Communications in Mathematics and Applications
Vol. 13, No. 2, pp. 477–491, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i2.1723

http://www.rgnpublications.com

Research Article

Generalized Monotone Mappings with an Application to Variational Inclusions

Tirth Ram* and Mohd Iqbal

Department of Mathematics, University of Jammu, Jammu, India
***Corresponding author:** tir1ram2@yahoo.com

Received: November 9, 2021 **Accepted:** April 19, 2022

Abstract. In this paper, we study a generalized monotone mapping, which is the sum of cocoercive and monotone mapping. The resolvent operator associated with generalized monotone mapping is defined, and some of its properties are discussed. We employ the equivalent formulation of generalized set-valued variational inclusion problems and resolvent equations to show the existence of a solution. In addition, we create an iterative algorithm for the convergence of resolvent equations and solving generalized set-valued variational inclusion problem. An example has also been provided to support the main result.

Keywords. Monotone mapping, Variational inclusions, Iterative algorithm, Resolvent operator, Semi-inner product space

Mathematics Subject Classification (2020). 47H05, 47H10, 49J40

1. Introduction

Variational inequalities and inclusions are important and fascinating mathematical problems that have been thoroughly investigated in recent years due to their wide range of applications in optimization and control, physics, mechanics, economics and transportation, equilibrium and engineering sciences, nonlinear programming, and so on (see, for example, [2–4, 6, 16]). Many iterative methods for solving variational inequalities have been developed, but for convergence, the underlying mapping over the feasible set must be strongly monotone. Strong monotonicity is a stronger concept than cocoercivity. A Lipschitz continuous and strongly monotone mapping is cocoercive, whereas a cocoercive mapping is monotone but not necessarily strongly or strictly monotone. When the underlying mapping is cocoercive and affine, Marcotte and Wu [15], and Tseng [19] explored the convergence of iterative methods. Zhu and Marcotte [21] investigated iterative schemes for solving nonlinear variational inequalities under the cocoercivity assumptions.

To study different variational inequalities and variational inclusions, Zou and Huang [22] presented and investigated in Banach spaces, the $H(\cdot,\cdot)$-accretive mapping and its resolvent operator, Ahmad *et al*. [1] presented and investigated $H(\cdot,\cdot)$-cocoercive mapping and its resolvent operator in real Hilbert spaces, Sahu *et al*. [18] proved the existence of solutions in semi-inner product spaces for a family of nonlinear implicit variational inclusion problems. Using the generalized resolvent technique, Bhat and Zahoor [5] recently introduced and analyzed the (H,φ)-η-monotone mapping in semi-inner product space and discovered the existence of solutions to generalized variational inclusion problems. They proposed an algorithm and performed a sequence convergence analysis. Gupta and Singh [8, 9] recently developed and analyzed $H(\cdot,\cdot,\cdot)$-φ-η-cocoercive mapping in semi-inner product spaces, demonstrating the existence of a solution to the set-valued variational-like inclusion and fixed point problem. Ram and Iqbal [17] introduced and studied the $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive operator and used it to solve a variational-like inclusion including an infinite family of set-valued mappings in semi-inner product spaces using the resolvent equation technique.

In light of recent exciting advances in this field, we investigate a mapping so-called $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive mapping in semi-inner product spaces. We define the $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive mapping's resolvent operator and show that it is single-valued and Lipschitz continuous. Finally, we use these new ideas to solve a variational inclusions problem in semi-inner product spaces, and also provide an example to support the main finding.

First and foremost, we must review the following definitions and key ideas that will be used throughout the paper.

Definition 1.1 ([13, 18]). Consider the vector space E over the field F $(=R$ or $C)$. If a functional $[\cdot,\cdot]: E\times E\to F$ meets the following criteria, it is called a semi-inner product:

(i) $[a_1+a_2,b_1]=[a_1,a_1]+[a_2,b_1]$, for all $a_1,a_2,b_1\in E$,

(ii) $[\alpha a_1,b_1]=\alpha[a_1,b_1]$, for all $\alpha\in F$, $a_1,b_1\in E$,

(iii) $[a_1,a_1]\ge 0$, for $a_1\neq 0$,

(iv) $|[a_1,b_1]|^2\le [a_1,a_1][b_1,b_1]$, for all $a_1,b_1\in E$.

The pair $(E,[\cdot,\cdot])$ is referred as semi-inner product space.

We can claim that every semi-inner product space is a normed linear space because $\|a_1\|=[a_1,a_1]^{\frac{1}{2}}$ is a norm on E. Every normed linear space, on the other hand, can be transformed into a semi-inner product space in an unlimited number of ways. Giles [7] demonstrated that defining a semi-inner product uniquely is possible if the underlying space E is a uniformly convex smooth Banach space. Giles [7], Lumer [13], and Koehler [12] provide a detailed research and foundational results on semi-inner product spaces.

Definition 1.2. [18, 20] Consider the real-Banach space E. The modulus of smoothness $\rho_E : [0,\infty) \to [0,\infty)$ of E is defined as

$$\rho_E(t) = \sup\left\{\frac{\|a_1+b_1\|+\|a_1-b_1\|}{2} - 1 : \|a_1\| = 1,\, \|b_1\| = t,\, t>0\right\}.$$

E is considered to be uniformly smooth if $\lim\limits_{t\to 0} \frac{\rho_E(t)}{t} = 0$.
For $p > 1$, E is said to be p-uniformly smooth if there exists a real constant $c > 0$ such that $\rho_E(t) \leq ct^p$.
If there is a real constant $c > 0$ such that $\rho_E(t) \leq ct^2$, then E is said to be 2-uniformly smooth.

Lemma 1.3 ([18,20])**.** *Let E be a smooth Banach space and $p > 1$ be a real number. The following statements are identical in this case:*

(i) *E is 2-uniformly smooth.*

(ii) *There is a constant $c > 0$, such that the following inequality holds for every $a_1, b_1 \in E$.*

$$\|a_1+b_1\|^2 \leq \|a_1\|^2 + 2\langle b_1, g_{a_1}\rangle + c\|b_1\|^2, \tag{1.1}$$

where $g_{a_1} \in J(a_1)$ and $J(a_1) = \{a_1{}^* \in E^* : \langle a_1, a_1{}^*\rangle = \|a_1\|^2$ and $\|a_1{}^*\| = \|a_1\|\}$ is the normalized duality mapping, where E^* denotes the dual space of E.

