le \frac{\tau}{(n-2)^s-(n-2)}\|x\|^s.$$

Hence the proof is complete. □

Corollary 2.6. *Let $f_o : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|Df_o(x_1,x_2,\cdots,x_n)\| \le \tau \prod_{i=1}^{n}\|x_i\|^s, \quad s \ne \frac{1}{n}, \tag{2.13}$$

for all $x_1,x_2,\cdots,x_n \in M$, then there exists a unique additive function $\Delta : M \to N$ such that

$$\|f_o(x) - \Delta(x)\| \le \tau \frac{(\|x\|^{ns})}{n|(n-2)-(n-2)^{ns}|}, \tag{2.14}$$

for all $x \in M$.

Proof. Setting $\Xi(x_1,x_2,\cdots,x_n) = \tau \prod_{i=1}^{n}\|x_i\|^s$, for all $x_1,x_2,\cdots,x_n \in M$. Now,

$$\frac{\Xi(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)}{\pi_i^k} = \frac{\tau}{\pi_i^k}\prod_{i=1}^{n}\|\pi_i^k x_i\|^s = \tau \pi_i^{k(ns-1)}\prod_{i=1}^{n}\|x_i\|^s$$
$$\to 0 \text{ as } k \to \infty.$$

Thus, (2.1) is holds.

But we have $\Gamma(x) = \frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right)$ has the property $\Gamma(x) \le L \cdot \frac{1}{\pi_i}\,\Gamma(\pi_i x)$ for all $x \in M$. Hence

$$\Gamma(x) = \frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right) = \frac{\tau}{n}\frac{1}{(n-2)^{ns}}\|x\|^{ns}.$$

Now,

$$\frac{1}{\pi_i}\Gamma(\pi_i x) = \frac{\tau}{\pi_i}\frac{1}{n(n-2)^{ns}}(\|\pi_i x\|^s) = \pi_i^{ns-1}\Gamma(x).$$

Now from (2.5), we prove the following cases:

Case 5: $L = (n-2)^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$,

$$\|f_o(x) - \Delta(x)\| \le \frac{\tau}{n(n-2)^s}\left(\frac{((n-2)^{(ns-1)})^{1-0}}{1-(n-2)^{(ns-1)}}\right)\|x\|^{ns}$$
$$\le \frac{\tau}{n((n-2)-(n-2)^{ns})}\|x\|^{ns}.$$

Case 6: $L = (n-2)^{1-ns}$ for $s > \frac{1}{n}$ if $i = 1$,

$$\|f_o(x) - \Delta(x)\| \le \frac{\tau}{n(n-2)^s}\left(\frac{\left((n-2)^{(1-ns)}\right)^{1-0}}{1-(n-2)^{(1-ns)}}\right)\|x\|^{ns}$$
$$\le \frac{\tau}{n((n-2)^{ns}-(n-2))}\|x\|^{ns}.$$

Hence the proof is complete. □

Corollary 2.7. *Let $f_o : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|Df_o(x_1,x_2,\cdots,x_n)\| \le \tau\left\{\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}, \quad s \neq \frac{1}{n}, \tag{2.15}$$

for all $x_1,x_2,\cdots,x_n \in M$, then there exists a unique additive function $\Delta : M \to N$ such that

$$\|f_o(x)-\Delta(x)\| \le \frac{((1+n)\tau\|x\|^{ns})}{n|(n-2)-(n-2)^{ns}|}, \tag{2.16}$$

for all $x \in M$.

Proof. Setting $\Xi(x_1,x_2,\cdots,x_n) = \tau\left\{\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}$, for all $x_1,x_2,\cdots,x_n \in M$. Now,

$$\frac{\Xi(\pi_i^k x_1,\pi_i^k x_2,\cdots,\pi_i^k x_n)}{\pi_i^k} = \frac{\tau}{\pi_i^k}\left\{\|\pi_i^n x_i\|^s + \sum_{i=1}^{n}\|\pi_i^k x_i\|^{ns}\right\} = \tau\pi_i^{k(ns-1)}\left\{\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}$$
$$\to 0 \ \text{ as } \ k \to \infty.$$

Thus, (2.1) is holds.

