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Research Article

Vorticity Gramian of Compact Riemannian Manifolds

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Abstract. The vorticity of a vector field on 3-dimensional Euclidean space is usually given by the curl of the vector field. In this paper, we extend this concept to n-dimensional compact and oriented Riemannian manifold. We analyse many properties of this operation. We prove that a vector field on a compact Riemannian manifold admits a unique Helmholtz decomposition and establish that every smooth vector field on an open neighbourhood of a compact Riemannian manifold admits a Stokes' type identity.

Keywords. Manifold, Curl, Metric tensor flow, Hodge star operator, Helmholtz decomposition

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1. Introduction

Vorticity is a pseudovector field describing the local spinning motion of a flow near some point on a manifold. In classical mechanics, the dynamics of a flow are described by its rotation and expansion. Calin and Chang [8] expressed the rotation component by the curl vector, while the expansion is described by the divergence function. The classical formulas involving rotation and expansion in the case of smooth functions and vector fields on Riemannian manifolds show that gradient vector fields do not rotate and that the curl vector field is incompressible. Varayu, Chew *et al*. [6] among others showed that on Riemannian manifolds, the curl of a vector field is not a vector field, but a tensor.

Many studies on vorticity of flows on manifolds have been ongoing for many decades. For example, Frankel [13] in fifties established how homology of manifolds influences vector and tensor fields on the manifolds. It showed that the vector fields give rise to one-parameter groups of point divergence-free transformations of the manifolds. The work of Xie and Mar [26] employed Poisson equation for stream and vorticity equation to study 2-dimensional vorticity and stream function expanded in general curvilinear coordinates. It constructed numerical algorithms of covariant, anti-covariant metric tensor and Christoffel symbols of the first and second kinds in curvilinear coordinates. Similarly, Perez-Garcia [24] studied exact solutions of the vorticity equation on the sphere as a manifold. In the work of Peng and Yang [23], the existence of the curl operator on higher dimensional Euclidean space, $\mathbb{R}^n$, $n > 3$, was proved.

More recently, Bauer, Kolev and Preston [5] carried out a geometric investigations of a vorticity model equation extending the works of [9, 12, 15, 16, 22, 25] and Kim [18] on vorticity to manifold study. Besides, Deshmukh, Pesta and Turki [11] went ahead to show that the presence of a geodesic vector field on a Riemannian manifold influences its geometry while Bär [4] and M$\ddot{u}$ller [20] extended the curl and divergence operators to odd-dimensional manifolds in arbitrary basis.

In this study, we extend the concept of vorticity to n-dimensional compact and oriented Riemannian manifolds and analyse many properties of this operation. We proceed with fixing our notations and briefly explaining some basic concepts required to follow the discussions.

Let (M,g) be a Riemannian manifold. By this we mean that M is a topological space that is locally similar to the Euclidean space and g is the Riemannian metric on M. We recall that a Riemannian metric g on a smooth manifold M is a symmetric, positive definite $(0,2)$-tensor field, see e.g. [1, 17, 19] and [14]. This means that for any point $p \in M$, the metric is the map $g_p : T_pM \times T_pM \to \mathbb{R}$ that is a positive definite scalar product for a tangent space T_pM. The Riemannian metric enables to measure distances, angles and lengths of vectors and curves on the manifold, see e.g. [3, 7, 10, 17] and [19], for details. We denote the Riemannian manifold (M,g) simply by M. The manifold M is called compact if it is compact as a topological space. If M is a smooth manifold, then [8] and [1] proved that there is at least one Riemannian metric on M.

We call a function $f : M \to \mathbb{R}$ smooth if for every chart (U,ϕ) on M, and the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. The set of all smooth functions on the manifold M will be denoted by $C^\infty(M)$. Let Ω^k denote the vector space of smooth k-forms on M, and let $d : \Omega^k \to \Omega^{k+1}$ be the exterior derivative. Note that the metric which gives an inner product on the tangent space T_pM at each $p \in M$ induces a natural metric on each cotangent space T_p^*M, as follows. At p, let $\{b_1, b_2, \cdots, b_n\}$ be an orthonormal basis for the tangent space. One obtains a metric on the cotangent space by declaring that the dual basis $\{b^1, b^2, \cdots, b^n\}$ is orthonormal. Hence given any two k-forms β and γ, we have that (β,γ) is a function on M. We call $(\cdot,\cdot)$ the pointwise inner product; see e.g. [7, 10, 14, 19] and [21]. For a coordinate chart on M,