Remark 1.4 ([18])**.** Every normed linear space is a semi-inner product space (see [13]). Indeed, according to the Hahn-Banach theorem, for each $a_1 \in E$, there exists at least one functional $g_{a_1} \in E^*$ such that $\langle a_1, g_{a_1}\rangle = \|a_1\|^2$. Given any such mapping g from E into E^*, we can verify that $[b_1, a_1] = \langle b_1, g_{a_1}\rangle$ defines a semi-inner product. As a result, the inequality (1.1) can be written as

$$\|a_1+b_1\|^2 \leq \|a_1\|^2 + 2[b_1, a_1] + c\|b_1\|^2, \quad \text{for all } a_1, b_1 \in E. \tag{1.2}$$

The constant c is chosen to be the smallest possible value. We refer to c as the constant of smoothness of E.

The rest part of the paper is laid out as follows:

We present several definitions in Section 2, which are essential for the subsequent sections. We offer various definitions and assumptions in Section 3 to verify the Lipschitz continuity of the resolvent operator. We prove lemmas and construct an algorithm to prove the strongly convergence and uniqueness of the solutions of the resolvent equation corresponding to the set-valued variational inclusion problem. As an application, an example demonstrating the validity of the main result is provided in Section 4.

2. Preliminaries

Let E be a Banach space that is 2-uniformly smooth. $\|\cdot\|$ and E^* are its norm and topological dual space, respectively. The dual pair of E and E^* is represented by the semi-inner product $[\cdot,\cdot]$. In order to proceed on the next level, we need to review some basic concepts that will be useful in the subsequent sections.

Definition 2.1 ([14, 18])**.** Let E represent a real 2-uniformly smooth Banach space, and $\eta : E \times E \to E$ and $L : E \to E$ represent single-valued mappings. Then L is said to be:

(i) (r,η)-strongly monotone, if there exists a constant $r > 0$ such that

$$[L(a) - L(b), \eta(a,b)] \geq r\|a-b\|^2, \quad \text{for all } a,b \in E,$$

(ii) (s,η)-cocoercive, if there exists a constant $s > 0$ such that

$$[L(a) - L(b), \eta(a,b)] \geq s\|L(a) - L(b)\|^2, \quad \text{for all } a,b \in E,$$

(iii) (s',η)-relaxed cocoercive, if there exists a constant $s' > 0$ such that

$$[L(a) - L(b), \eta(a,b)] \geq -s'\|L(a) - L(b)\|^2, \quad \text{for all } a,b \in E,$$

(iv) α-expansive, if there exists constant $\alpha > 0$ such that

$$\|L(a) - L(b)\| \geq \alpha\|a-b\|, \quad \text{for all } a,b \in E,$$

(v) η is said to be τ-Lipschitz continuous, if there exists constant $\tau > 0$ such that

$$\|\eta(a,b)\| \leq \tau\|a-b\|, \quad \text{for all } a,b \in E.$$

Definition 2.2 ([11])**.** Let $L,M,N,O : E \to E$, $\eta : E \times E \to E$, $H : E \times E \times E \times E \to E$ are single-valued mappings, then

(i) $H(L,\cdot,\cdot,\cdot)$ is (μ_1,η)-cocoercive with respect to L, if there exists constant $\mu_1 > 0$ such that

$$[H(La_1,u,u,u) - H(La_2,u,u,u), \eta(a_1,a_2)] \geq \mu_1\|La_1 - La_2\|^2, \quad \text{for all } u,a_1,a_2 \in E,$$

(ii) $H(\cdot,M,\cdot,\cdot)$ is (μ_2,η)-cocoercive with respect to M, if there exists constant $\mu_2 > 0$ such that

$$[H(u,Ma_1,u,u) - H(u,Ma_2,u,u), \eta(a_1,a_2)] \geq \mu_2\|Ma_1 - Ma_2\|^2, \quad \text{for all } u,a_1,a_2 \in E,$$

(iii) $H(\cdot,\cdot,N,\cdot)$ is (γ,η)-relaxed cocoercive with respect to N, if there exists constant $\gamma > 0$ such that

$$[H(u,u,Na_1,u) - H(u,u,Na_2,u), \eta(a_1,a_2)] \geq -\gamma\|Na_1 - Na_2\|^2, \quad \text{for all } u,a_1,a_2 \in E,$$

(iv) $H(\cdot,\cdot,\cdot,O)$ is (δ,η)-strongly monotone with respect to O, if there exists constant $\delta > 0$ such that

$$[H(u,u,u,Oa_1) - H(u,u,u,Oa_2), \eta(a_1,a_2)] \geq \delta\|a_1 - a_2\|, \quad \text{for all } u,a_1,a_2 \in E,$$

(v) $H(L,\cdot,\cdot,\cdot)$ is κ_1-Lipschitz continuous with respect to L, if there exists constant $\kappa_1 > 0$ such that

$$\|H(La_1,u,u,u) - H(La_2,u,u,u)\| \leq \kappa_1\|a_1 - a_2\|, \quad \text{for all } u,a_1,a_2 \in E.$$

For other components, we can define the Lipschitz continuity for $H(\cdot,\cdot,\cdot,\cdot)$ in the same way.