But we have $\Gamma(x) = \frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right)$ has the property $\Gamma(x) \le L\cdot\frac{1}{\pi_i}\,\Gamma(\pi_i x)$ for all $x \in M$. Hence

$$\Gamma(x) = \frac{1}{n}\Xi\left(\frac{x}{n-2},\frac{x}{n-2},\cdots,\frac{x}{n-2}\right) = \frac{\tau}{n}\left(\frac{n}{(n-2)^s} + \frac{1}{(n-2)^{ns}}\right)\|x\|^{ns}.$$

Now,

$$\frac{1}{\pi_i}\Gamma(\pi_i x) = \frac{\tau}{\pi_i}\frac{1+n}{n(n-2)^{ns}}\left(\|\pi_i x\|^{ns}\right) = \pi_i^{ns-1}\Gamma(x).$$

Now from (2.5), we prove the following cases:

Case 7: $L = (n-2)^{ns-1}$ for $s > \frac{1}{n}$ if $i = 0$,

$$\|f_o(x)-\Delta(x)\| \le \frac{((1+n)\tau\|x\|^{ns})}{n\left((n-2)-(n-2)^{ns}\right)}.$$

Case 8: $L = (n-2)^{1-ns}$ for $s < \frac{1}{n}$ if $i = 1$,

$$\|f_o(x)-\Delta(x)\| \le \frac{((1+n)\tau\|x\|^{ns})}{n\left((n-2)^{ns}-(n-2)\right)}.$$

Hence the proof is complete. □

2.2 Quadratic Stability Results

This subsection deals with the Ulam-Hyers stability results of quadratic functional equation using fixed point method in Banach spaces.

Theorem 2.8. *Let $f_e : M \to N$ be a mapping for which there exist a function $\Xi : M^n \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty}\frac{1}{\pi_i^{2k}}\Xi(\pi_i^k x_1,\pi_i^k x_2,\cdots,\pi_i^k x_n) = 0, \tag{2.17}$$

where $\pi_i = n-2$ if $i = 0$ and $\pi_i = \frac{1}{n-2}$ if $i = 1$ such that the functional inequality with

$$\|D\ f_o(x_1,x_2,\cdots,x_n)\| \le \Xi(x_1,x_2,\cdots,x_n), \tag{2.18}$$

for all $x_1, x_2, \cdots, x_n \in M$. *If there exists* $L = L(i) < 1$ *such that the function*

$$x \to \Gamma(x) = \frac{1}{n}\Xi\left(\frac{x}{n-2}, \frac{x}{n-2}, \cdots, \frac{x}{n-2}\right), \tag{2.19}$$

has the property

$$\Gamma(x) \le L\frac{1}{\pi_i}\Gamma(\pi_i x), \tag{2.20}$$

for all $x \in M$, *then there exists a unique quadratic mapping* $\nabla : M \to N$ *satisfying the functional equation* (1.7) *and*

$$\|f_e(x) - \nabla(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x), \tag{2.21}$$

for all $x \in U$.

Proof. Consider the set

$$\Omega = \{p/p : M \to N, p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = d_\Gamma(p,q) = \inf\{K \in (0,\infty) : \|p(x) - q(x)\| \le K\Gamma(x), x \in M\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(x) = \frac{1}{\pi_i^2}p(\pi_i x),$$

for all $x \in M$. Now $p, q \in \Omega$, we have

$$\begin{aligned}
& d(p,q) \le K \\
\Rightarrow\quad & \|p(x) - q(x)\| \le K\Gamma(x), && x \in M \\
\Rightarrow\quad & \left\|\frac{1}{\pi_i^2}p(\pi_i x) - \frac{1}{\pi_i^2}q(\pi_i x)\right\| \le \frac{1}{\pi_i^2}K\Gamma(\pi_i x), && x \in M \\
\Rightarrow\quad & \left\|\frac{1}{\pi_i^2}p(\pi_i x) - \frac{1}{\pi_i^2}q(\pi_i x)\right\| \le LK\Gamma(x), && x \in M \\
\Rightarrow\quad & \|Tp(x) - Tq(x)\| \le LK\Gamma(x), && x \in M \\
\Rightarrow\quad & d_\Gamma(p,q) \le LK
\end{aligned}$$

