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Dirac Operators with Generalized Coefficients and Their Line Integration

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Abstract Dirac operators with generalized coefficients are studied. Their non-commutative line integration over Cayley-Dickson algebras is investigated.

1. Introduction

In the previous article non-commutative line integration of Dirac operators over octonions and Cayley-Dickson algebras was investigated [14]. This paper is devoted to Dirac operators with generalized coefficients. Their non-commutative line integration over Cayley-Dickson algebras is investigated. This is very important, because many *partial differential operators* (PDO) can be decomposed into compositions of Dirac operators with usual or more frequently with generalized coefficients [15].

Recall that Dirac had used complexified quaternions to solve Klein-Gordon's hyperbolic differential equation with constant coefficients. The aim of this paper consists of a development and an extension of Dirac's approach on partial differential equations with generalized coefficients.

The technique presented below can be used for solutions of partial differential equations of the second order of arbitrary signatures and with variable coefficients. Using iterated antiderivatives it is possible to write solutions of *partial differential equations* (PDE). In this article notations and definitions of previous article [14] are used.

2. Dirac Operators with Generalized Coefficients and PDE

1. Definition. Let X and Y be two $\mathbf{R}$ linear normed spaces which are also left and right $\mathscr{A}_r$ modules, where $1 \leq r$. Let Y be complete relative to its norm.

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We put $X^{\otimes k} := X \otimes_{\mathbf{R}} \dots \otimes_{\mathbf{R}} X$ is the k times ordered tensor product over $\mathbf{R}$ of X. By $L_{q,k}(X^{\otimes k}, Y)$ we denote a family of all continuous k times $\mathbf{R}$ poly-linear and $\mathcal{A}_r$ additive operators from $X^{\otimes k}$ into Y. Then $L_{q,k}(X^{\otimes k}, Y)$ is also a normed $\mathbf{R}$ linear and left and right $\mathcal{A}_r$ module complete relative to its norm. In particular, $L_{q,1}(X,Y)$ is denoted also by $L_q(X,Y)$.

We present X as the direct sum $X = X_0 i_0 \oplus \dots \oplus X_{2^r-1} i_{2^r-1}$, where $X_0, \dots, X_{2^r-1}$ are pairwise isomorphic real normed spaces. If $A \in L_q(X,Y)$ and $A(xb) = (Ax)b$ or $A(bx) = b(Ax)$ for each $x \in X_0$ and $b \in \mathcal{A}_r$, then an operator A we call right or left $\mathcal{A}_r$-linear respectively.

An $\mathbf{R}$ linear space of left (or right) k times $\mathcal{A}_r$ poly-linear operators is denoted by $L_{l,k}(X^{\otimes k}, Y)$ (or $L_{r,k}(X^{\otimes k}, Y)$ respectively).

As usually a support of a function $g : S \to \mathcal{A}_r$ on a topological space S is by the definition $\mathrm{supp}(g) = \mathrm{cl}\{t \in S : g(t) \neq 0\}$, where the closure is taken in S.

We consider a space of test function $\mathcal{D} := \mathcal{D}(\mathbf{R}^n, Y)$ consisting of all infinite differentiable functions $f : \mathbf{R}^n \to Y$ on $\mathbf{R}^n$ with compact supports. A sequence of functions $f_n \in \mathcal{D}$ tends to zero, if all f_n are zero outside some compact subset K in the Euclidean space $\mathbf{R}^n$, while on it for each $k = 0, 1, 2, \dots$ the sequence $\{f_n^{(k)} : n \in \mathbf{N}\}$ converges to zero uniformly. Here as usually $f^{(k)}(t)$ denotes the k-th derivative of f, which is a k times $\mathbf{R}$ poly-linear symmetric operator from $(\mathbf{R}^n)^{\otimes k}$ to Y, that is $f^{(k)}(t) \cdot (h_1, \dots, h_k) = f^{(k)}(t) \cdot (h_{\sigma(1)}, \dots, h_{\sigma(k)}) \in Y$ for each $h_1, \dots, h_k \in \mathbf{R}^n$ and every transposition $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$, σ is an element of the symmetric group S_k, $t \in \mathbf{R}^n$. For convenience one puts $f^{(0)} = f$. In particular, $f^{(k)}(t) \cdot (e_{j_1}, \dots, e_{j_k}) = \partial^k f(t) / \partial t_{j_1} \dots \partial t_{j_k}$ for all $1 \leq j_1, \dots, j_k \leq n$, where $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{R}^n$ with 1 on the j-th place.

Such convergence in $\mathcal{D}$ defines closed subsets in this space $\mathcal{D}$, their complements by the definition are open, that gives the topology on $\mathcal{D}$. The space $\mathcal{D}$ is $\mathbf{R}$ linear and right and left $\mathcal{A}_r$ module.

By a generalized function of class $\mathcal{D}' := [\mathcal{D}(\mathbf{R}^n, Y)]'$ is called a continuous $\mathbf{R}$-linear $\mathcal{A}_r$-additive function $g : \mathcal{D} \to \mathcal{A}_r$. The set of all such functionals is denoted by $\mathcal{D}'$. That is, g is continuous, if for each sequence $f_n \in \mathcal{D}$, converging to zero, a sequence of numbers $g(f_n) =: [g, f_n) \in \mathcal{A}_r$ converges to zero for n tending to the infinity.

A generalized function g is zero on an open subset V in $\mathbf{R}^n$, if $[g, f) = 0$ for each $f \in \mathcal{D}$ equal to zero outside V. By a support of a generalized function g is called the family, denoted by $\mathrm{supp}(g)$, of all points $t \in \mathbf{R}^n$ such that in each neighborhood of each point $t \in \mathrm{supp}(g)$ the functional g is different from zero. The addition of generalized functions g, h is given by the formula:

(1) $$[g + h, f) := [g, f) + [h, f).$$

The multiplication $g \in \mathcal{D}'$ on an infinite differentiable function w is given by the equality:

(2) $[gw, f) = [g, wf)$ either for $w : \mathbf{R}^n \to \mathcal{A}_r$ and each test function $f \in \mathcal{D}$ with a real image $f(\mathbf{R}^n) \subset \mathbf{R}$,

where $\mathbf{R}$ is embedded into Y; or $w : \mathbf{R}^n \to \mathbf{R}$ and $f : \mathbf{R}^n \to Y$.

A generalized function g' prescribed by the equation:

(3) $[g', f) := -[g, f')$ is called a derivative g' of a generalized function g, where $f' \in \mathcal{D}(\mathbf{R}^n, L_q(\mathbf{R}^n, Y))$, $g' \in [\mathcal{D}(\mathbf{R}^n, L_q(\mathbf{R}^n, Y))]'$.

Another space $\mathcal{B} := \mathcal{B}(\mathbf{R}^n, Y)$ of test functions consists of all infinite differentiable functions $f : \mathbf{R}^n \to Y$ such that the limit $\lim_{|t|\to+\infty} |t|^m f^{(j)}(t) = 0$ exists for each $m = 0, 1, 2, \dots,$ $j = 0, 1, 2, \dots$. A sequence $f_n \in \mathcal{B}$ is called converging to zero, if the sequence $|t|^m f_n^{(j)}(t)$ converges to zero uniformly on $\mathbf{R}^n \setminus B(\mathbf{R}^n, 0, R)$ for each $m, j = 0, 1, 2, \dots$ and each $0 < R < +\infty$, where $B(Z, z, R) := \{y \in Z : \rho(y, z) \le R\}$ denotes a ball with center at z of radius R in a metric space Z with a metric ρ. The family of all $\mathbf{R}$-linear and $\mathcal{A}_r$-additive functionals on $\mathcal{B}$ is denoted by $\mathcal{B}'$.

In particular we can take $X = \mathcal{A}_r^\alpha$, $Y = \mathcal{A}_r^\beta$ with $1 \le \alpha,\ \beta \in \mathbf{Z}$. Analogously spaces $\mathcal{D}(U, Y)$, $[\mathcal{D}(U, Y)]'$, $\mathcal{B}(U, Y)$ and $[\mathcal{B}(U, Y)]'$ are defined for domains U in $\mathbf{R}^n$. For definiteness we write $\mathcal{B}(U, Y) = \{f|_U : f \in \mathcal{B}(\mathbf{R}^n, Y)\}$ and $\mathcal{D}(U, Y) = \{f|_U : f \in \mathcal{D}(\mathbf{R}^n, Y)\}$.

A function $g : U \to \mathcal{A}_v$ is called locally integrable, if it is absolutely integrable on each bounded λ measurable sub-domain V in U, i.e.

$$\int_V |g(z)| \lambda(dz) < \infty,$$

where λ denotes the Lebesgue measure on U.