Let $Q : E \to 2^E$ be a set-valued mapping, and $\mathscr{G}(Q) = \{(a,b) : b \in Q(a)\}$ be the graph of Q. The domain of Q is defined as

$$\text{Dom}(Q) = \{a \in E : \exists\, b \in E : (a,b) \in \mathscr{G}(Q)\}.$$

The Range of Q is defined as

$$\text{Ran}(Q) = \{b \in E : \exists\, a \in E : (a,b) \in \mathscr{G}(Q)\}.$$

The inverse of Q is defined as

$$Q^{-1} = \{(b,a) : (a,b) \in \mathscr{G}(Q)\}.$$

Let R and Q be any two set-valued mappings, and ρ is any real integer, we define

$$R + Q = \{(a, b_1 + b_2) : (a, b_1) \in \mathscr{G}(R), (a, b_2) \in \mathscr{G}(Q)\},$$

$$\rho Q = \{(a, \rho b) : (a,b) \in \mathscr{G}(Q)\}.$$

For any type of mapping A and $Q : E \to 2^E$, which is a set-valued mapping, we define

$$A + Q = \{(a, b_1 + b_2) : Aa = b_1, (a, b_2) \in \mathscr{G}(Q)\}. \quad \text{(see [5])}$$

Definition 2.3 ([14, 18]). A set-valued mapping $Q : E \to 2^E$ is said to be (m,η)-relaxed monotone, if $\exists$ a constant $m > 0$ such that

$$[a^* - b^*, \eta(a,b)] \geq -m\|a-b\|^2, \text{ for all } a,b \in E, a^* \in Q(a), b^* \in Q(b).$$

Definition 2.4. Let $G, \eta : E \times E \to E$ be the mappings. Then $G(\cdot,\cdot)$ is said to be:

(i) (t,η)-relaxed monotone with respect to first component, if there exist a constant $t > 0$ such that

$$[G(a,z) - G(b,z), \eta(a,b)] \geq -t\|a-b\|^2, \quad \text{for all } a,b,z \in E.$$

(ii) ϵ_1-Lipschitz continuous with respect to first component, if there exist a constant $\epsilon_1 > 0$ such that

$$\|G(a,z) - G(b,z)\| \leq \epsilon_1 \|a-b\|^2, \quad \text{for all } a,b,z \in E.$$

Definition 2.5 ([8]). The Hausdorff metric $\mathscr{D}(\cdot,\cdot)$ on $CB(E)$, is defined by

$$\mathscr{D}(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}, \quad A,B \in CB(E),$$

where $d(\cdot,\cdot)$ represents the induced metric on E and $CB(E)$ represents the family of all nonempty closed and bounded subsets of E.

Definition 2.6 ([8]). A set-valued mapping $S : E \to CB(E)$ is $\mathscr{D}$-Lipschitz continuous with constant $\lambda_S > 0$, if

$$\mathscr{D}(Sa, Sb) \leq \lambda_S \|a-b\|, \quad \text{for all } a,b \in E.$$

3. Generalized $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive Mapping

This section contains various definitions and assumptions which are used to prove main results associated with the generalized $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive operator.

Let E be a Banach space that is 2-uniformly smooth. Assume $\eta : E \times E \to E$, $H : E \times E \times E \times E \to E$ and $\varphi, L, M, N, O : E \to E$ represents single-valued mappings and $Q : E \to 2^E$ represents a set-valued mapping.

Definition 3.1. Let $H(\cdot,\cdot,\cdot,\cdot)$ is (μ_1,η)-cocoercive with respect to L with non-negative constant μ_1, (μ_2,η)-cocoercive with respect to M with non-negative constant μ_2, (γ,η)-relaxed cocoercive with respect to N with non-negative constant γ and (δ,η)-strongly monotone with respect to O

with non-negative constant δ. Then Q is called generalized $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive with respect to L, M, N and O if

(i) $\varphi\circ Q$ is (m,η)-relaxed monotone,

(ii) $(H(\cdot,\cdot,\cdot,\cdot)+\lambda\varphi\circ Q)(E)=E$, $\lambda>0$.

We now need to make the following assumptions:

(A1) Let $H(\cdot,\cdot,\cdot,\cdot)$ is (μ_1,η)-cocoercive with respect to L with non-negative constant μ_1, (μ_2,η)-cocoercive with respect to M with non-negative constant μ_2, (γ,η)-relaxed cocoercive with respect to N with non-negative constant γ and (δ,η)-strongly monotone with respect to O with non-negative constant δ with $\mu_1,\mu_2>\gamma$.

(A2) Let L is α_1-expansive, M is α_2-expansive and N is β-Lipschitz continuous with $\alpha_1,\alpha_2>\beta$.

(A3) Let η is τ-Lipschitz continuous.

(A4) Let Q is generalized $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive operator with respect to L,M,N and O.

Theorem 3.2. *Suppose Assumptions* (A1), (A2) *and* (A3) *hold good with* $\ell=\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta>m\lambda$, *then* $(H(L,M,N,O)+\lambda\varphi\circ Q)^{-1}$ *is single-valued.*

Proof. Let $v,w\in(H(L,M,N,O)+\lambda\varphi\circ Q)^{-1}(u)$ for any given $u\in E$. It is obvious that
$$\begin{cases}-H(Lv,Mv,Nv,Ov)+u\in\lambda\varphi\circ Q(v),\\ -H(Lw,Mw,Nw,Ow)+u\in\lambda\varphi\circ Q(w).\end{cases}$$
Since $\varphi\circ Q$ is (m,η)-relaxed monotone in the first component, we have
$$\begin{aligned}-m\lambda\|v-w\|^2&\le[-H(Lv,Mv,Nv,Ov)+u-(-H(Lw,Mw,Nw,Ow)+u),\eta(v,w)]\\&=[-H(Lv,Mv,Nv,Ov)+H(Lw,Mw,Nw,Ow),\eta(v,w)]\\&=-[H(Lv,Mv,Nv,Ov)-H(Lw,Mv,Nv,Ov),\eta(v,w)]\\&\quad-[H(Lw,Mv,Nv,Ov)-H(Lw,Mw,Nv,Ov),\eta(v,w)]\\&\quad-[H(Lw,Mw,Nv,Ov)-H(Lw,Mw,Nw,Ov),\eta(v,w)]\\&\quad-[H(Lw,Mw,Nw,Ov)-H(Lw,Mw,Nw,Ow),\eta(v,w)].\end{aligned}$$
Since Assumption (A1) holds, we have
$$-m\lambda\|v-w\|^2\le-\mu_1\|Lv-Lw\|^2-\mu_2\|Mv-Mw\|^2+\gamma\|Nv-Nw\|^2-\delta\|v-w\|^2.$$
Since Assumption (A2) holds, we have
$$\begin{aligned}-m\lambda\|v-w\|^2&\le-\mu_1{\alpha_1}^2\|v-w\|^2-\mu_2{\alpha_2}^2\|v-w\|^2+\gamma\beta^2\|v-w\|^2-\delta\|v-w\|^2\\&=-(\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta)\|v-w\|^2,\\0&\le-(\ell-m\lambda)\|v-w\|^2\le0,\quad\text{where } \ell=\mu_1{\alpha_1}^2+\mu_2{\alpha_2}^2-\gamma\beta^2+\delta.\end{aligned}$$
Since $\mu_1,\mu_2>\gamma$, $\alpha_1,\alpha_2>\beta$, $\delta>0$, it follows that $\|v-w\|\le0$ and hence $v=w$. Therefore, $(H(L,M,N,O)+\lambda\varphi\circ Q)^{-1}$ is single-valued.