This implies $d(Tp, Tq) \le Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, \cdots, x)$ in (1.7), we arrive

$$\left\|f_e(x) - \frac{f_e((n-2)x)}{(n-2)^2}\right\| \le \frac{1}{n(n-2)^2}\Xi(x, x, \cdots, x), \tag{2.22}$$

for all $x \in M$. Using (2.19) and (2.20) for the case $i = 0$, it reduces to

$$\left\|f_e(x) - \frac{f_e((n-2)x)}{(n-2)^2}\right\| \le L\Gamma(x),$$

for all $x \in M$, i.e.,

$$d_\Xi(f, Tf) \le L \quad \Rightarrow \quad d(f, Tf) \le L \le L^1 < \infty.$$

Again replacing $x = \frac{x}{n-2}$ in (2.22), we get,

$$\left\|(n-2)^2 f_e\left(\frac{x}{n-2}\right) - f_e(x)\right\| \le \frac{1}{n}\Xi\left(\frac{x}{n-2}, \frac{x}{n-2}, \cdots, \frac{x}{n-2}\right), \tag{2.23}$$

for all $x \in M$. Using (2.19) and (2.20) for the case $i = 1$ it reduces to

$$\left\|f_e(x) - (n-2)^2 f_e\left(\frac{x}{n-2}\right)\right\| \le \Gamma(x),$$

for all $x \in M$, i.e.,

$$d_\Xi(f, Tf_e) \le 1 \quad \Rightarrow \quad d(f, Tf_e) \le 1 \le L^0 < \infty.$$

In both cases, we arrive

$$d(f, Tf_e) \le L^{1-i}.$$

Therefore (A1) holds.

By (A2), it follows that there exists a fixed point ∇ of T in Ω such that

$$\nabla(x) = \lim_{k\to\infty} \frac{1}{\pi_i^{2k}} (f_e(\pi_i^k x)) \tag{2.24}$$

for all $x \in M$.

To prove $\nabla : M \to N$ is quadratic. Replacing $(x_1, x_2, \cdots, x_n)$ by $(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)$ in (2.18) and dividing by π_i^{2k}, it follows from (2.17) that

$$\begin{aligned}\|\nabla(x_1, x_2, \cdots, x_n)\| &= \lim_{k\to\infty} \frac{\|D\ f_e(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)\|}{\pi_i^{2k}} \\ &\le \lim_{k\to\infty} \frac{\Xi(\pi_i^k x_1, \pi_i^k x_2, \cdots, \pi_i^k x_n)}{\pi_i^{2k}} = 0,\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in M$. i.e., ∇ satisfies the functional equation (1.7).

By (A3), ∇ is the unique fixed point of T in the set $\Delta = \{\nabla \in \Omega : d(f, \nabla) < \infty\}$, ∇ is the unique function such that

$$\|f_e(x) - \nabla(x)\| \le K\Gamma(x),$$

for all $x \in M$ and $K > 0$. Finally, by (A4), we obtain

$$d(f_e, \nabla) \le \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f_e, \nabla) \le \frac{L^{1-i}}{1-L}$$

which yields

$$\|f_e(x) - \nabla(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x).$$

This completes the proof of the theorem. □

The following corollaries express the instant significance of the Theorem 2.8 concerning the Ulam-Hyers, Hyers-Ulam-Rassias, Ulam-Gavruta-Rassias and Rassias stability results of the functional equation (1.7).

Corollary 2.9. *Let $f_e : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|D\ f_e(x_1, x_2, \cdots, x_n)\| \le \tau, \tag{2.25}$$

for all $x_1, x_2, \cdots, x_n \in M$, then there exists a unique quadratic function $\nabla : M \to N$ such that

$$\|f_e(x) - \nabla(x)\| \le \frac{\tau}{n|n-3|}, \tag{2.26}$$

for all $x \in M$.

Corollary 2.10. *Let $f_e : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|D\ f_e(x_1, x_2, \cdots, x_n)\| \le \tau \sum_{i=1}^{n} \|x_i\|^s, \quad s \neq 2, \tag{2.27}$$

for all $x_1, x_2, \cdots, x_n \in M$, then there exists a unique quadratic function $\nabla : M \to N$ such that

$$\|f_e(x) - \nabla(x)\| \le \frac{\tau \|x\|^s}{|(n-2)^2 - (n-2)^s|}, \tag{2.28}$$

for all $x \in M$.