A generalized function f is called regular if locally integrable functions ${}_{j,k}f^1, {}_l f^2 : U \to \mathcal{A}_v$ exist such that

$$[f, \omega) = \int_U \left\{ \sum_{j,k,l} {}_{j,k}f^1(z)_k\omega(z)_j f^2(z) \right\}_{q(3)} \lambda(dz),$$

for each test function $\omega \in \mathcal{B}(U, Y)$ or $\omega \in \mathcal{D}(U, Y)$ correspondingly, where $\omega = ({}_1\omega, \dots, {}_\beta\omega)$, ${}_k\omega(z) \in \mathcal{A}_v$ for each $z \in U$ and all k, $q(3)$ is a vector indicating on an order of the multiplication in the curled brackets and it may depend on the indices $j, l = 1, \dots, \alpha$, $k = 1, \dots, \beta$.

We supply the space $\mathcal{B}(\mathbf{R}^n, Y)$ with the countable family of semi-norms

(4) $$p_{\alpha,k}(f) := \sup_{x \in \mathbf{R}^n} |(1 + |x|)^k \partial^\alpha f(x)|$$

inducing its topology, where $k = 0, 1, 2, \dots;$ $\alpha = (\alpha_1, \dots, \alpha_n)$, $0 \le \alpha_j \in \mathbf{Z}$. On this space we take the space $\mathcal{B}'(\mathbf{R}^n, Y)_l$ of all Y valued continuous generalized functions (functionals) which are left $\mathcal{A}_r$-linear of the form

(5) $\quad f = f_0 i_0 + \ldots + f_{2^v-1} i_{2^v-1} \quad$ and $\quad g = g_0 i_0 + \ldots + g_{2^v-1} i_{2^v-1},$

where f_j and $g_j \in \mathcal{B}'(\mathbf{R}^n, Y)$, with restrictions on $\mathcal{B}(\mathbf{R}^n, \mathbf{R})$ being real- or $\mathbf{C_i} = \mathbf{R} \oplus \mathbf{iR}$- valued generalized functions $f_0, \ldots, f_{2^v-1}$, $g_0, \ldots, g_{2^v-1}$ respectively. Let $\phi = \phi_0 i_0 + \ldots + \phi_{2^v-1} i_{2^v-1}$ with $\phi_0, \ldots, \phi_{2^v-1} \in \mathcal{B}(\mathbf{R}^n, \mathbf{R})$, then

(6) $$[f, \phi) = \sum_{k,j=0}^{2^v-1} [f_j, \phi_k) i_k i_j .$$

Let their convolution be defined in accordance with the formula:

(7) $$[f * g, \phi) = \sum_{j,k=0}^{2^v-1} ([f_j * g_k, \phi) i_j) i_k \quad \text{for each } \phi \in \mathcal{B}(\mathbf{R}^n, Y).$$

Particularly,

(8) $\quad [f * g)(x) = f(x-y) * g(y) = f(y) * g(x-y) \quad$ for all $x, y \in \mathbf{R}^n$ due to (7),

since the latter equality is satisfied for each pair f_j and g_k.

2. Partial Differential Operators with Generalized Coefficients. Let an operator Q be given by Formula

(α) $$Qf(z) = \sum_{j,k} [(\partial f(z)/\partial z_j)(\partial \phi_j^1(z)^*/\partial z_k)] \phi_k^*(z)$$
$$+ \sum_j [(\partial f(z)/\partial z_j) \phi_j^1(z)^*] \beta(z) + \sum_k [f(z)(\partial \beta^1(z)/\partial z_k)] \phi_k^*(z)$$

on a domain U. Initially it is considered as a domain in the Cayley-Dickson algebra $\mathcal{A}_v$. But in the case when Q and f depend on smaller number of real coordinates $z_0, \ldots, z_{n-1}$ we can take the real shadow of U and its sub-domain V of variables $(z_0, \ldots, z_{n-1})$, where z_k are marked for example being zero for all $n \le k \le 2^v - 1$. Thus we take a domain V which is a canonical closed subset in the Euclidean space $\mathbf{R}^n$, $2^{v-1} \le n \le 2^v - 1$, $v \ge 2$.

A canonical closed subset P of the Euclidean space $X = \mathbf{R}^n$ is called a quadrant if it can be given by the condition $P := \{x \in X : q_j(x) \ge 0\}$, where $(q_j : j \in \Lambda_P)$ are linearly independent elements of the topologically adjoint space X^*. Here $\Lambda_P \subset \mathbf{N}$ (with $\mathrm{card}(\Lambda_P) = k \le n$) and k is called the index of P. If $x \in P$ and exactly j of the q_i's satisfy $q_i(x) = 0$ then x is called a corner of index j.

That is P is affine diffeomorphic with $P^n = \prod_{j=1}^n [\mathsf{a}_j, b_j]$, where $-\infty \le \mathsf{a}_j < b_j \le \infty$, $[\mathsf{a}_j, b_j] := \{x \in \mathbf{R} : \mathsf{a}_j \le x \le b_j\}$ denotes the segment in $\mathbf{R}$. This means that there exists a vector $p \in \mathbf{R}^n$ and a linear invertible mapping C on $\mathbf{R}^n$ so that $C(P) - p = P^n$. We put $t^{j,1} := (t_1, \ldots, t_j, \ldots, t_n : t_j = \mathsf{a}_j)$, $t^{j,2} := (t_1, \ldots, t_j, \ldots, t_n : t_j = b_j)$. Consider $t = (t_1, \ldots, t_n) \in P^n$.

This permits to define a manifold M with corners. It is a metric separable space modelled on $X = \mathbf{R}^n$ and is supposed to be of class C^s, $1 \le s$. Charts on M are denoted (U_l, u_l, P_l), that is, $u_l : U_l \to u_l(U_l) \subset P_l$ is a C^s-diffeomorphism for each l, U_l is open in M, $u_l \circ u_j^{-1}$ is of C^s class of smoothness from the domain $u_j(U_l \cap U_j) \neq \emptyset$ onto $u_l(U_l \cap U_j)$, that is, $u_j \circ u_l^{-1}$ and $u_l \circ u_j^{-1}$ are bijective, $\bigcup_j U_j = M$.

A point $x \in M$ is called a corner of index j if there exists a chart (U, u, P) of M with $x \in U$ and $u(x)$ is of index $\mathrm{ind}_M(x) = j$ in $u(U) \subset P$. A set of all corners of index $j \geq 1$ is called a border ∂M of M, x is called an inner point of M if $\mathrm{ind}_M(x) = 0$, so $\partial M = \bigcup_{j \geq 1} \partial^j M$, where $\partial^j M := \{x \in M : \mathrm{ind}_M(x) = j\}$ (see also [16]). We consider that

(D1) V is a canonical closed subset in the Euclidean space $\mathbf{R}^n$, that is $V = \mathrm{cl}(\mathrm{Int}(V))$, where $\mathrm{Int}(V)$ denotes the interior of V and $\mathrm{cl}(V)$ denotes the closure of V.

Particularly, the entire space $\mathbf{R}^n$ may also be taken.

Let a manifold W be satisfying the following conditions $(i - v)$.

(i) The manifold W is continuous and piecewise C^α, where C^l denotes the family of l times continuously differentiable functions. This means by the definition that W as the manifold is of class $C^0 \cap C^\alpha_{\mathrm{loc}}$. That is W is of class C^α on open subsets $W_{0,j}$ in W and $W \setminus \left(\bigcup_j W_{0,j}\right)$ has a codimension not less than one in W.

(ii) $W = \bigcup_{j=0}^m W_j$, where $W_0 = \bigcup_k W_{0,k}$, $W_j \cap W_k = \emptyset$ for each $k \neq j$, $m = \dim_{\mathbf{R}} W$, $\dim_{\mathbf{R}} W_j = m - j$, $W_{j+1} \subset \partial W_j$.

(iii) Each W_j with $j = 0, \dots, m-1$ is an oriented C^α-manifold, W_j is open in $\bigcup_{k=j}^m W_k$. An orientation of W_{j+1} is consistent with that of ∂W_j for each $j = 0, 1, \dots, m-2$. For $j > 0$ the set W_j is allowed to be void or non-void.

(iv) A sequence W^k of C^α orientable manifolds embedded into $\mathbf{R}^n$, $\alpha \geq 1$, exists such that W^k uniformly converges to W on each compact subset in $\mathbf{R}^n$ relative to the metric dist.

For two subsets B and E in a metric space X with a metric ρ we put

$$\text{(1)} \qquad \mathrm{dist}(B, E) := \max\left\{\sup_{b \in B} \mathrm{dist}(\{b\}, E), \sup_{e \in E} \mathrm{dist}(B, \{e\})\right\},$$

where $\mathrm{dist}(\{b\}, E) := \inf_{e \in E} \rho(b, e)$, $\mathrm{dist}(B, \{e\}) := \inf_{b \in B} \rho(b, e)$, $b \in B$, $e \in E$.

Generally, $\dim_{\mathbf{R}} W = m \leq n$. Let $(e_1^k(x), \dots, e_m^k(x))$ be a basis in the tangent space $T_x W^k$ at $x \in W^k$ consistent with the orientation of W^k, $k \in \mathbf{N}$.