Definition 3.3. Let Assumptions (A1), (A2) and (A4) hold good with $\ell = \mu_1 {\alpha_1}^2 + \mu_2 {\alpha_2}^2 - \gamma\beta^2 + \delta > m\lambda$, then the resolvent operator $R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi} : E \to E$ is given as

$$R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(a) = (H(L,M,N,O) + \lambda\varphi \circ Q)^{-1}(a), \quad \text{for all } a \in E. \tag{3.1}$$

The next step is to show that the resolvent operator defined by (3.1) is Lipschitz continuous.

Theorem 3.4. *Suppose Assumptions* (A1)-(A4) *hold good with* $\ell = \mu_1 {\alpha_1}^2 + \mu_2 {\alpha_2}^2 - \gamma\beta^2 + \delta > m\lambda$, *and* η *is* τ*-Lipschitz, then* $R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi} : E \to E$ *is* $\frac{\tau}{\ell - m\lambda}$*-Lipschitz continuous, that is,*

$$\left\| R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(v) - R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(w) \right\| \le \frac{\tau}{\ell - m\lambda} \|v - w\|, \quad \textit{for all } v, w \in E.$$

Proof. Suppose $v, w \in E$ be any given points, then from (3.1), we have

$$R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(v) = (H(L,M,N,O) + \lambda\varphi \circ Q)^{-1}(v),$$
$$R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(w) = (H(L,M,N,O) + \lambda\varphi \circ Q)^{-1}(w).$$

Let $a_0 = R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(v)$ and $a_1 = R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(w)$.

$$\begin{cases} \lambda^{-1}(v - H(L(a_0), M(a_0), N(a_0), O(a_0))) \in \varphi \circ Q(a_0), \\ \lambda^{-1}(w - H(L(a_1), M(a_1), N(a_1), O(a_1))) \in \varphi \circ Q(a_1). \end{cases}$$

Since $\varphi \circ Q$ is (m,η)-relaxed monotone in the first component, we have

$$[(v - H(L(a_0), M(a_0), N(a_0), O(a_0))) - (w - H(L(a_1), M(a_1), N(a_1), O(a_1))), \eta(a_0, a_1)] \ge -m\lambda \|a_0 - a_1\|^2,$$

which implies

$$[v - w, \eta(a_0, a_1)] \ge [H(L(a_0), M(a_0), N(a_0), O(a_0)) - H(L(a_1), M(a_1), N(a_1), O(a_1)), \eta(a_0, a_1)] - m\lambda \|a_0 - a_1\|^2.$$

Now, we have

$$\begin{aligned} \|v - w\| \, \|\eta(a_0, a_1)\| &\ge [v - w, \eta(a_0, a_1)] \\ &\ge [H(L(a_0), M(a_0), N(a_0), O(a_0)) - H(L(a_1), M(a_1), N(a_1), O(a_1)), \eta(a_0, a_1)] \\ &\quad - m\lambda \|a_0 - a_1\|^2. \end{aligned}$$

Since Assumption (A1), (A2), (A3) hold and η is τ-Lipschitz continuous, we have

$$\|v - w\| \tau \|a_0 - a_1\| \ge (\ell - m\lambda) \|a_0 - a_1\|^2 \quad \text{or}$$
$$\left\| R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(v) - R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(w) \right\| \le \frac{\tau}{\ell - m\lambda} \|v - w\|, \quad \text{for all } v, w \in E.$$

Hence, we get the required result. □

4. Formulation of the Problem and Existence of Solution

In this section, our main aim is to formulate a generalized set-valued variational inclusion problem and establish the existence of a solution by using generalized $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive operator, under certain assumptions.

Let E be 2-uniformly smooth Banach space. Let $V,T : E \to CB(E)$ be the set-valued mappings, and let $L,M,N,O,g,\varphi : E \to E$, $\eta,G : E\times E \to E$ and $H(\cdot,\cdot,\cdot,\cdot) : E\times E\times E\times E \to E$ be single-valued mappings. Suppose that the set-valued mapping $Q : E \to 2^E$ be a generalized $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive operator with respect to L, M, N and O and $\mathrm{Ran}(g)\cap \mathrm{Dom}Q \neq \phi$. We consider the following generalized set-valued variational inclusion problem:

Find $a\in E$, $b\in V(a)$ and $z\in T(a)$ such that

$$0\in G(b,z)+Q(g(a)). \tag{4.1}$$

If E is a real Hilbert space and Q is a maximal monotone operator, then the problem (4.1) is identical to that investigated by Huang *et al*. [10].

Lemma 4.1. *Let us consider the mapping $\varphi : E \to E$ such that $\varphi(b+z)=\varphi(b)+\varphi(z)$ and $\mathrm{Ker}(\varphi)=\{0\}$, where $\mathrm{Ker}(\varphi)=\{b\in E : \varphi(b)=0\}$. Then (a,b,z), where $a\in E$, $b\in V(a)$ and $z\in T(a)$ is a solution of problem (4.1) if and only if (a,b,z) satisfies the following relation:*

$$g(a)=R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}[H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z)]. \tag{4.2}$$

The resolvent equation corresponding to generalized set-valued variational inclusion problem (4.1).

$$\varphi\circ G(b,z)+\lambda^{-1}J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)=0, \tag{4.3}$$

where $\lambda>0$,

$$J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)=[I-H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)),M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)),N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)),O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)))],$$

I is the identity mapping and

$$\begin{aligned}&H(L,M,N,O)[R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)]\\&\quad=H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)),M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)),N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)),O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x))).\end{aligned}$$

Now, we show that the problem (4.1) is equivalent to the resolvent equation problem (4.3).