$$(x^1, \cdots, x^n) : U \to \mathbb{R}^n,$$

we represent g by the Gram matrix (g_{ij}) where $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$, and $\langle , \rangle$ is the inner product on the tangent space. The volume form dV is defined be $b^1 \wedge b^2 \wedge \cdots \wedge b^n$, and it is a well-known fact from linear algebra that $dV = \sqrt{|g|}dx$ where $dx = dx^1 \wedge \cdots \wedge dx^n$. Using the point-wise inner product above, one writes the L^2-inner product on $\Omega^k(M)$ as

$$\langle \beta, \gamma \rangle = \int_M (\beta, \gamma) dV, \quad \text{for all } \beta, \gamma \in \Omega^k(M).$$

2. Vector Fields and Differential Operators

Vector fields and differential operators are the main tools used in the analysis of vorticity in this work. We employ these tools to construct the curl operator on M and analyse its many properties.

A vector field on M is a family $\{X(p)\}_{p \in M}$ of tangent vectors such that $X(p) \in T_pM$ for any $p \in M$. In local coordinates chart $(x^1, \cdots, x^n)$,

$$X(p) = X^i(p) \frac{\partial}{\partial x^i|_{x=p}}.$$

The vector field $X(p)$ is called smooth if all functions X^i are smooth in any chart in M; see e.g. [1,7] and [8]. We denote the set of all vector fields on M by $\Gamma(M)$.

Definition 2.1 ([10, 14])**.** For every $p \in M$ the differential map df at p is defined by

$$df_p : T_pM \to T_{f(p)}N \ \text{ with } \ df_p(V)(h) = V(h \circ f), \quad \text{for all } V \in T_pM, \text{ for all } h \in C^\infty(N).$$

Locally, it is given by

$$df_p \left(\frac{\partial}{\partial x_j|_p} \right) = \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_j|_p} \frac{\partial}{\partial y^k},$$

where $f = (f^1, f^2, \cdots, f^n)$. The matrix $\left(\frac{\partial f^k}{\partial x_j}\right)_{k,j}$ is the Jacobian of f with respect to the charts $(x^1, x^2, \cdots, x^n)$ and $(y^1, y^2, \cdots, y^n)$ on M and N, respectively.

Definition 2.2. Let $f \in C^\infty(M)$ be a smooth function. The gradient of f, denoted by $\operatorname{grad} f$, is a vector field on M metrically equivalent to the differential df of f :

$$g(\operatorname{grad} f, X) = df(X) = X(f), \quad \text{for all } X \in \Gamma(M).$$

Definition 2.3. Let $X \in \Gamma(M)$ on M. The divergence of X at the point $p \in M$ denoted as ∂X is defined locally as

$$\partial X = \sum_{i=1}^{n} X^i_{;i} = \sum_{i=1}^{n} \left(\frac{\partial X^i}{\partial x_i} + \sum_j \Gamma^i_{ij} X^j \right),$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x_k} + \frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l} \right)$$

is the Christoffel symbol. In local coordinates,

$$\partial X = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} (\sqrt{|g|} X^j)$$

with summation over $j = 1, \cdots, n$.

Definition 2.4. The Lie bracket $[,]:\Gamma(M)\times\Gamma(M)\to\Gamma(M)$ is defined by

$$[X,Y]=X(Y)-Y(X),\quad \text{for all } X,Y\in\Gamma(M).$$

Locally,

$$[X,Y]=\sum_{i,j=1}^{n}\left(\frac{\partial Y^i}{\partial x_j}X^j-\frac{\partial X^i}{\partial x_j}Y^j\right)\frac{\partial}{\partial x_i}.$$

If $[U,V]=0$, we say that the vector fields commute.

An extension of the usual directional derivative on the Euclidean space to smooth manifold is linear connection.

Definition 2.5. A linear connection ∇ on M is a map $\nabla:\Gamma(M)\times\Gamma(M)\to\Gamma(M)$ such that ∇_XY is $C^\infty(M)$ in X and linear in Y over the real field with $\nabla_X(fY)=(Xf)Y+f\nabla_XY$, for all $f\in C^\infty(M)$.