We suppose that the sequence of orientation frames $(e_1^k(x_k), \dots, e_m^k(x_k))$ of W^k at x_k converges to $(e_1(x), \dots, e_m(x))$ for each $x \in W_0$, where $\lim_k x_k = x \in W_0$, while $e_1(x), \dots, e_m(x)$ are linearly independent vectors in $\mathbf{R}^n$.

(v) Let a sequence of Riemann volume elements λ_k on W^k (see §XIII.2 [17]) induce a limit volume element λ on W, that is, $\lambda(B \cap W) = \lim_{k\to\infty}(B \cap W^k)$ for each compact canonical closed subset B in $\mathbf{R}^n$, consequently, $\lambda(W \setminus W_0) = 0$.

(vi) We shall consider surface integrals of the second kind, i.e. by the oriented surface W (see (iv)), where each W_j, $j = 0, \dots, m-1$ is oriented (see also §XIII.2.5 [17]).

Suppose that a boundary ∂U of U satisfies Conditions $(i - v)$ and

(vii) Let the orientations of ∂U^k and U^k be consistent for each $k \in \mathbf{N}$ (see Proposition 2 and Definition 3 [17]).

Particularly, the Riemann volume element λ_k on ∂U^k is consistent with the Lebesgue measure on U^k induced from $\mathbf{R}^n$ for each k. This induces the measure λ on ∂U as in (v). This consideration certainly encompasses the case of a domain U with a C^α boundary ∂U as well.

Suppose that $U_1, \dots, U_l$ are domains in $\mathbf{R}^n$ satisfying conditions $(D1, i - vii)$ and such that $U_j \cap U_k = \partial U_j \cap \partial U_k$ for each $j \neq k$, $U = \bigcup_{j=1}^{l} U_j$. Consider a function $g : U \to \mathcal{A}_v$ such that each its restriction $g|_{U_j}$ is of class C^s, but g on the entire domain U may be discontinuous or not C^k, where $0 \leq k \leq s$, $1 \leq s$. If $x \in \partial U_j \cap \partial U_k$ for some $j \neq k$, $x \in \mathrm{Int}(U)$, such that x is of index $m \geq 1$ in U_j (and in U_k also). Then there exist canonical C^α local coordinates $(y_1, \dots, y_n)$ in a neighborhood V_x of x in U such that $S := V_x \cap \partial^m U_j = \{y : y \in V_x; y_1 = 0, \dots, y_m = 0\}$. Using locally finite coverings of the locally compact topological space $\partial U_j \cap \partial U_k$ without loss of generality we suppose that C^α functions $P_1(z), \dots, P_m(z)$ on $\mathbf{R}^n$ exist with $S = \{z : z \in \mathbf{R}^n; P_1(z) = 0, \dots, P_m(z) = 0\}$. Therefore, on the surface S the delta-function $\delta(P_1, \dots, P_m)$ exists, for $m = 1$ denoting them $P = P_1$ and $\delta(P)$ respectively (see §III.1 [4]). It is possible to choose $y_j = P_j$ for $j = 1, \dots, m$. Using generalized functions with definite supports, for example compact supports, it is possible without loss of generality consider that $y_1, \dots, y_n \in \mathbf{R}$ are real variables. Let $\theta(P_j)$ be the characteristic function of the domain $\mathcal{P}_j := \{z : P_j(z) \geq 0\}$, $\theta(P_j) := 1$ for $P_j \geq 0$ and $\theta(P_j) = 0$ for $P_j < 0$. Then the generalized function $\theta(P_1, \dots, P_m) := \theta(P_1) \dots \theta(P_m)$ can be considered as the direct product of generalized functions $\theta(y_1), \dots, \theta(y_m)$, $1(y_{m+1}, \dots, y_n) \equiv 1$, since variables $y_1, \dots, y_n$ are independent. Then in the class of generalized functions the following formulas are valid:

(2) $$\partial\theta(P_j)/\partial z_k = \delta(P_j)(\partial P_j/\partial z_k) \quad \text{for each } k = 1, \dots, n,$$

consequently,

(3) $$\mathrm{grad}[\theta(P_1, \dots, P_m)] = \sum_{j=1}^{m} [\theta(P_1) \dots \theta(P_{j-1})\delta(P_j)(\mathrm{grad}(P_j))\theta(P_{j+1}) \dots \theta(P_m)],$$

where, $\operatorname{grad} g(z) := (\partial g(z)/\partial z_1, \dots, \partial g(z)/\partial z_n)$ (see Formulas III.1.3(1,7,7′,9) and III.1.9(6) [4]).

Let for the domain U in the Euclidean space $\mathbf{R^n}$ the set of internal surfaces $\mathrm{cl}_U[\mathrm{Int}_{\mathbf{R}^n}(U) \cap \bigcup_{j\neq k}(\partial U_j \cap \partial U_k)]$ in U on which a function $g : U \to \mathcal{A}_v$ or its derivatives may be discontinuous is presented as the disjoint union of surfaces Γ^j, where each surface Γ^j is the boundary of the sub-domain

$$\mathcal{P}^j := \{P_{j,1}(z) \geq 0, \dots, P_{j,m_j}(z) \geq 0\}, \quad \Gamma^j = \partial \mathcal{P}^j = \bigcup_{k=1}^{m_j} \partial^k \mathcal{P}^j, \tag{4}$$

$m_j \in \mathbf{N}$ for each j, $\mathrm{cl}_X(V)$ denotes the closure of a subset V in a topological space X, $\mathrm{Int}_X(V)$ denotes the interior of V in X. By its construction the family $\{\mathcal{P}^j : j\}$ is the covering of U which is the refinement of the covering $\{U_k : k\}$, i.e. for each $\mathcal{P}^j$ a number k exists so that $\mathcal{P}^j \subset U_k$ and $\partial \mathcal{P}^j \subset \partial U_k$ and $\bigcup_j \mathcal{P}^j = \bigcup_k U_k = U$. We put

$$h_j(z(x)) = h(x)|_{x\in\Gamma^j} := \lim_{y_{j,1}\downarrow 0, \dots, y_{j,n}\downarrow 0} g(z(x+y)) - \lim_{y_{j,1}\downarrow 0, \dots, y_{j,n}\downarrow 0} g(z(x-y)) \tag{5}$$

in accordance with the supposition made above that g can have only discontinuous of the first kind, i.e. the latter two limits exist on each Γ^j, where x and y are written in coordinates in $\mathcal{P}^j$, $z = z(x)$ denotes the same point in the global coordinates z of the Euclidean space $\mathbf{R}^n$. We take new continuous function

$$g^1(z) = g(z) - \sum_j h_j(z)\theta(P_{j,1}(z), \dots, P_{j,m_j}(z)). \tag{6}$$

Let the partial derivatives and the gradient of the function g^1 be calculated piecewise one each U_k. Since g^1 is the continuous function, it is the regular generalized function by the definition, consequently, its partial derivatives exist as the generalized functions. If $g^1|_{U_j} \in C^1(U_j, \mathcal{A}_v)$, then $\partial g^1(z)/\partial z_k$ is the continuous function on U_j, i.e. in this case $\partial g^1(z)\chi_{U_j}(z)/\partial z_k$ is the regular generalized function on U_j for each k, where $\chi_G(z)$ denotes the characteristic function of a subset G in $\mathcal{A}_v$, $\chi_G(z) = 1$ for each $z \in G$, while $\chi_G(z) = 0$ for $z \in \mathcal{A}_v \setminus G$. Therefore, $g^1(z) = \sum_j g^1(z)\chi_{U_j \setminus \bigcup_{k<j} U_k}(z)$, where $U_0 = \emptyset$, $j, k \in \mathbf{N}$.

On the other hand, the function $g(z)$ is locally continuous on U, consequently, it defines the regular generalized function on the space $\mathcal{D}(U, \mathcal{A}_v)$ of test functions as

$$[g, \omega) := \int_U \omega(z)g(z)\lambda(dz),$$

where λ is the Lebesgue measure on U induced by the Lebesgue measure on the real shadow $\mathbf{R}^{2^v}$ of the Cayley-Dickson algebra $\mathcal{A}_v$, $\omega \in \mathcal{D}(U, \mathcal{A}_v)$. Thus partial derivatives of g exist as generalized functions.

In accordance with formulas (2, 3, 6) we infer that the gradient of the function $g(z)$ on the domain U is the following:

$$(7)\qquad \operatorname{grad} g(z) = \operatorname{grad} g^1(z) + \sum_j h_j(z) \operatorname{grad} \theta(P_{j,1}, \dots, P_{j,m_j}).$$

Thus formulas (3, 7) permit calculations of coefficients of the partial differential operator Q given by formula $2(\alpha)$.