Lemma 4.2. *(a,b,z), where $a\in E$, $b\in V(a)$ and $z\in T(a)$ is a solution of problem (4.1) if and only if the resolvent equation problem (4.3) has a solution (x,a,b,z) with $x,a\in E$, $b\in V(a)$ and $z\in T(a)$, where*

$$g(a)=R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x) \tag{4.4}$$

and $x=H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z)$.

Proof. Let (a,b,z) be the solution of problem (4.1), and from Lemma 4.1 using the fact that

$$J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}=[I-H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}))],$$

$$\begin{aligned}J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)&=J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}\left[H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z)\right]\\&=[I-H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}))]\\&\quad\cdot[H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z)]\\&=[H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z)]\end{aligned}$$

$$
\begin{aligned}
&-(H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi})))\\
&\cdot(H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z))\\
=&[H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z)]\\
&-H(L(ga),M(ga),N(ga),O(ga))\\
=&-\lambda\varphi\circ G(b,z).
\end{aligned}
$$

This implies that

$$\varphi\circ G(b,z)+\lambda^{-1}J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)=0.$$

Conversely, let (x,a,b,z) is a solution of resolvent equation problem (4.3), then

$$
\begin{aligned}
&J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)=-\lambda\varphi\circ G(b,z)\\
&\left[I-H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}),O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}))\right](x)=-\lambda\varphi\circ G(b,z)\\
&x-H(L(ga),M(ga),N(ga),O(ga))=-\lambda\varphi\circ G(b,z).
\end{aligned}
$$

This implies that

$$x=H(L(ga),M(ga),N(ga),O(ga))-\lambda\varphi\circ G(b,z).$$

Hence (a,b,z) is a solution of variational inclusion problem (4.1).

From numerical point of view, Lemma 4.1 and Lemma 4.2 are quite essential. They allow us to propose an iterative algorithm for finding the approximate solution of (4.3) as follows:

Algorithm 1 (Iterative Algorithm)**.**
For any given (x_0,a_0,b_0,z_0), we can choose $x_0,a_0\in E$, $b_0\in V(a_0)$, and $z_0\in T(a_0)$ and $0<\epsilon<1$ such that sequences $\{x_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{z_n\}$ satisfy

$$
\begin{cases}
g(a_n)=R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x_n),\\
b_n\in V(a_n),\ \|b_n-b_{n+1}\|\le\mathscr{D}(V(a_n),V(a_{n+1}))+\epsilon^{n+1}\|a_n+a_{n+1}\|,\\
z^n\in T(a_n),\ \|z^n-z^{n+1}\|\le\mathscr{D}(T(a_n),T(a_{n+1}))+\epsilon^{n+1}\|a_n-a_{n+1}\|,\\
x_{n+1}=H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))-\lambda\varphi\circ G(b_n,z_n),
\end{cases}
$$

where $\lambda>0$, $k\ge0$ and $\mathscr{D}(\cdot,\cdot)$ is a Hausdorff metric on $CB(X)$.

Next, we find the convergence of the iterative algorithm for the resolvent equation problem (4.3), which corresponds to the generalized set-valued variational inclusion problem (4.1).

Theorem 4.3. *Let us consider the problem* (4.1) *with Assumptions* (A1)-(A4) *and* $\varphi:E\to E$ *be a single-valued mapping with* $\varphi(b+z)=\varphi(b)+\varphi(z)$ *and* $\mathrm{Ker}(\varphi)=\{0\}$. *Assume that*

(i) *V and T are λ_V and λ_T $\mathscr{D}$-Lipschitz continuous, respectively,*

(ii) *$\varphi\circ G$ is (t,η)-relaxed monotone with respect to first component,*

(iii) *$\varphi\circ G$ is ϵ_1, ϵ_2-Lipschitz continuous with respect to first and second component, respectively,*

(iv) *$H(L,M,N,O)$ is $\kappa_1,\kappa_2,\kappa_3,\kappa_4$-Lipschitz continuous with respect to L, M, N and O, respectively,*

(v) *g is r-strongly monotone and λ_g-Lipschitz continuous,*

(vi) $0<\sqrt{\{\lambda_g{}^2\kappa^2+2t\lambda\lambda_V\left(\lambda_g\kappa+\tau\lambda_V\right)+\epsilon_1{}^2\lambda^2\lambda_V{}^2\}}<\frac{(1-\sqrt{1-2r+\lambda_g{}^2})(\ell-m\lambda)}{\tau}-\epsilon_2\lambda\lambda_T$, *where* $\kappa=\kappa_1+\kappa_2+\kappa_3+\kappa_4$.

Then the iterative sequences $\{x_n\}$, $\{a_n\}$, $\{b_n\}$ *and* $\{z_n\}$ *generated by Algorithm* 1 *converges strongly to the unique solution* (x,a,b,z) *of the resolvent equation problem* (4.3).

Proof. Using λ_V,λ_T-$\mathscr{D}$ Lipschitz continuity of V, T and Algorithm 1 , we have

$$\|b_n-b_{n-1}\|\le\mathscr{D}(V(a_n),V(a_{n-1}))+\epsilon^n\|a_n-a_{n-1}\|\le\{\lambda_V+\epsilon^n\}\|a_n-a_{n-1}\|, \tag{4.5}$$

$$\|z_n-z_{n-1}\|\le\mathscr{D}(T(a_n),T(a_{n-1}))+\epsilon^n\|a_n-a_{n-1}\|\le\{\lambda_T+\epsilon^n\}\|a_n-a_{n-1}\|, \tag{4.6}$$

where $n=1,2,3,\ldots$.