Corollary 2.11. *Let $f_e : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|D\ f_e(x_1, x_2, \cdots, x_n)\| \le \tau \prod_{i=1}^{n} \|x_i\|^s, \quad s \neq \frac{2}{n}, \tag{2.29}$$

for all $x_1, x_2, \cdots, x_n \in M$, then there exists a unique quadratic function $\nabla : M \to N$ such that

$$\|f_e(x) - \nabla(x)\| \le \frac{\tau \|x\|^{ns}}{n|(n-2)^2 - (n-2)^{ns}|}, \tag{2.30}$$

for all $x \in M$.

Corollary 2.12. *Let $f_e : M \to N$ be a mapping and there exist real numbers τ and s such that*

$$\|D\ f_e(x_1, x_2, \cdots, x_n)\| \le \tau \left\{ \prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right\}, \quad s \neq \frac{2}{n}, \tag{2.31}$$

for all $x_1, x_2, \cdots, x_n \in M$, then there exists a unique quadratic function $\nabla : M \to N$ such that

$$\|f_e(x) - \nabla(x)\| \le \frac{((1+n)\tau \|x\|^{ns})}{n|(n-2)^2 - (n-2)^{ns}|}, \tag{2.32}$$

for all $x \in M$.

2.3 Additive-quadratic Stability Results

This subsection deals with the Ulam-Hyers stability results of mixed type additive-quadratic functional equation using fixed point method in Banach spaces.

Theorem 2.13. *Let $\Xi : M^n \to [0, \infty)$ be a function satisfying* (2.1) *and* (2.17) *for all $x_1, x_2, \cdots, x_n \in M$. Let $f : M \to N$ be a function satisfies the inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\| \le \Xi(x_1, x_2, \cdots, x_n), \tag{2.33}$$

for all $x_1, x_2, \cdots, x_n \in M$. If there exists $L = L(i) < 1$ such that the functions (2.3) *and* (2.19) *has the properties* (2.4) *and* (2.20) *for all $x \in M$, then there exists a unique additive mapping $\Delta : M \to N$ and a unique quadratic mapping $\nabla : M \to N$ satisfying the functional equation* (1.7) *and*

$$\|f(x) - \Delta(x) - \nabla(x)\| \le \frac{L^{1-i}}{1-L} \Gamma(x), \tag{2.34}$$

for all $x \in M$.

Proof. Let $f_e(x) = \frac{1}{2}\{f(x)+f(-x)\}$ for all $x \in M$. Then $f_e(0)=0$, $f_e(x) = f_e(-x_1)$. Hence

$$\begin{aligned}\|Df_e(x_1,x_2,\cdots,x_n)\| &= \frac{1}{2}\{\|Df(x_1,x_2,\cdots,x_n)+Df(-x_1,-x_2,\cdots,-x_n)\|\}\\ &\le \frac{1}{2}\{\|Df(x_1,x_2,\cdots,x_n)\|+\|Df(-x_1,-x_2,\cdots,-x_n)\|\}\\ &\le \frac{1}{2}\{\Xi(x_1,x_2,\cdots,x_n)+\Xi(-x_1,-x_2,\cdots,-x_n)\},\end{aligned}$$

for all $x \in M$. Hence from Theorem 2.8, there exits a unique quadratic function $\nabla : M \to N$ such that

$$\|f_e(x)-\nabla(x)\| \le \frac{1}{2}\frac{L^{1-i}}{1-L}\{\Gamma(x)+\Gamma(-x)\}, \tag{2.35}$$

for all $x \in M$. Again $f_o(x) = \frac{1}{2}\{f(x)-f(-x)\}$ for all $x \in M$. Then $f_o(0)=0$, $f_o(x) = -f_o(-x)$. Hence

$$\begin{aligned}\|Df_o(x_1,x_2,\cdots,x_n)\| &= \frac{1}{2}\{\|Df(x_1,x_2,\cdots,x_n)+Df(-x_1,-x_2,\cdots,-x_n)\|\}\\ &\le \frac{1}{2}\{\|Df(x_1,x_2,\cdots,x_n)\|+\|Df(-x_1,-x_2,\cdots,-x_n)\|\}\\ &\le \frac{1}{2}\{\Xi(x_1,x_2,\cdots,x_n)+\Xi(-x_1,-x_2,\cdots,-x_n)\},\end{aligned}$$

for all $x \in M$. Hence from Theorem 2.3, there exits a unique additive function $\Delta : M \to N$ such that

$$\|f_o(x)-\Delta(x)\| \le \frac{1}{2}\frac{L^{1-i}}{1-L}\{\Gamma(x)+\Gamma(-x)\}, \tag{2.36}$$

for all $x \in M$.