3. Line Generalized Functions. Let U be a domain satisfying Conditions [14] 1(D1, D2) and 2(D1,i-vii). We embed the Euclidean space $\mathbf{R}^n$ into the Cayley-Dickson algebra $\mathcal{A}_v$, $2^{v-1} \le n \le 2^v - 1$, as the $\mathbf{R}$ affine sub-space putting $\mathbf{R}^n \ni x = (x_1, \dots, x_n) \mapsto x_1 i_{j_1} + \dots + x_n i_{j_n} + x^0 \in \mathcal{A}_v$, where $j_k \ne j_l$ for each $k \ne l$, x^0 is a marked Cayley-Dickson number, for example, $j_k = k$ for each k, $x^0 = 0$. Moreover, each z_j can be written in the z-representation in accordance with Formulas [14] 1(1-3).

We denote by $\mathbf{P} = \mathbf{P}(U)$ the family of all rectifiable paths $\gamma : [a_\gamma, b_\gamma] \to U$ supplied with the metric

$$(1)\qquad \rho(\gamma, \omega) := |\gamma(a) - \omega(a_\omega)| + \inf_\phi V_a^b(\gamma(t) - \omega(\phi(t))$$

where the infimum is taken by all diffeomorphisms $\phi : [a_\gamma, b_\gamma] \to [a_\omega, b_\omega]$ so that $\phi(a_\gamma) = a_\omega$, $a = a_\gamma < b_\gamma = b$ (see [14] §3).

Let us introduce a continuous mapping $g : \mathcal{B}(U, \mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U, \mathcal{A}_v) \to Y$ such that its values are denoted by $[g; \omega, \gamma; \nu]$, where Y is a module over the Cayley-Dickson algebra $\mathcal{A}_v$, $\omega \in \mathcal{B}(U, \mathcal{A}_v)$, $\gamma \in \mathbf{P}(U)$, $\mathcal{V}(U, \mathcal{A}_v)$ denotes the family of all functions on U with values in the Cayley-Dickson algebra of bounded variation (see [14] §3), $\nu \in \mathcal{V}(U, \mathcal{A}_v)$. For the identity mapping $\nu(z) = \operatorname{id}(z) = z$ values of this functional will be denoted shortly by $[g; \omega, \gamma]$. Suppose that this mapping g satisfies the following properties (G1-G5):

(G1) $[g; \omega, \gamma; \nu]$ is bi- $\mathbf{R}$ homogeneous and $\mathcal{A}_v$ additive by a test function ω and by a function of bounded variation ν;

(G2) $[g; \omega, \gamma; \nu] = [g; \omega, \gamma^1; \nu] + [g; \omega, \gamma^2; \nu]$ for each γ, γ^1 and $\gamma^2 \in \mathbf{P}(U)$ such that $\gamma(t) = \gamma^1(t)$ for all $t \in [a_{\gamma^1}, b_{\gamma^1}]$ and $\gamma(t) = \gamma^2(t)$ for any $t \in [a_{\gamma^2}, b_{\gamma^2}]$ and $a_{\gamma^1} = a_\gamma$ and $a_{\gamma^2} = b_{\gamma^1}$ and $b_\gamma = b_{\gamma^2}$.

(G3) If a rectifiable curve γ does not intersect a support of a test function ω or a function of bounded variation ν, $\gamma([a, b] \cap (\operatorname{supp}(\omega) \cap \operatorname{supp}(\nu)) = \emptyset$, then $[g; \omega, \gamma; \nu] = 0$, where $\operatorname{supp}(\omega) := \operatorname{cl}\{z \in U : \omega(z) \ne 0\}$.

Further we put

(G4) $[\partial^{|m|} g(z) / \partial z_0^{m_0} \dots \partial z_{2^v-1}^{m_{2^v-1}}; \omega, \gamma] := (-1)^{|m|} [g; \partial^{|m|} \omega(z) / \partial z_0^{m_0} \dots \partial z_{2^v-1}^{m_{2^v-1}}, \gamma]$ for each $m = (m_0, \dots, m_{2^v-1})$, m_j is a non-negative integer $0 \le m_j \in \mathbf{Z}$ for each

j, $|m| := m_0 + \dots + m_{2^v-1}$. In the case of a super-differentiable function ω and a generalized function g we also put

(G5) $[(d^k g(z)/dz^k).(h_1,\dots,h_k);\omega,\gamma] := (-1)^k [g;(d^k\omega(z)/dz^k).(h_1,\dots,h_k),\gamma]$ for any natural number $k \in \mathbf{N}$ and Cayley-Dickson numbers $h_1,\dots,h_k \in \mathcal{A}_v$.

Then g is called the Y valued line generalized function on $\mathcal{B}(U,\mathcal{A}_v)\times \mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v)$. Analogously it can be defined on $\mathcal{D}(U,\mathcal{A}_v)\times \mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v)$ (see also §1). If $Y = \mathcal{A}_v$ we call it simply the line generalized function, while for $Y = L_q(\mathcal{A}_v^k, \mathcal{A}_v^l)$ we call it the line generalized operator valued function, $k,l \ge 1$, omitting "on $\mathcal{B}(U,\mathcal{A}_v)\times \mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v)$" or "line" for short, when it is specified. Their spaces we denote by $L_q(\mathcal{B}(U,\mathcal{A}_v)\times \mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v);Y)$. Thus if g is a generalized function, then Formula (G5) defines the operator valued generalized function $d^k g(z)/dz^k$ with $k \in \mathbf{N}$ and $l = 1$.

If g is a continuous function on U (see §3), then the formula

(G6) $[g;\omega,\gamma;v] = \int_\gamma \omega(y) g(y) dv(y)$

defines the generalized function.If $\hat{f}(z)$ is a continuous $L_q(\mathcal{A}_v,\mathcal{A}_v)$ valued function on U, then it defines the generalized operator valued function with $Y = L_q(\mathcal{A}_v,\mathcal{A}_v)$ such that

(G7) $[\hat{f};\omega,\gamma;v] = \int_\gamma \{\hat{f}(z).\omega(z)\} dv(z)$.

Particularly, for $v = \mathrm{id}$ we certainly have $dv(z) = dz$.

We consider on $L_q(\mathcal{B}(U,\mathcal{A}_v)\times \mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v);Y)$ the strong topology:

(G8) $\lim_l f^l = f$ means that for each marked test function $\omega \in \mathcal{B}(U,\mathcal{A}_v)$ and rectifiable path $\gamma \in \mathbf{P}(U)$ and function of bounded variation $v \in \mathcal{V}(U,\mathcal{A}_v)$ the limit relative to the norm in Y exists

$$\lim_l [f^l;\omega,\gamma;v] = [f;\omega,\gamma;v].$$

4. Line Integration of Generalized Functions. Let $C^m_{ab}(V,\mathcal{A}_v)$ denote the $\mathbf{R}$ linear space and right and left $\mathcal{A}_v$ module of all functions $\gamma : V \to \mathcal{A}_v$ such that $\gamma(z)$ and each its derivative $\partial^{|k|} g(z)/\partial z_1^{m_1}\dots\partial z_n^{m_n}$ for $1 \le |k| \le m$ is absolutely continuous on V (see §§3 [14] and 3 above). This definition means that $C^{m+1}(V,\mathcal{A}_v) \subset C^m_{ab}(V,\mathcal{A}_v)$, where $C^m(V,\mathcal{A}_v)$ denotes the family of all m times continuously differentiable functions on a domain V either open or canonical closed in $\mathbf{R}^n$, which may be a a real shadow of U as well.

5. Lemma. *Let $\gamma \in C^m_{ab}([a,b],\mathcal{A}_v)\cap \mathbf{P}(U)$ and $\omega \in \mathcal{B}(U,\mathcal{A}_v)$ and $v \in C^0_{ab}(U,\mathcal{A}_v)$ for $m = 0$ or $v = \mathrm{id}$ for $m \ge 1$, where $0 \le m \in \mathbf{Z}$, then a line generalized function $[g;\omega,\gamma|_{[a,x]};v]$ is continuous for $m = 0$ or of class C^m by the parameter $x \in [a,b]$ for $m \ge 1$.*

Proof. For absolutely continuous functions $\gamma(t)$ and ν (i.e. when $m = 0$) the continuity by the parameter x follows from the definition of the line generalized function, since

$$\lim_{\Delta x \to 0} \rho(\gamma|_{[a,x]}, \gamma|_{[a,x+\Delta x]}) = 0 \quad \text{and} \quad \lim_{\Delta x \to 0} \rho(\nu \circ \gamma|_{[a,x]}, \nu \circ \gamma|_{[a,x+\Delta x]}) = 0.$$

Consider now the case $m \geq 1$ and $\nu = \mathrm{id}$. In view of properties 3(G1, G2) for any $\Delta x \neq 0$ so that $x \in (a,b] := \{t \in \mathbf{R} : a < t \leq b\}$ and $x + \Delta x \in (a,b) := \{t \in \mathbf{R} : a < t < b\}$ the difference quotient satisfies the equalities:

$$\text{(1)} \qquad \{[g;\omega,\gamma|_{[a,x+\Delta x]}] - [g;\omega,\gamma|_{[a,x]}]\}/\Delta x$$
$$= [g;\omega/\Delta x, \gamma \circ \phi|_{[a,x]}] - [g;\omega/\Delta x, \gamma|_{[a,x]}],$$

where $\phi : [a,x] \to [a,x+\Delta x]$ is a diffeomorphism of $[a,x]$ onto $[a,x+\Delta x]$ with $\phi(a) = a$. Therefore, $\Delta\omega := \omega(z + \Delta z) - \omega(z)$ for $z = \gamma(t)$ and $z + \Delta z = \gamma(\phi(t))$, $t \in [a,x]$ in the considered case. Using Conditions (G1, G3) one can mention that if $\omega = \omega^1$ on an open neighborhood V of γ in U, then

$$\text{(2)} \qquad [g;\omega,\gamma] = [g;\omega^1,\gamma],$$

since $\omega - \omega^1 = 0$ on V and $\gamma \cap \mathrm{supp}(\omega - \omega^1) = 0$.