We now compute

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))\\ &\quad -H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\\ &\quad -\lambda(\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_{n-1}))\|\\ &\le \|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))\\ &\quad -H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\\ &\quad -\lambda(\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n))\|\\ &\quad +\lambda\left\|\varphi\circ G(b_{n-1},z_n)-\varphi\circ G(b_{n-1},z_{n-1})\right\|.\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))-H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\\ &-\lambda(\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n))\|^2\\ &\le\|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))-H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\|^2\\ &\quad -2\lambda[\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n),\eta(b_n,b_{n-1})]+2\lambda\|\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n)\|\\ &\quad\times\{\|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))-H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\|\\ &\quad+\|\eta(b_n,b_{n-1})\|\}+\lambda^2\|\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n)\|^2.\end{aligned} \tag{4.8}$$

Since $H(L,M,N,O)$ is $\kappa_1,\kappa_2,\kappa_3,\kappa_4$-Lipschitz continuous with respect to L,M,N,O, respectively, we have

$$\begin{aligned}&\|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))-H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\|^2\\ &\quad\le\lambda_g{}^2\kappa^2\|a_n-a_{n-1}\|^2,\end{aligned} \tag{4.9}$$

where $\kappa=\kappa_1+\kappa_2+\kappa_3+\kappa_4$.

Since $\varphi\circ G$ is (t,η)-relaxed monotone, then we have

$$[\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n),\eta(b_n,b_{n-1})]\ge -t\{\lambda_V+\epsilon^n\}^2\|a_n-a_{n-1}\|^2. \tag{4.10}$$

As $\varphi\circ G(\cdot,\cdot)$ is ϵ_1,ϵ_2-Lipschitz continuous in the first, second components, respectively and using (4.5), (4.6), we have

$$\|\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n)\|\le\epsilon_1\{\lambda_V+\epsilon^n\}\|a_n-a_{n-1}\|, \tag{4.11}$$

$$\|\varphi\circ G(b_{n-1},z_n)-\varphi\circ G(b_{n-1},z_{n-1})\|\le\epsilon_2\{\lambda_T+\epsilon^n\}\|a_n-a_{n-1}\|. \tag{4.12}$$

By using Assumption (A3) and (4.9)-(4.12) in (4.8), we have

$$\begin{aligned}&\|H(L(ga_n),M(ga_n),N(ga_n),O(ga_n))-H(L(ga_{n-1}),M(ga_{n-1}),N(ga_{n-1}),O(ga_{n-1}))\\&-\lambda(\varphi\circ G(b_n,z_n)-\varphi\circ G(b_{n-1},z_n))\|\\&\le\sqrt{[\lambda_g{}^2\kappa^2+2t\lambda\{\lambda_V+\epsilon^n\}^2+2\epsilon_1\lambda\{\lambda_V+\epsilon^n\}\{\lambda_g\kappa+\tau\{\lambda_V+\epsilon^n\}\}+\epsilon_1{}^2\lambda^2\{\lambda_V+\epsilon^n\}^2]}\\&\quad\times\|a_n-a_{n-1}\|.\end{aligned}\tag{4.13}$$

Using (4.12) and (4.13) in (4.7), we have

$$\begin{aligned}\|x_{n+1}-x_n\|\le\Big[&\sqrt{\big[\lambda_g{}^2\kappa^2+2t\lambda\{\lambda_V+\epsilon^n\}^2+2\epsilon_1\lambda\{\lambda_V+\epsilon^n\}\big\{\lambda_g\kappa+\tau\{\lambda_V+\epsilon^n\}\big\}+\epsilon_1{}^2\lambda^2\{\lambda_V+\epsilon^n\}^2\big]}\\&+\epsilon_2\lambda\{\lambda_T+\epsilon^n\}\Big]\times\|a_n-a_{n-1}\|.\end{aligned}\tag{4.14}$$

By Lipschitz continuity of resolvent operator and condition (v), we have

$$\begin{aligned}\|a_n-a_{n-1}\|&=\|a_n-a_{n-1}-(g(a_n)-g(a_{n-1}))+R_{Q,\lambda,\varphi}^{H(\cdot,\cdot,\cdot,\cdot)-\eta}(x_n)-R_{Q,\lambda,\varphi}^{H(\cdot,\cdot,\cdot,\cdot)-\eta}(x_{n-1})\|\\&\le\|a_n-a_{n-1}-(g(a_n)-g(a_{n-1}))\|+\frac{\tau}{(\ell-m\lambda)}\|x_n-x_{n-1}\|,\end{aligned}\tag{4.15}$$

$$\|a_n-a_{n-1}-(g(a_n)-g(a_{n-1}))\|^2\le(1-2r+\lambda_g{}^2)\|a_n-a_{n-1}\|^2.\tag{4.16}$$

Using (4.16) in (4.15), we have

$$\|a_n-a_{n-1}\|\le\sqrt{1-2r+\lambda_g{}^2}\|a_n-a_{n-1}\|+\frac{\tau}{(\ell-m\lambda)}\|x_n-x_{n-1}\|$$

$$\|a_n-a_{n-1}\|\le\left[\frac{\tau}{(1-\sqrt{1-2r+\lambda_g{}^2})(\ell-m\lambda)}\right]\|x_n-x_{n-1}\|\tag{4.17}$$

Using (4.17) in (4.14), then (4.14) becomes

$$\|x_{n+1}-x_n\|\le\Psi(\epsilon^n)\|x_n-x_{n-1}\|,\tag{4.18}$$

where

$$\Psi(\epsilon^n)=\frac{\begin{pmatrix}\tau\sqrt{\{\lambda_g{}^2\kappa^2+2t\lambda\{\lambda_V+\epsilon^n\}^2+2\epsilon_1\lambda\{\lambda_V+\epsilon^n\}\big(\lambda_g\kappa+\tau\{\lambda_V+\epsilon^n\}\big)+\epsilon_1{}^2\lambda^2\{\lambda_V+\epsilon^n\}^2\}}\\+\tau\epsilon_2\lambda\{\lambda_T+\epsilon^n\}\end{pmatrix}}{(1-\sqrt{1-2r+\lambda_g{}^2})(\ell-m\lambda)}.$$

Since $0<\epsilon<1$, this implies that $\Psi(\epsilon^n)\to\Psi$ as $n\to\infty$, where

$$\Psi=\frac{\tau\big[\sqrt{\{\lambda_g{}^2\kappa^2+2t\lambda\lambda_V{}^2+2\epsilon_1\lambda\lambda_V\big(\lambda_g\kappa+\tau\lambda_V\big)+\epsilon_1{}^2\lambda^2\lambda_V{}^2\}}+\epsilon_2\lambda\lambda_T\big]}{\big(1-\sqrt{1-2r+\lambda_g{}^2}\big)(\ell-m\lambda)}.$$

From condition (vi), $\Psi<1$, then $\{x_n\}$ is a Cauchy sequence in Banach space E, hence $x_n\to x$ as $n\to\infty$.