We note that ∇_XY is a new vector field which, roughly speaking, is the vector rate of change of Y in the direction of X. A particular connection on Riemannian manifolds that is torsion-free is the Levi-Civita connection. The Levi-Civita connection is defined in local coordinates as

$$\nabla_XY=\sum_{i,k}^{n}X^i\left(\frac{\partial Y^k}{\partial x_i}+\sum_j\Gamma_{ij}^kW^j\right)\frac{\partial}{\partial x_k},$$

where

$$X=\sum_{i=1}^{n}X^i\frac{\partial}{\partial x_i},\quad Y=\sum_{k=1}^{n}Y^k\frac{\partial}{\partial x_k}$$

and Γ_{ij}^k are the Christoffel symbols defined by

$$\Gamma_{ij}^k=\frac{1}{2}\sum_m g^{km}\left(\frac{\partial g_{jm}}{\partial x_i}+\frac{\partial g_{im}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_m}\right),$$

where (g^{km}) is the inverse of (g_{ij}). Note that X is a Killing vector field if $\mathscr{L}_Xg=0$.

Let M be a compact n-dimensional Riemannian manifold. A vector field on M which generates isometries of the Riemannian metrics is represented by a Killing vector field $v=v^ie_i$, where $e_i=\frac{\partial}{\partial x_i}$. A Killing vector satisfies the differential equation $v_i|_j=-v_j|_i$ where solidus indicates covariant differentiation. In particular, a Killing vector is divergence free with respect to the volume density $\sqrt{|g|}$, that is,

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}v^i\right)=v^i|_i=0$$

since a distance preserving map conserves volume automatically. Thus, a Killing vector, in particular, is an isometric flow. We also need to clarify the notion of tensor on M.

Definition 2.6. A tensor of type (r,s) at $p\in M$ is a multi-linear function

$$T:(T_p^*M)^r\times(T_pM)^s\to\mathbb{R}.$$

A tensor field $\mathscr{T}$ of type (r,s) is a smooth map, which assigns to each point $p\in M$ an (r,s)-tensor $\mathscr{T}_p$ on M at the point p. In local coordinates,

$$\mathscr{T}=\mathscr{T}_{j_1j_2\cdots j_r}^{i_1i_2\cdots i_s}dx^{j_1}\otimes dx^{j_2}\otimes\cdots\otimes dx^{j_r}\otimes\frac{\partial}{\partial x_{i_1}}\otimes\frac{\partial}{\partial x_{i_2}}\otimes\cdots\otimes\frac{\partial}{\partial x_{i_s}}.$$

$\mathscr{T}$ acts on r one-forms and s vector fields thus

$$\begin{aligned}&\mathscr{T}(\omega_1,\omega_2,\cdots,\omega_r,X_1,X_2,\cdots,X_s)\\&=\mathscr{T}^{i_1i_2\cdots i_s}_{j_1j_2\cdots j_r}dx_{j_1}(X_1)dx_{j_2}(X_2)\cdots dx_{j_r}(X_r)\frac{\partial}{\partial x_{i_1}}(\omega_1)\frac{\partial}{\partial x_{i_2}}(\omega_2)\cdots\frac{\partial}{\partial x_{i_s}}(\omega_s)\\&=\mathscr{T}^{i_1i_2\cdots i_s}_{j_1j_2\cdots j_r}X_1^{j_1}X_2^{j_2}\cdots X_r^{j_r}\omega_1^{i_1}\omega_2^{i_2}\cdots\omega_s^{i_s}.\end{aligned}$$

We say the tensor $\mathscr{T}$ is s covariant and r contravariant.

Let M be a compact n-dimensional manifold and let ρ be a positive scalar density on M. A p-tensor

$$\omega=\omega^{i_1\cdots i_p}e_{i_1}\wedge\cdots\wedge e_{i_p}$$

is classically a skew-symmetric contravariant tensor of order p, where $e_i=\frac{\partial}{\partial x_i}$ are the vectors of the coordinate frame (basis vectors). Let T_p be the linear space of all p-tensors on all of the manifold, M. We can now define the divergence of a tensor field.