From Conditions 3(G1, G4) and formula (2) we deduce that

$$\text{(3)} \qquad \lim_{\Delta x \to 0} \{[g;\omega,\gamma|_{[a,x+\Delta x]}] - [g;\omega,\gamma|_{[a,x]}]\}/\Delta x$$
$$= \sum_{j=0}^{2^\nu - 1} [g; (\partial\omega(z)/\partial z_j), (d\gamma_j(t)/dt)\gamma|_{[a,x]}],$$

where $z_j{}' = d\gamma_j(t)/dt$ for $z = \gamma(t)$, $t \in [a,b]$, since each partial derivative of the test function ω is again the test function. From the first part of the proof we get that $[g;\omega,\gamma|_{[a,x]}]$ is of class C^1 by the parameter x, since the product $(d\gamma_j(t)/dt)\gamma(t)$ of absolutely continuous functions $(d\gamma_j(t)/dt)$ and $\gamma(t)$ is absolutely continuous for each j. Repeating this proof by induction for $k = 1, \dots, m$ one gets the statement of this lemma for $\gamma \in C^m_{ab}([a,b], \mathcal{A}_\nu) \cap \mathbf{P}(U)$. $\square$

6. Lemma. *If γ is a rectifiable path, then a line generalized function $[g;\omega,\gamma|_{[a,x]}]$ is of bounded variation by the parameter $x \in [a,b]$.*

Proof. Let $\gamma \in \mathbf{P}(U)$ be a rectifiable path in U, $\gamma : [a,b] \to U$. We can present γ in the form

$$\text{(1)} \qquad \gamma(t) = \sum_{j=0}^{2^\nu - 1} \gamma_j(t) i_j,$$

where each function $\gamma_j(t)$ is real-valued. Therefore, $\gamma_j(t)$ is continuous and of bounded variation for each j, since γ is such. Thus the function $\omega(\gamma(t))$ is of bounded variation $V_a^b \omega(\gamma) < \infty$, since ω is infinite differentiable and $\gamma([a,b])$ is compact.

On the other hand, each function $f:[a,b]\to \mathbf{R}$ of bounded variation can be written as the difference $f=f^1-f^2$ of two monotone non-decreasing functions f^1 and f^2 of bounded variations: $f^1(t):=V_a^t f$ and $f^2(t)=f^1(t)-f(t)$ for each $t\in[a,b]$ (see [9]). This means that $f^k=g^k+h^k$, where a function g^k is continuous monotone and of bounded variation, while h^k is a monotone step function, where $k=1,2$. When the function f is continuous one gets $f=g^1-g^2$. For a monotone non-decreasing function p one has $V_a^t p=p(t)-p(a)$.

In view of Property 3(G1) we infer that

$$(2) \qquad [g;\omega,\gamma|_{[a,x]}]=\sum_{j=0}^{2^v-1}[g_j;\omega,\gamma|_{[a,x]}]i_j,$$

where the function $[g_j;\omega,\gamma|_{[a,x]}]$ by x is real-valued for any $\omega\in\mathscr{B}(U,\mathscr{A}_v)$ and $\gamma\in\mathbf{P}(U)$ for all $j=0,\dots,2^v-1$.

The metric space $\mathbf{P}(\bar U)$ is complete, where $\bar U=\mathrm{cl}(U)$. Indeed, let g^n be a sequence of rectifiable paths in $\bar U$ fundamental relative to the metric ρ given by Formula 3(1). Using diffeomorphism preserving orientations of segments we can consider without loss of generality that each path g^n is defined on the unit segment $[0,1]$, $a=0$, $b=1$. It is lightly to mention that

$$(3) \qquad |g(a)-f(a))|+V_a^b(g-f)\ge \sup_{t\in[a,b]}|g(t)-f(t)|$$

for any two functions of bounded variation, $f,g:[a,b]\to\bar U$. For each $\epsilon>0$ a natural number $n_0=n_0(\epsilon)$ exists so that $\rho(g^n,g^m)<\epsilon/2$ for all $n,m\ge n_0$. That is $\phi^n:[0,1]\to[0,1]$ diffeomorphisms exist such that

$$|g^n(a)-g^m(a))|+V_a^b(g^n\circ\phi^n-g^m\circ\phi^m)<\epsilon \quad \text{for all } n,m\ge n_0,$$

since $\phi^m\circ(\phi^n)^{-1}$ is also the diffeomorphism preserving the orientation of the segment. Using induction by $\epsilon=1/l$ with $l\in\mathbf{N}$ one chooses a sequence of diffeomorphisms ϕ^n such that for each $l\in\mathbf{N}$ a natural number $n_0=n_0(l)$ exists such that

$$|g^n(a)-g^m(a))|+V_a^b(g^n\circ\phi^n-g^m\circ\phi^m)<1/l \quad \text{for all } n,m\ge n_0(l),$$

consequently,

$$\sup_{t\in[a,b]}|g^n(\phi^n(t))-g^m(\phi^m(t))|<1/l \quad \text{for all } n,m\ge n_0(l).$$

Thus the sequence $g^n\circ\phi^n$ is fundamental in $C^0([a,b],\bar U)$. The latter metric space is complete relative to the metric

$$d(f,g):=\sup_{t\in[a,b]}|f(t)-g(t)|,$$

since from the completeness of the Cayley-Dickson algebra $\mathscr{A}_v$ considered as the normed space over the real field the completeness of the closed subset $\bar U$ follows (see also Chapter 8 in [3]). Therefore, the sequence $g^n\circ\phi^n$ converges to a continuous function $f:[a,b]\to\bar{U}$. On the other hand, $\lim_{m\to\infty}\rho(g^n\circ\phi^n, g^m\circ\phi^m)=\rho(g^n\circ\phi^n, f)\le 1/l$ for each $n>n_0(l)$, $l\in\mathbf{N}$. The function $g^n\circ\phi^n$ is of bounded variation, consequently, the function f is also of bounded variation. That is $f\in\mathbf{P}(\bar{U})$. Thus $\mathbf{P}(\bar{U})$ is complete. □

Take any sequence γ^n of $C^2_{ab}([a,b],\mathscr{A}_v)$ paths in U converging to γ relative to the metric ρ on $\mathbf{P}(\bar{U})$ and the latter metric space is complete as it was demonstrated above. In view of formula 5(3) and Property 3(G3) the sequence $[g;\omega,\gamma^n|_{[a,x]}]$ is fundamental in $\mathbf{P}(\bar{U})$. On the other hand, the generalized function g is continuous on $\mathscr{B}(U,\mathscr{A}_r)\times\mathbf{P}(\bar{U})$, consequently, the sequence $[g;\omega,\gamma^n|_{[a,x]}]$ converges in $\mathscr{B}(U,\mathscr{A}_r)\times\mathbf{P}(\bar{U})$ to $[g;\omega,\gamma|_{[a,x]}]$ for each $a<x\le b$, hence $[g;\omega,\gamma|_{[a,x]}]=\lim_n[g;\omega,\gamma^n|_{[a,x]}]$ in $\mathbf{P}(\bar{U})$. By the conditions of this lemma $[g;\omega,\gamma|_{[a,x]}]\in\mathbf{P}(U)$, since $\gamma([a,b])\subset U$. Thus the function $[g;\omega,\gamma|_{[a,x]}]$ by $x\in[a,b]$ is of bounded variation:

$$V^b_a[g;\omega,\gamma|_{[a,x]}]<\infty.$$

7. Definition. Let f and η be two line generalized functions on $\mathscr{B}(U,\mathscr{A}_v)\times\mathbf{P}(U)\times\mathscr{V}(U,\mathscr{A}_v)$. We define a line functional with values denoted by

$$[\int_\gamma f\,d\eta,\omega^1\otimes\omega):=[\hat{f};\omega^1,\gamma;[\eta;\omega,\kappa]]|_{\kappa=\gamma}$$
$$=[\hat{f};\omega^1,*;[\eta;\omega,*]](\gamma),$$

where $\gamma\in\mathbf{P}(U)$ is a rectifiable path in U, $\omega,\omega^1\in\mathscr{B}(U,\mathscr{A}_v)$ are any test functions. The functional $\int_\gamma f\,d\eta$ is called the non-commutative line integral over the Cayley-Dickson algebra $\mathscr{A}_v$ of line generalized functions f by η. Quite analogously such integral is defined for line generalized functions f and η on $\mathscr{D}(U,\mathscr{A}_v)\times\mathbf{P}(U)\times\mathscr{V}(U,\mathscr{A}_v)$.