From (4.17), $\{a_n\}$ is also a Cauchy sequence in Banach space E, then there exist a such that $a_n\to a$.

From (4.5)-(4.6) and Algorithm 1, the sequences $\{b_n\}$ and $\{z_n\}$ are also Cauchy sequences in Banach space E. Thus, there exists b and z such that $b_n\to b$ and $z_n\to z$ as $n\to\infty$.

Next we prove that $b\in V(a)$. Since $b_n\in V(a)$, then

$$d(b,V(a))\le\|b-b_n\|+d(b_n,V(a))$$

$$\leq \|b-b_n\| + \mathscr{D}(V(a_n),V(a))$$
$$\leq \|b-b_n\| + \lambda_V \|a_n - a\| \to 0, \text{ as } n\to\infty,$$

which gives $d(b,V(a))=0$. Since $V(a)\in CB(E)$, we have $b\in V(a)$. Similarly, we can show that $z\in T(a)$.

By the continuity of $R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}$, L, M, N, O, V, T, $\varphi\circ G$, g, η and Q and Algorithm 1, we know that a, b, z and x satisfy

$$x_{n+1} = [H(L(ga_n),M(ga_n),N(ga_n),O(ga_n)) - \varphi\circ G(b_n,z_n)],$$
$$\to x = [H(L(ga),M(ga),N(ga),O(ga)) - \varphi\circ G(b,z)] \text{ as } n\to\infty$$
$$R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x_n) = g(a_n) \to g(a) = R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x) \text{ as } n\to\infty.$$

Now by using Lemma 4.2, we have

$$\varphi\circ G(b,z) + \lambda^{-1}(x - H(L(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)), M(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)), N(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)), O(R^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x)))) = 0.$$

Thus, we have

$$\varphi\circ G(b,z) + \lambda^{-1} J^{H(\cdot,\cdot,\cdot,\cdot)-\eta}_{Q,\lambda,\varphi}(x) = 0. \tag{4.19}$$

Hence (x,a,b,z) is a solution of the problem (4.3).

Example 4.4. Let $E=R^2$ with usual inner product. Let $V,T: R^2\to 2^{R^2}$ are defined by

$$Va = Ta = \left\{ \begin{pmatrix} \frac{1}{n}a_1 \\ \frac{1}{n}a_2 \end{pmatrix} : \text{for all } n\in N,\ a=(a_1,a_2)\in R^2 \right\}.$$

Then, it is easy to check that

(i) V and T are $\frac{1}{10}$ $\mathscr{D}$-Lipschitz continuous for n=10.

Let $\varphi: R^2\to R^2$ be defined by

$$\varphi(a) = \begin{pmatrix} \frac{1}{5}a_1 \\ \frac{1}{5}a_2 \end{pmatrix}, \text{ for all } a=(a_1,a_2)\in R^2$$

and $G: R^2\times R^2\to R^2$ be defined by

$$G(a^1,a^2) = \begin{pmatrix} \frac{1}{2}a^i_1 \\ \frac{1}{2}a^i_2 \end{pmatrix}, \ i=1,2, \text{ for all } a^1=(a^1_1,a^1_2),\ a^2=(a^2_1,a^2_2)\in R^2.$$

Then it is easy to show that

(ii) $\varphi\circ G$ is $(\frac{1}{10},\eta)$-relaxed monotone with respect to first component,

(iii) $\varphi\circ G$ is $\frac{1}{10}$, $\frac{1}{10}$-Lipschitz continuous with respect to first and second component, respectively.

Let $L,M,N,O,g: R^2\to R^2$ be defined by

$$L(a) = \begin{pmatrix} \frac{1}{20}a_1 \\ \frac{1}{20}a_2 \end{pmatrix},\ M(a) = \begin{pmatrix} \frac{1}{21}a_1 \\ \frac{1}{21}a_2 \end{pmatrix},\ N(a) = \begin{pmatrix} \frac{1}{22}a_1 \\ \frac{1}{22}a_2 \end{pmatrix},$$
$$O(a) = \begin{pmatrix} \frac{1}{23}a_1 \\ \frac{1}{23}a_2 \end{pmatrix},\ g(a) = \begin{pmatrix} \frac{1}{20}a_1 \\ \frac{1}{20}a_2 \end{pmatrix}, \text{ for all } a=(a_1,a_2)\in R^2.$$

Suppose that $H : R^2 \times R^2 \times R^2 \times R^2 \to R^2$ is defined by

$$H(La, Ma, Na, Oa) = La + Ma + Na + Oa, \quad \text{for all } a \in R^2.$$

Then, it is easy to check that

(iv) $H(L,M,N,O)$ is $\frac{1}{20}$, $\frac{1}{21}$, $\frac{1}{22}$, $\frac{1}{23}$-Lipschitz continuous with respect to L, M, N and O, respectively,

(v) g is $\frac{1}{20}$-strongly monotone and $\frac{1}{20}$-Lipschitz continuous.

Next, it is easy to check that

(vi) $0 < \sqrt{\{\lambda_g{}^2\kappa^2 + 2t\lambda\lambda_V\left(\lambda_g\kappa + \tau\lambda_V\right) + \epsilon_1{}^2\lambda^2\lambda_V{}^2\}} < \frac{(1-\sqrt{1-2r+\lambda_g{}^2})(\ell - m\lambda)}{\tau} - \epsilon_2\lambda\lambda_T$, where $\kappa = \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} = 0.18$.