Definition 2.7. The divergence of a p-tensor ω written $\partial\omega$ is the $(p-1)$-tensor

$$\partial\omega=\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\rho\omega^{i_1\cdots i_p}e_{i_1}\wedge\cdots\wedge e_{i_p}\right).$$

Let $V_p=\{\omega\in T_p:\partial\omega=0\}$ be the linear space of divergence-free p-tensors and let the linear space of p-tensor divergences be

$$D_p=\{\omega\in T_p:\omega=\partial\omega' \text{ for some } \omega'\in T_{p+1}\}.$$

For $p=0$ we have that the p-tensor is the ordinary scalar function. For such functions f, we have that $\partial f=0$. An easy calculation show that $\partial^2-\partial\partial=0$, hence, D_p is a linear subspace of V_p. We have the following preliminary results in form of lemmas.

Lemma 2.8. Let $X\in\Gamma(M)$, T be an $(n,0)$-tensor field and dV be the volume form on M. Let $\mathscr{L}_X$ be the Lie derivative of T, then $\mathscr{L}_XdV=(\partial X)dV$.

Proof. We recall that $T=dV=\sqrt{|g|}dx_1\wedge dx_2\wedge\cdots\wedge dx_n$ is an $(n,0)$-tensor field on M. The Lie derivative $\mathscr{L}_X$ of T given by $\mathscr{L}_X(T)=T_{12\cdots n}dx_1\wedge dx_2\wedge\cdots\wedge dx_n$ is also an $(n,0)$-tensor or an n-form $\mathscr{L}_X(T)=(\mathscr{L}_XT)_{12\cdots n}dx_1\wedge dx_2\wedge\cdots\wedge dx_n$. We need to show that $(\mathscr{L}_XT)_{12\cdots n}=(\partial X)\sqrt{g}$. Indeed, using the formula which gives the components of the Lie derivative of a tensor, we have

$$(\mathscr{L}_XT)_{12\cdots n}=\frac{\partial T_{12\cdots n}}{\partial x_i}X^i+T^{j_12\cdots n}\frac{\partial X^1}{\partial x_{j_1}}+T^{2j_2\cdots n}\frac{\partial X^2}{\partial x_{j_2}}+\cdots+T^{12\cdots j_n}\frac{\partial X^n}{\partial x_{j_n}}.$$

As $T_{12\cdots j_p\cdots n}=\delta_{p,j_p}T_{12\cdots p\cdots n}$, we get

$$\begin{aligned}(\mathscr{L}_XT)_{12\cdots n}&=\frac{\partial T_{12\cdots n}}{\partial x_i}X^i+T_{12\cdots n}\left(\frac{\partial X^1}{\partial x_1}+\cdots+\frac{\partial X^n}{\partial x_n}\right)\\&=\frac{\partial T_{12\cdots n}}{\partial x_i}X^i+T_{12\cdots n}\frac{\partial X^i}{\partial x_i}\\&=\frac{\sqrt{g}}{\partial x_i}+\sqrt{g}\frac{\partial X^i}{\partial x_i}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x_i}\left(\sqrt{g}X^i\right)\\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}X^i\right)\sqrt{g}\\
&= (\partial X\sqrt{g}).
\end{aligned}
$$

Hence, $\mathscr{L}_X T = \partial X\sqrt{g}dx_1\wedge dx_1\wedge\cdots\wedge dx_n = \partial X dV$. □

Lemma 2.9. Let $f\in C^\infty(M)$ and $X\in\Gamma(M)$. Then $\partial(fX) = f\partial X + g(\operatorname{grad} f, X)$.

Proof. From the definition of the ∂ operator, we have

$$
\begin{aligned}
\partial(fX) &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}\left(\sqrt{g}fX^j\right)\sqrt{g}\\
&= \frac{1}{\sqrt{g}}\frac{\partial f}{\partial x_j}\sqrt{g}X^j + f\frac{1}{\sqrt{g}}\frac{\partial f}{\partial x_j}(\sqrt{g}X^j)\\
&= \frac{\partial f}{\partial x_j}X^j + f\partial X\\
&= g_{kj}(\operatorname{grad} f)^k X^j + f\partial X\\
&= g(\operatorname{grad} f, X) + f\partial X.
\end{aligned}
$$

Using that $\mathscr{L}_X dV = (\partial X)dV$, for all $X\in\Gamma(M)$, the result follows. □