8. Theorem. *Let F and Ξ be two generalized functions on U, $F,\Xi\in\mathscr{B}'(U,\mathscr{A}_v)$ or $F,\Xi\in\mathscr{D}'(U,\mathscr{A}_v)$, then the the non-commutative line integral over the Cayley-Dickson algebra $\mathscr{A}_v$ of line generalized functions f by ξ exists, where f is induced by F and ξ by Ξ.*

Proof. At first it is easy to mention that Definition 7 is justified by Definition 3 and Lemma 6, since the function $[\eta;\omega,\kappa|_{[a,x]}]$ is of bounded variation by the variable x for each rectifiable path $\kappa\in\mathbf{P}(U)$ and any test function ω (see Properties 3(G1-G3)), while the operator $\hat{f}$ always exists in the class of generalized line operators, $\hat{f}=dg/dz$, $(dg(z)/dz).1=f(z)$ (see Property 3(G5)).

Each generalized function $f\in\mathscr{B}(U,\mathscr{A}_v)$ can be written in the form:

$$(1)\qquad [f,\omega)=\sum_{j,k=0}^{2^v-1}[f_{j,k},\omega_k)i_j,$$

where each $f_{j,k}$ is a real valued generalized function, $f_{j,k} \in \mathscr{B}'(U,\mathbf{R})$, $\omega = \sum_k \omega_k i_k$, $\omega_k \in \mathscr{B}(U,\mathbf{R})$ is a real valued test function, $[f_{j,k},\omega_k) = [f_j,\omega_k i_k)$, $[f,\omega) = \sum_j [f_j,\omega)i_j$, $[f_j,\omega) \in \mathbf{R}$ for each $j = 0,\dots,2^v - 1$ and $\omega \in \mathscr{B}(U,\mathscr{A}_v)$, $i_0,\dots,i_{2^v-1}$ is the standard base of generators of the Cayley-Dickson algebra $\mathscr{A}_v$. It is well-known that in the space $\mathscr{B}'(U,\mathbf{R})$ of generalized functions the space $\mathscr{B}(U,\mathbf{R})$ of test functions is everywhere dense (see [4] and §1 above). Each functional $f_{j,k}$ induces a left $\mathscr{A}_v$-linear functional and hence using Properties 3$(G1,G7)$ it remains to consider the case, when F and Ξ are left $\mathscr{A}_v$-linear. In view of the decomposition given by Formula (1) we get that $\mathscr{B}(U,\mathscr{A}_v)$ is everywhere dense in $\mathscr{B}'(U,\mathscr{A}_v)_l$. Thus sequences of test functions F^l and Ξ^l exist converging to F and Ξ correspondingly in $\mathscr{B}'(U,\mathscr{A}_v)_l$.

Without loss of generality we can embed U into $\mathscr{A}_v$ taking its ϵ-enlargement (open neighborhood) in case of necessity. So it is sufficient to treat the case of a domain U in $\mathscr{A}_v$. In view of the analog of the Stone-Weierstrass theorem (see [11, 12]) in $C^0(Q,\mathscr{A}_v)$ for each compact canonical closed subset in $\mathscr{A}_v$ the family of all super-differentiable on Q functions is dense, consequently, the space $\mathscr{H}(U,\mathscr{A}_v)$ of all super-differentiable functions on U is everywhere dense in $\mathscr{D}(U,\mathscr{A}_v)$. For each rectifiable path γ in the domain U a compact canonical closed domain Q exists $Q \subset U$ so that $\gamma([a,b]) \subset Q$. Therefore, it is sufficient to consider test functions with compact supports in Q. Thus we take super-differentiable functions F^n and Ξ^n.

Let γ^l be a sequence of rectifiable paths continuously differentiable, $\gamma^l \in C^1([a,b],\mathscr{A}_v)$, converging to γ in $\mathbf{P}(U)$ relative to the metric ρ.

Then for any super-differentiable functions p and q we have

$$(2) \qquad \int_{\gamma^l} p(z)dq(z) = \int_a^b (d\zeta(z)/dz)\cdot[(dq(z)/dz)\cdot d\gamma^l(t)]|_{z=\gamma^l(t)}$$
$$= \int_a^b \sum_{k=0}^{2^v-1}(\partial\zeta(z)/\partial z_k)[\sum_{j=0}^{2^v-1}(\partial q_k(z)/\partial z_j)d\gamma^l_j(t)],$$

since each super-differentiable function is Fréchet differentiable, $d\gamma^l_j(t) = \gamma^{l\prime}_j(t)dt$, where $(d\zeta(z)/dz)\cdot 1 = p(z)$ and for the corresponding phrases of them for each $z \in U$. On the other hand, the functional

$$(3) \qquad \int_a^b \sum_{k=0}^{2^v-1}(\partial\zeta(z)/\partial z_k)\left[\sum_{j=0}^{2^v-1}(\partial q_k(z)/\partial z_j)d\gamma^l_j(t)\right]$$

is continuous on $\mathscr{B}(U,\mathscr{A}_v)^2 \times \mathbf{P}(U)$, i.e. for $\zeta, q \in \mathscr{B}(U,\mathscr{A}_v)$ and $\gamma \in \mathbf{P}(U)$ as well.

For a rectifiable path γ in U it is possible to take a sequence of open ϵ neighborhoods $\Gamma^\epsilon := \bigcup_{z\in\gamma([a,b])} \breve{B}(\mathscr{A}_v,z,\epsilon)$, $\epsilon = \epsilon(l) = 1/l$, where $\breve{B}(\mathscr{A}_v,z,\epsilon) := \{y : y \in \mathscr{A}_v; |y-z| < \epsilon\}$. Therefore, for each function v of bounded variation on

U and each rectifiable path γ in U a sequence of test functions θ^l with supports contained in $\Gamma^{1/l}$ exists such that

$$\lim_l \int_U [(d\zeta(z)/dz)\cdot\theta^l(z)]\lambda(dz) = \int_\gamma p(z)d\nu(z)$$

for each super-differentiable test functions $p,\zeta \in \mathcal{H}(U,\mathcal{A}_v)$ with $(d\zeta(z)/dz)\cdot 1 = p(z)$ on U, where λ denotes the Lebesgue measure on U induced by the Lebesgue measure on the real shadow $\mathbf{R}^{2^v}$ of the Cayley-Dickson algebra $\mathcal{A}_v$, where $\mathcal{H}(U,\mathcal{A}_v)$ denotes the family of all super-differentiable functions on the domain U with values in the Cayley-Dickson algebra $\mathcal{A}_v$.

Using the latter property and in accordance with Formulas (1-3) and 3(G6, G7) we put:

$$(4)\qquad [\xi;\omega,\gamma] := \lim_l [\Xi^l;\omega,\gamma] = \lim_l \int_\gamma \omega(y)\Xi^l(y)dy$$

and

$$(5)\qquad [\hat{f};\omega^1,\gamma;\nu] = \lim_l [dG^l/dz;\omega^1,\gamma;\nu] = \lim_l \int_\gamma \{(dG^l(z)/dz).\omega^1(z)\}d\nu(z)$$

for any $\nu \in \mathcal{V}(U,\mathcal{A}_v)$, where $(dG^l/dz)\cdot 1 = F^l(z)$ on U.

Therefore Ξ^l converges to ξ and dG^l/dz converges to $\hat{f}$, where $[\xi;\omega,*](\kappa|_{[a,x]}) = [\xi;\omega,\kappa|_{[a,x]}]$ for each $\kappa \in \mathbf{P}(U)$, $a < x \le b$ (see Lemma 6). Therefore, from Formulas (2-5) and Lemmas 5 and 6 we infer that

$$(6)\qquad [\int_\gamma f\,d\xi,\omega^1\otimes\omega) = \lim_l [dG^l/dz;\omega^1,*;[\Xi^l;\omega,*]](\gamma^l)$$

$$= \lim_l \int_{\gamma^l} [dG^l/dz;\omega^1,*;d[\Xi^l;\omega,*](z)],$$

where $z = \gamma^l(t)$, $a \le t \le b$.

9. Corollary. *If $F: U \to \mathcal{A}_v$ is a continuous function on U and Ξ is a generalized function on U, then the non-commutative line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ of line generalized functions f by ξ*

$$(1)\qquad [\int_\gamma f\,d\xi,\omega^1\otimes\omega)$$

exists, where f is induced by F and ξ by Ξ.