Therefore, for the constants $\ell = 3$, $m = 2$, $r = \lambda_g = \frac{1}{20}$, $\lambda_V = \lambda_T = \epsilon_1 = \epsilon_2 = t = \frac{1}{10}$, $\kappa_1 = \frac{1}{20}$, $\kappa_2 = \frac{1}{21}$, $\kappa_3 = \frac{1}{22}$, $\kappa_4 = \frac{1}{23}$, $\tau = 10$, obtained in (i)-(vi), all the hypotheses of Theorem 4.3 are satisfied for $\lambda = 0.01$.

5. Conclusion

The goal of this paper is to develop a novel mapping called $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive mapping that combines cocoercive and monotone mappings and defines its resolvent operator. We further show that the resolvent operator associated with $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive mapping is Lipschitz continuous and single-valued. Finally, we establish an existence and convergence result for a generalized set-valued variational inclusion problem using these ideas. An example is also developed to support our findings. The obtained results generalize the majority of the findings in the literature, implying a broad range of potential applications in future research on sensitivity analysis, variational inequality problems, and variational inclusion problems in Banach spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R. Ahmad, M. Dilshad, M.M. Wong and J.C. Yao, $H(\cdot,\cdot)$-cocoercive operator and an application for solving generalized variational inclusions, *Abstract and Applied Analysis* **2011** (2011) Article ID 261534, DOI: 10.1155/2011/261534.

[2] R. Ahmad, A.H. Siddiqi and M. Dilshad, Application of $H(\cdot,\cdot)$-cocoercive operators for solving a set-valued variational inclusion problem via a resolvent equation problem, *Indian Journal of Industrial and Applied Mathematics* **4**(2) (2013), 160 – 169, DOI: 10.5958/j.1945-919X.4.2.018.

[3] R. Ahmad, A.H. Siddiqi and Z. Khan, Proximal point algorithm for generalized multi-valued nonlinear quasi-variational-like inclusions in Banach spaces, *Applied Mathematics and Computation* **163**(1) (2005), 295 – 308, DOI: 10.1016/j.amc.2004.02.021.

[4] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin (1984).

[5] M.I. Bhat and B. Zahoor, Existence of solution and iterative approximation of a system of generalized variational-like inclusion problems in semi-inner product spaces, *Filomat* **31**(19) (2017), 6051 – 6070, DOI: 10.2298/FIL1719051B.

[6] S.S. Chang, J.K. Kim and K.H. Kim, On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces, *Journal of Mathematical Analysis and Applications* **268**(1) (2002), 89 – 108, DOI: 10.1006/jmaa.2001.7800.

[7] J.R. Giles, Classes of semi-inner product spaces *Transactions of the American Mathematical Society* **129** (1967), 436 – 446, DOI: 10.2307/1994599.

[8] S. Gupta and M. Singh, Variational-like inclusion involving infinite family of set-valued mappings governed by resolvent equations, *Journal of Mathematical and Computational Science* **11**(1) (2021), 874 – 892, DOI: 10.28919/jmcs/5177.

[9] S. Gupta and M. Singh, Generalized monotone mapping and resolvent equation technique with an application, *Journal of Mathematical and Computational Science* **11**(2) (2021), 1767–1783, DOI: 10.28919/jmcs/5343.

[10] N.J. Huang, M.R. Bai, Y.J. Cho and S.M. Kang, Generalized nonlinear mixed quasi-variational inequalities, *Computers & Mathematics with Applications* **40**(2-3) (2000), 205 – 215, DOI: 10.1016/S0898-1221(00)00154-1.

[11] S. Husain, S. Gupta and V.N. Mishra, Generalized $H(\cdot,\cdot,\cdot)$-η-cocoercive operators and generalized set-valued variational-like inclusions, *Journal of Mathematics* **2013** (2013), Article ID 738491, DOI: 10.1155/2013/738491.

[12] D.O. Koehler, A note on some operator theory in certain semi-inner-product space, *Proceedings of the American Mathematical Society* **30** (1971), 363 – 366, DOI: 10.1090/S0002-9939-1971-0281024-6.

[13] G. Lumer, Semi-inner product spaces, *Transactions of the American Mathematical Society* **100** (1961), 29 – 43, DOI: 10.2307/1993352.

[14] X.P. Luo and N.J. Huang, (H,ϕ)-η-monotone operators in Banach spaces with an application to variational inclusions, *Applied Mathematics and Computation* **216** (2010), 1131 – 1139, DOI: 10.1016/j.amc.2010.02.005.

[15] P. Marcotte and J.H. Wu, On the convergence of projection methods: Application to the decomposition of affine variational inequalities, *Journal of Optimization Theory and Applications* **85** (1995), 347 – 362, DOI: 10.1007/BF02192231.

[16] T. Ram, Parametric generalized nonlinear quasi-variational inclusion problems, *International Journal of Mathematical Archive* **3** (3) (2012), 1273 – 1282, http://www.ijma.info/index.php/ijma/article/view/1119/634.

[17] T. Ram and M. Iqbal, $H(\cdot,\cdot,\cdot,\cdot)$-$\varphi$-$\eta$-cocoercive operator with an application to variational inclusions, *International Journal of Nonlinear Analysis and Applications* **13**(1) (2022), 1311 – 1327, DOI: 10.22075/IJNAA.2021.23245.2506.

[18] N.K. Sahu, R.N. Mohapatra, C. Nahak and S. Nanda, Approximation solvability of a class of A-monotone implicit variational inclusion problems in semi-inner product spaces, *Applied Mathematics and Computation* **236** (2014), 109 – 117, DOI: 10.1016/j.amc.2014.02.095.

[19] P. Tseng, Further applications of a splitting algorithm to decomposition in variational inequalities and convex programming, *Mathematical Programming* **48**(1) (1990), 249 – 263, DOI: 10.1007/BF01582258.

[20] H.K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Analysis* **16**(12) (1991), 1127–1138, DOI: 10.1016/0362-546X(91)90200-K.

[21] D.L. Zhu and P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, *SIAM Journal on Optimization* **6**(3) (1996), 714 – 726, DOI: 10.1137/S1052623494250415.

[22] Y.Z. Zou and N.J. Huang, $H(\cdot,.)$-accretive operator with an application for solving variational inclusions in Banach spaces, *Applied Mathematics and Computation* **204**(2) (2008), 809 – 816, DOI: 10.1016/j.amc.2008.07.024.