We define the adjoint operator $\delta:\Omega^k\to\Omega^{k+1}$ of d by requiring that $\langle\omega,\delta\beta\rangle = \langle d\omega,\beta\rangle$, for all $\omega\in\Omega^{k-1}$, and $\beta\in\Omega^k$ using the L^2 inner product on $\Omega^k(M)$. From the metric g, we can also define the Hodge star operator $*:\Omega^k\to\Omega^{n-k}$ by requiring that for all $\beta,\gamma\in\Omega^k$, $\beta\wedge*\gamma = (\beta,\gamma)dV$. Notice that the Hodge star operator $*$ is linear and point-wise. Therefore, the inner product on Ω^k can be written as $\langle\beta,\gamma\rangle = \int_M \beta\wedge*\gamma$. So, one can find an expression for δ acting on one-forms. Given $f\in\Omega^0$ and $\omega\in\Omega^1$, we have

$$
\begin{aligned}
\langle f,\delta\omega\rangle &= \langle df,\omega\rangle\\
&= \int_M df\wedge*\omega\\
&= \int_M [d(f\wedge*\omega) - f\wedge d*\omega]\\
&= \int_M -f\wedge d*\omega\\
&= -\int_M f\wedge *^2 d*\omega\\
&= \int_M f\wedge*(-*d*)\omega\\
&= \langle f,(-*d*)\omega\rangle.
\end{aligned}
$$

Now, we can go on to construct vorticity through the curl operator on the compact Riemannian manifold M and study their flow vorticity.

3. Vorticity of Flows

Let X be a vector field on an open subset of $U \subset M$ and ω_X be the associated 1-form on U dual to X. So, for each $p \in U$, the linear functional $\omega_X(p) \in T^*M$ on T_pM is $\langle X(p), \cdot \rangle$. It follows that the assignment $X \mapsto \omega_X$ is additive in X and linear with respect to multiplication by smooth function on U. Let V be a finite dimensional vector space with inner product $\langle , \rangle$, then the dual vector space V^* is naturally isomorphic to V under the map

$$\alpha : V \to V^*, \text{ with } \alpha(v) = v^* \in V^*, \text{ satisfying } v^*(w) = \langle v, w \rangle, \quad \text{for all } v, w \in V.$$

These lead to define the curl operator taking vector fields to vector fields to be

$$\operatorname{curl} X = \alpha^{-1} * d\alpha X = g^{-1} * dg(X).$$

In coordinate form, the curl of a vector field X on a Riemannian manifold M is a 2-covariant antisymmetric tensor A with the components A_{ij} given by

$$A_{ij} = X_{i;j} - X_{j;i} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}.$$

In particular, a Riemannian metric on a manifold M is an assignment of inner product on each cotangent space T_p^*M under the isomorphism α. The inner product g induces an inner product on each of the tensor product $T_pM \otimes \cdots \otimes T_pM$.

Definition 3.1. Let $X \in \Gamma(U)$ with U open in M, the function $\operatorname{div}(X) \in C^\infty(U)$ is characterised by

$$d(*(\omega_X)) = \operatorname{div}(X) dV_M\big|_U.$$

To see this, let $M = \mathbb{R}^n$ with the standard flat Riemannian metric and orientation in the standard linear coordinates $\{x_1, \cdots, x_n\}$. If $X = \sum X_j \partial_{x_j} \in \Gamma(U)$ is a vector field, since the volume form determined by this metric and orientation is $dx_1 \wedge \cdots \wedge dx_n$, we have

$$\begin{aligned}
d(*(\omega_X)) &= d\left(*\left(\sum X_j dX_j\right)\right) \\
&= d\left(\sum X_j * (dX_j)\right) \\
&= d\left(\sum (-1)^{j-1} X_j dx_1 \wedge \cdots \wedge \widehat{dx_n} \wedge \cdots \wedge dx_n\right) \\
&= \sum (-1)^{j-1} dX_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_n} \wedge \cdots \wedge dx_n \\
&= \sum \left(\frac{\partial X_j}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n \\
&= \sum \left(\frac{\partial X_j}{\partial x_j}\right) dV_M \\
&= \sum_{j=1}^{n} \frac{\partial X_j}{\partial x_j} \\
&= \operatorname{div}(X).
\end{aligned}$$

Our definition of divergence of a vector field was intrinsic to the Riemannian structure and orientation, thus, we can likewise compute the divergence in any oriented coordinate system on M. We make the next definition.