Proof. This follows from Theorem 8 and the fact that each continuous function F on U gives the corresponding regular line operator valued generalized function on the space of test functions ω^1 in $\mathcal{B}(U,\mathcal{A}_v)$ or $\mathcal{D}(U,\mathcal{A}_v)$:

$$[\hat{F};\omega^1,\gamma] = \int_\gamma (\hat{F}(z).\omega^1(z))dz.$$

In this case one can take the marked function $\omega^1 = \chi_V$, where V is a compact canonical closed sub-domain in U, since $\gamma([a,b])$ is compact for each rectifiable path γ in U so that $\gamma([a,b]) \subset V$ for the corresponding compact sub-domain V. This gives $\hat{F}.\chi_V(z) = F(z)$ for each $z \in V$ and $\hat{F}.\chi_V(z) = 0$ for each $z \in U \setminus V$. □

10. Corollary. *If $F \in \mathscr{B}'(U, \mathscr{A}_v)$ or $F \in \mathscr{D}'(U, \mathscr{A}_v)$ is a generalized function on U and Ξ is a function of bounded variation on U, then the non-commutative line integral over the Cayley-Dickson algebra $\mathscr{A}_v$ of line generalized functions f by ξ*

$$[\int_\gamma f\, d\xi, \omega^1 \otimes \omega)$$

exists, where f is induced by F and ξ by Ξ.

Proof. In this case we put

$$[\xi; \omega, \kappa] := \int_\kappa \omega(z) d\Xi(z)$$

for each test function ω and each rectifiable path κ in U. It is sufficient to take marked test function $\omega(z) = 1$ for each $z \in U$ that gives $d[\xi; 1, *] = d\Xi$. Thus this corollary follows from Theorem 8. □

11. Corollary. *If F is a continuous function on U and Ξ is a function of bounded variation on U, then the non-commutative line integral over the Cayley-Dickson algebra $\mathscr{A}_v$ of line generalized functions f by ξ.*

(1) $$[\int_\gamma f\, d\xi, \omega^1 \otimes \omega)$$

exists, where f is induced by F and ξ by Ξ. Moreover, this integral coincides with the non-commutative line integral from [14] §3 *for the unit test functions $\omega(z) = \omega^1(z) = 1$ for each $z \in U$:*

(2) $$[\int_\gamma f\, d\xi, 1 \otimes 1) = \int_\gamma f\, d\xi .$$

Proof. This follows from the combination of two preceding corollaries, since for a rectifiable path γ its image in U is contained in a compact sub-domain V in U, i.e. $\gamma([a,b]) \subset V$. □

12. Convolution Formula for Solutions of Partial Differential Equations. Using convolutions of generalized functions a solution of the equation

(C1) $(\Upsilon^s + \beta) f = g$ in $\mathscr{B}(\mathbf{R}^n, Y)$ or in the space $\mathscr{B}'(\mathbf{R}^n, Y)_l$ is:

(C2) $f = \mathscr{E}_{\Upsilon^s + \beta} * g$,

where $\mathscr{E}_{\Upsilon^s + \beta}$ denotes a fundamental solution of the equation

(C3) $(\Upsilon^s + \beta)\mathscr{E}_{\Upsilon + \beta} = \delta$,

$(\delta, \phi) = \phi(0)$ (see §9). The fundamental solution of the equation

(C4) $A_0\mathscr{V} = \delta$ with $A_0 = (\Upsilon^s + \beta)(\Upsilon_1^{s_1} + \beta_1)$
can be written as the convolution

(C5) $\mathscr{V} =: \mathscr{V}_{A_0} = \mathscr{E}_{\Upsilon^s+\beta} * \mathscr{E}_{\Upsilon_1^{s_1}+\beta_1}$.
In view of Formulas [14] 4(7-9) each generalized function $\mathscr{E}_{\Upsilon^s+\beta}$ can also be found from the elliptic partial differential equation

(C6) $\Xi_\beta \Psi_{\Upsilon^s+\beta} = \delta$ by the formula:

(C7) $\mathscr{E}_{\Upsilon^s+\beta} = [(\Upsilon^s + \beta)^*]\Psi_{\Upsilon^s+\beta}$,
where

(C8) $\Xi_\beta := (\Upsilon^s + \beta)(\Upsilon^s + \beta)^*$
(see §33 [15]).

13. Poly-functionals. Let $\mathbf{a}_k : \mathscr{B}(U, \mathscr{A}_r)^k \to \mathscr{A}_r$ or $\mathbf{a}_k : \mathscr{D}(U, \mathscr{A}_r)^k \to \mathscr{A}_r$ be a continuous mapping satisfying the following three conditions:

(P1) $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$ is $\mathbf{R}$ homogeneous

$$[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes (\omega^l t) \otimes \ldots \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k)t$$
$$= [\mathbf{a}_k t, \omega^1 \otimes \ldots \otimes \omega^k)$$

for each $t \in \mathbf{R}$ and $\mathscr{A}_r$ additive

$$[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes (\omega^l + \kappa^l) \otimes \ldots \otimes \omega^k)$$
$$= [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k) + [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \kappa^l \otimes \ldots \otimes \omega^k)$$

by any $\mathscr{A}_r$ valued test functions ω^l and κ^l, when others are marked, $l = 1, \ldots, k$, i.e. it is k $\mathbf{R}$ linear and k $\mathscr{A}_r$ additive, where $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$ denotes a value of $\mathbf{a}_k$ on given test $\mathscr{A}_r$ valued functions $\omega^1, \ldots, \omega^k$;

(P2) $[\mathbf{a}_k\alpha, \omega^1 \otimes \ldots \otimes (\omega^l\beta) \otimes \ldots \otimes \omega^k) = ([\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k)\alpha)\beta = [(\mathbf{a}_k\alpha)\beta, \omega^1 \otimes \ldots \otimes \omega^l \otimes \ldots \otimes \omega^k)$ for all real-valued test functions and $\alpha, \beta \in \mathscr{A}_r$;

(P3) $[\mathbf{a}_k, \omega^{\sigma(1)} \otimes \ldots \otimes \omega^{\sigma(k)}) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$ for all real-valued test functions and each transposition σ, i.e. bijective surjective mapping $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$.

Then $\mathbf{a}_k$ will be called the symmetric k $\mathbf{R}$ linear k $\mathscr{A}_r$ additive continuous functional, $1 \le k \in \mathbf{Z}$. The family of all such symmetric functionals is denoted by $\mathscr{B}'_{k,s}(U, \mathscr{A}_v)$ or $\mathscr{D}'_{k,s}(U, \mathscr{A}_r)$ correspondingly. A functional satisfying Conditions (P1, P2) is called a continuous k-functional over $\mathscr{A}_r$ and their family is denoted by $\mathscr{B}'_k(U, \mathscr{A}_r)$ or $\mathscr{D}'_k(U, \mathscr{A}_r)$. When a situation is outlined we may omit for short “continuous” or “k $\mathbf{R}$ linear k $\mathscr{A}_v$ additive”.

The sum of two k-functionals over the Cayley-Dickson algebra $\mathscr{A}_r$ is prescribed by the equality:

(P4) $[\mathbf{a}_k + \mathbf{b}_k, \omega^1 \otimes \ldots \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k) + [\mathbf{b}_k, \omega^1 \otimes \ldots \otimes \omega^k)$
for each test functions. Using Formula $(P4)$ each k-functional can be written as

(1) $$[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)$$
$$= [\mathbf{a}_{k,0} i_0, \omega^1 \otimes \ldots \otimes \omega^k) + \ldots + [\mathbf{a}_{k,2^r-1} i_{2^r-1}, \omega^1 \otimes \ldots \otimes \omega^k),$$

where $[\mathbf{a}_{k,j}, \omega^1 \otimes \ldots \otimes \omega^k) \in \mathbf{R}$ is real for all real-valued test functions $\omega^1, \ldots, \omega^k$ and each j; $i_0, \ldots, i_{2^r-1}$ denote the standard generators of the Cayley-Dickson algebra $\mathscr{A}_r$.

The direct product $\mathbf{a}_k \otimes \mathbf{b}_p$ of two functionals $\mathbf{a}_k$ and $\mathbf{b}_p$ for the same space of test functions is a $k+p$-functional over $\mathscr{A}_r$ given by the following three conditions:

(P5) $[\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes \omega^{k+p}) = [\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k)[\mathbf{b}_p, \omega^{k+1} \otimes \ldots \otimes \omega^{k+p})$
for any real-valued test functions $\omega^1, \ldots, \omega^{k+p}$;

(P6) if $[\mathbf{b}_p, \omega^{k+1} \otimes \ldots \otimes \omega^{k+p}) \in \mathbf{R}$ is real for any real-valued test functions, then
$$[(\mathbf{a}_k N_1) \otimes (\mathbf{b}_p N_2), \omega^1 \otimes \ldots \otimes \omega^{k+p}) = ([\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes \omega^{k+p}) N_1) N_2$$
for any real-valued test functions $\omega^1, \ldots, \omega^{k+p}$ and Cayley-Dickson numbers $N_1, N_2 \in \mathscr{A}_r$;

(P7) if $[\mathbf{a}_k, \omega^1 \otimes \ldots \otimes \omega^k) \in \mathbf{R}$ and $[\mathbf{b}_p, \omega^{k+1} \otimes \ldots \otimes \omega^{k+p}) \in \mathbf{R}$ are real for any real-valued test functions, then
$$[\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes (\omega^l N_1) \otimes \ldots \otimes \omega^{k+p}) = [\mathbf{a}_k \otimes \mathbf{b}_p, \omega^1 \otimes \ldots \otimes \omega^{k+p}) N_1$$
for any real-valued test functions $\omega^1, \ldots, \omega^{k+p}$ and each Cayley-Dickson number $N_1 \in \mathscr{A}_r$ for each $l = 1, \ldots, k+p$.