Since $f(x) = f_e(x)+f_o(x)$, then it follows from (2.35) and (2.36), we arrive

$$\begin{aligned}\|f(x)-\Delta(x)-\nabla(x)\| &= \|f_e(x)+f_o(x)-\Delta(x)-\nabla(x)\|\\ &\le \|f_e(x)-\nabla(x)\|+\|f_o(x)-\Delta(x)\|\\ &\le \frac{1}{2}\frac{L^{1-i}}{1-L}\{\Gamma(x)+\Gamma(-x)+\Gamma(x)+\Gamma(-x)\}\\ &\le \frac{L^{1-i}}{1-L}\{\Gamma(x)+\Gamma(-x)\},\end{aligned}$$

for all $x \in M$. Hence this completes the proof. □

The following Corollary is an immediate consequence of Theorem 2.13 concerning the stability of (1.7).

Corollary 2.14. *Let τ and s be nonnegative real numbers. If a function $f : M \to N$ satisfies the inequality*

$$\|Df(x_1,x_2,\cdots,x_n)\| \le \begin{cases}\tau, & \\ \tau\sum_{i=1}^{n}\|x_i\|^s, & s \ne 1,2,\\ \tau\prod_{i=1}^{n}\|x_i\|^s, & s \ne \frac{1}{n},\frac{2}{n},\\ \tau\left\{\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right\}, & s \ne \frac{1}{n},\frac{2}{n},\end{cases} \tag{2.37}$$

for all $x_1, x_2, \cdots, x_n$ in M. Then there exist a unique quadratic function $\nabla : M \to N$ and a unique additive function $\Delta : M \to N$ such that

$$\|f(x) - \Delta(x) - \nabla(x)\| \leq \begin{cases} \tau\left(\frac{1}{n(n-3)} + \frac{1}{n\left((n-2)^2-1\right)}\right), \\ \tau\left(\frac{1}{|(n-2)-(n-2)^s|} + \frac{1}{|(n-2)^2-(n-2)^s|}\right)\|x\|^s, \\ \tau\left(\frac{1}{n|(n-2)-(n-2)^s|} + \frac{1}{n|(n-2)^2-(n-2)^s|}\right)\|x\|^{ns}, \\ \tau\left(\frac{1}{n|(n-2)-(n-2)^{ns}|} + \frac{1}{n|(n-2)^2-(n-2)^{ns}|}\right)\|x\|^{ns}, \end{cases} \tag{2.38}$$

for all $x \in M$.

3. Conclusion

This article has proved the Hyers-Ulam, Hyers-Ulam-Rassias, generalized Hyers-Ulam-Rassias, and Rassias stability results of the additive functional equation, quadratic functional equation and additive-quadratic mixed type functional equations in Banach spaces by the fixed point method.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, *Cambridge University Press*, (1989), DOI: 10.1017/CBO9781139086578.

[2] M. Adam, Alienation of the quadratic and additive functional equations, *Analysis Mathematica* **45** (2019), 449 – 460, DOI: 10.1007/s10476-019-0869-1.

[3] T. Aoki, On the stability of the linear transformation in Banach spaces, *Journal of the Mathematical Society of Japan* **2** (1950), 64 – 66, DOI: 10.2969/jmsj/00210064.

[4] M. Arunkumar and S. Karthikeyan, Solution and stability of n-dimensional additive functional equation, *International Journal of Applied Mathematics* **25**(2) (2012), 163 – 174, URL: https://www.diogenes.bg/ijam/contents/2012-25/v25_2/2_IJAM_25_2_2012.pdf.

[5] M. Arunkumar and S. Karthikeyan, Solution and intuitionistic fuzzy stability of n-dimensional quadratic functional equation: Direct and fixed point methods, *International Journal of Advanced Mathematical Sciences* **2**(1) (2014), 21 – 33, DOI: 10.14419/ijams.v2i1.1498.

[6] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ (2002), DOI: 10.1142/4875.