Therefore, we can now consider a partial differential operator of order u acting on a generalized function $f \in \mathscr{B}'(U, \mathscr{A}_r)$ or $f \in \mathscr{D}'(U, \mathscr{A}_r)$ and with generalized coefficients either $\mathbf{a}_\alpha \in \mathscr{B}'_{|\alpha|}(U, \mathscr{A}_r)$ or all $\mathbf{a}_\alpha \in \mathscr{D}'_{|\alpha|}(U, \mathscr{A}_r)$ correspondingly:

(1) $$Af(x) = \sum_{|\alpha| \le u} (\partial^\alpha f(x)) \otimes [(\mathbf{a}_\alpha(x)) \otimes 1^{\otimes(u-|\alpha|)}],$$

where $\partial^\alpha f = \partial^{|\alpha|} f(x) / \partial x_0^{\alpha_0} \ldots \partial x_n^{\alpha_n}$, $x = x_0 i_0 + \ldots x_n i_n$, $x_j \in \mathbf{R}$ for each j, $1 \le n = 2^r - 1$, $\alpha = (\alpha_0, \ldots, \alpha_n)$, $|\alpha| = \alpha_0 + \ldots + \alpha_n$, $0 \le \alpha_j \in \mathbf{Z}$, $[1, \omega) := \int_U \omega(y) \lambda(dy)$, λ denotes the Lebesgue measure on U, for convenience $1^{\otimes 0}$ means the multiplication on the unit $1 \in \mathbf{R}$. The partial differential equation

(2) $$Af = g$$

in terms of generalized functions has a solution f means by the definition that

(3) $$[Af, \omega^{\otimes(u+1)}) = [g, \omega^{\otimes(u+1)})$$

for each real-valued test function ω on U, where $\omega^{\otimes k} = \omega \otimes \ldots \otimes \omega$ denotes the k times direct product of a test function ω.

14. Anti-derivatives of First Order Partial Differential Operators with Generalized Coefficients.

Theorem. *Let Υ be a first order partial differential operator given by the formula*

$$(1)\qquad \Upsilon f = \sum_{j=0}^{n} (\partial f / \partial z_j) \otimes [i_j^* \psi_j(z)]$$

or

$$(2)\qquad \Upsilon f = \sum_{j=0}^{n} (\partial f / \partial z_j) \otimes \phi_j^*(z),$$

where $\mathrm{supp}(\psi_j(z)) = U$ *or* $\mathrm{supp}(\phi_j(z)) = U$ *for each* j *respectively,* f *and* $\psi_j(z)$ *or* $\psi_j(z)$ *are* $\mathcal{A}_v$*-valued generalized functions in* $\mathcal{B}'(U, \mathcal{A}_r)$ *or* $\mathcal{D}'(U, \mathcal{A}_r)$ *on the domain* U *satisfying Conditions* [14] 1(D1, D2), $\mathrm{alg}_{\mathbf{R}}\{[\phi_j, \omega), [\phi_k, \omega), [\phi_l, \omega)\}$ *is alternative for all* $0 \le j,\ k, l \le 2^v - 1$ *and* $\mathrm{alg}_{\mathbf{R}}\{[\phi_0, \omega), \dots, [\phi_{2^v-1}, \omega)\} \subset \mathcal{A}_v$ *for each real-valued test function* ω *on* U*. Then its anti-derivative operator* $\mathcal{I}_\Upsilon$ *exists such that* $\Upsilon \mathcal{I}_\Upsilon f = f$ *for each continuous generalized function* $f : U \to \mathcal{A}_v$ *and it has an expression through line integrals of generalized functions.*

Proof. When an operator with generalized coefficients is given by Formula (1), we shall take unknown generalized functions $v_j(z) \in \mathcal{A}_v$ as solutions of the system of partial differential equations by real variables z_k:

$$(3)\qquad [(\partial v_j(z)/\partial z_j) \otimes \psi_j(z), \omega^{\otimes 2}) = [1, \omega^{\otimes 2}) \text{ for all } 1 \le j \le n;$$

$$(4)\qquad [\psi_k(z) \otimes (\partial v_j(z)/\partial z_k), \omega^{\otimes 2}) = [\psi_j(z) \otimes (\partial v_k(z)/\partial z_j), \omega^{\otimes 2}) \quad \text{for all } 1 \le j < k \le n$$

and real-valued test functions ω on U.

If the operator is given by Formula (2) we consider the system of partial differential equations:

$$(5)\qquad [((dg(z)/dz) \cdot [\partial v_j(z)/\partial z_k]) \otimes \phi_k^*(z) + ((dg(z)/dz) \cdot [\partial v_k(z)/\partial z_j]) \otimes \phi_j^*(z), \omega^{\otimes 2}) = 0 \text{ for all } 0 \le j < k \le n;$$

$$(6)\qquad \partial v_j(z)/\partial z_j = \phi_j(z) \text{ for all } j = 0, \dots, n;$$

$$(7)\qquad [((dg(z)/dz) \cdot \phi_j(z)) \otimes \phi_j^*(z), \omega^{\otimes 2}) = [f(z) \otimes 1, \omega^{\otimes 2}) \text{ for each } j = 0, \dots, n$$

and every real-valued test function ω.

Certainly the system of differential equations given by Formulas (3, 4) or (5–7) have solutions in the spaces of test functions $\mathcal{B}(U, \mathcal{A}_r)$ or $\mathcal{D}(U, \mathcal{A}_r)$, when all functions ψ_j or ϕ_j are in the same space respectively. Applying §§4 or 5 [14] we find generalized functions v_j resolving these system of partial differential equations correspondingly, when all functions ψ_j or ϕ_j are generalized functions, since the spaces of test functions are dense in the spaces of left $\mathcal{A}_r$-linear generalized functions (see §§3 and 8). Substituting line integrals $\int_\gamma q(y)dv_j(y)$ from [14] §§4 and 5 on line integrals $[\int_\gamma q(y)dv_j(y), \omega^1 \otimes \omega)$ from §8 one gets the statement of this theorem, since test functions ω^1 and ω in the line integrals of generalized functions can also be taken real-valued and the real field is the center of the Cayley-Dickson algebra $\mathcal{A}_v$. Therefore, we infer that

$$(8) \qquad \partial[\int_{\gamma^\alpha|_{<a_\alpha,t_z]}} f(y)dv_j(y), \omega\otimes\omega)/\partial z_k = [\hat{f}(z)\cdot[dv_j(z)/dz_k], \omega\otimes\omega)$$

for each real-valued test function ω and $z\in U$, where $\gamma^\alpha(t_z)=z$, $t_z\in\langle a_\alpha, b_\alpha\rangle$, $\alpha\in\Lambda$. Equality (8), Theorem 8 and Corollaries 9-11 and Conditions 13(P1-P7) give the formula for an anti-derivative operator:

$$(9) \qquad [\mathcal{I}_\Upsilon f, \omega\otimes\omega) = [_\Upsilon \int f(z)dz, \omega\otimes\omega)$$

$$= (n+1)^{-1}\sum_{j=0}^{n}\{[\int_{\gamma^\alpha|_{[a_\alpha,t]}} q(y)dv_j(y), \omega\otimes\omega)$$

for each real-valued test-function ω, where $\alpha\in\Lambda$, $a_\alpha\le t\le b_\alpha$, $t=t_z$, $z=\gamma(t)$, consequently,

$$(10) \qquad [\Upsilon_\Upsilon \int f(y)dy, \omega^{\otimes 3}) = [f\otimes 1\otimes 1, \omega^{\otimes 3}).$$

15. Note. Certainly, the case of the partial differential operator

$$(1) \qquad \Upsilon f = \sum_{j=0}^{n}(\partial f/\partial z_{k(j)})\otimes\phi^*_{k(j)}(z),$$

where $0\le k(0)<k(1)<\dots<k(n)\le 2^v-1$ reduces to the considered in §14 case by a suitable change of variables $z\mapsto y$ so that $z_{k(j)}=y_j$. In the next paper it is planned to give examples of solutions of PDE with the help of formulas presented in this article.

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