[7] I.S. Chang, E.H. Lee and H.M. Kim, On the Hyers-Ulam-Rassias stability of a quadratic functional equations, *Mathematical Inequalities & Applications* **6**(1) (2003), 87 – 95, URL: http://files.ele-math.com/abstracts/mia-06-08-abs.pdf.

[8] J.B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bulletin of the American Mathematical Society* **74** (1968), 305 – 309, DOI: 10.1090/S0002-9904-1968-11933-0.

[9] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *Journal of Mathematical Analysis and Applications* **184**(3) (1994), 431 – 436, DOI: 10.1006/jmaa.1994.1211.

[10] D.H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences* **27**(4) (1941), 222 – 224, DOI: 10.1073/pnas.27.4.222.

[11] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Basel (1998), DOI: 10.1007/978-1-4612-1790-9.

[12] K.W. Jun and H.M. Kim, On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation, *Bulletin of the Korean Mathematical Society* **42**(1) (2005), 133 – 148, DOI: 10.4134/BKMS.2005.42.1.133.

[13] K.W. Jun and H.M. Kim, On the stability of an n-dimensional quadratic and additive type functional equation, *Mathematical Inequalities & Applications* **9**(1) (2006), 153 – 165, DOI: 10.7153/mia-09-16.

[14] S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *Journal of Mathematical Analysis and Applications* **222** (1998), 126 – 137, DOI: 10.1006/jmaa.1998.5916.

[15] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor (2001).

[16] P.L. Kannappan, Quadratic functional equation inner product spaces, *Results in Mathematics volume* **27**(3-4) (1995), 368 – 372, DOI: 10.1007/BF03322841.

[17] S. Karthikeyan, C. Park, P. Palani and T.R.K. Kumar, Stability of an additive-quartic functional equation in modular spaces, *Journal of Mathematics and Computer Science* **26**(1) (2022), 22 – 40, DOI: 10.22436/jmcs.026.01.04.

[18] S. Karthikeyan, G. Ganapathy, P. Palani, M. Suresh and S. Jaikumar, Stability of a quadratic functional equation, *Advances in Dynamical Systems and Applications* **16**(2) (2021), 1167 — 1181, DOI: 10.37622/ADSA/16.2.2021.1167-1181.

[19] H.M. Kim and H.Y. Shin, Refined stability of additive and quadratic functional equations in modular spaces, *Journal of Inequalities and Applications* **2017** (2017), Article number: 146, DOI: 10.1186/s13660-017-1422-z.

[20] A. Najati and M.B. Moghimi, On the stability of a quadratic and additive functional equation, *Journal of Mathematical Analysis and Applications* **337**(1) (2008), 399 – 415, DOI: 10.1016/j.jmaa.2007.03.104.

[21] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, *Journal of Functional Analysis* **46** (1982), 126 – 130, DOI: 10.1016/0022-1236(82)90048-9.

[22] J.M. Rassias, P. Narasimman, R. Saadati and M. de la Sen, Approximation of mixed Euler-Lagrange σ-cubic-quartic functional equation in Felbin's type f-NLS, *Journal of Function Spaces* **2021** (2021), Article ID 8068673, 7 pages, DOI: 10.1155/2021/8068673.

[23] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society* **72** (1978), 297 – 300, DOI: 10.2307/2042795.

[24] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Applicandae Mathematica* **62** (2000), 23 – 130, DOI: 10.1023/A:1006499223572.

[25] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht — Bostan — London (2003), DOI: 10.1007/978-94-017-0225-6.

[26] Th.M. Rassias and P. Semrl, On the behavior of mappings which do not satisfy Hyers-Ulam stability, *Proceedings of the American Mathematical Society* **114** (1992), 989 – 993, DOI: 10.2307/2159617.

[27] K. Ravi, M. Arunkumar and J.M. Rassias, Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematics and Statistics* **3** (2008), 36 – 47, URL: https://go.gale.com/ps/i.do?id=GALE%7CA178412133&sid=googleScholar&v=2.1&it=r&linkaccess=abs&issn=09738347&p=AONE&sw=w&userGroupName=anon%7E664874ac.

[28] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, New York (1964).

[29] S. Vijayakumar, S. Karthikeyan, J.M. Rassias and B. Baskaran, A quartic functional equation originating from the sum of the medians of a triangle in fuzzy normed space, *AIP Conference Proceedings* **2282** (2020), Article ID 020006, DOI: 10.1063/5.0028290.