Definition 3.2. Let M be a Riemannian manifold with corners. For an open $U \subseteq M$ and $f \in C^{\infty(U)}$, the smooth vector, grad(f), on U is defined by the condition: $\omega_{\text{grad}(f)} = df \in \Omega^1_M(U)$. That is, for each $p \in U$, we have $\langle \text{grad}(f), \cdot \rangle = df(p)$ is a linear functional on T_pM.

Now, let M be a Riemannian manifold without boundary and N be an oriented submanifold with boundary inside of M with constant dimension 1. Let N be given the induced Riemannian metric of M. So the boundary ∂N is assigned a collection of signs $\mathscr{E}(p) \in \{\pm 1\}$ for each $p \in \partial N$ where $\mathscr{E}$ is the usual Levi-Civita symbol defined by

$$\mathscr{E}_{a_1,\cdots,a_n} = \begin{cases} +1, & \text{if } (a_1,\cdots,a_n) \text{ is an even permutation of } 1,2,\cdots,n, \\ -1, & \text{if } (a_1,\cdots,a_n) \text{ is an odd permutation of } 1,2,\cdots,n, \\ 0, & \text{if otherwise;} \end{cases}$$

see e.g. [3] and [7]. Let dl be the length form on N and T be the tangent field dual to dl. It can be proved that for any $f \in C^\infty(M)$ for which $f\big|_N \in C^\infty(N)$ is compactly supported, the smooth inner product function $\langle \text{grad}(f)\big|_N, T\rangle$ is compactly supported on N and

$$\int_N \langle \text{grad}(f)\big|_N, T\rangle dl = \sum_{p \in \partial N} \mathscr{E}(p) f(p).$$

In this way, we see that vector fields give rise to one-parameter groups of point transformations of the the manifold and one may be interested in those point transformations that are divergence-free. We call such vectors and their associated transformations simply "flows". The next lemma ensures the existence of vector fields generating flows on M.

Lemma 3.3. Let V be a nonzero vector field at a point p on the manifold M. Then there exists a system of coordinates $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$ about p such that there is $j \in \{1,\cdots,n\}$ for which $V = \frac{\partial}{\partial \bar{x}_j}$. We call this *rectification lemma*.

Proof. This lemma follows from the fact that in a compact Riemannian manifold, M, if $p \in M$ there is an open neighbourhood V of p in the ambient manifold $\mathbb{R}^{n+1}$ such that if $U \subset \mathbb{R}^n$ is open and $\phi : U \to \mathbb{R}^{n+1}$ is smooth, then $\phi(U)$ is a homomorphism. Besides, any transition Jacobian on U for change of coordinates has full rank for every $p \in U$. This proves the lemma. □

Given a vector field X, consider the system

$$\frac{dc^k}{dt} = X^k(c(t)), \quad k = 1,2,\cdots,n, \tag{3.1}$$

where $c(t)$ is the integral curve associated with X. The next result shows that the system (3.1) can be solved locally around the point $x_0 = c(0)$, for $0 < t < \epsilon$.

Proposition 3.4 (Existence and Uniqueness)**.** Given $x_0 \in M$ and let X be a nonzero vector field on an open set $U \subset M$ of x_0, then there is $\epsilon > 0$ such that the system (3.1) has a unique solution $c : [0,\epsilon) \to U$ such that $c(0) = x_0$.

Proof. By the rectification lemma, there is a local change of coordinates $\bar{x} = \phi(x)$ such that the system (3.1) becomes

$$\frac{dc^k}{dt} = \delta_{kn}, \quad k = 1,2,\cdots,n \ \text{ where } \ \bar{c} = \phi(c).$$

This system has a unique solution through the point $\bar{x}_0 = \phi(x_0)$ given by

$$\bar{c}^k(t) = \bar{x}_0^k, \quad k = 1, 2, \cdots, n-1 \quad \text{and} \quad \bar{c}^n(t) = t + \bar{x}_0^n.$$

Hence this will hold also for the system (3.1) in a neighbourhood of $x_0 = \phi^{-1}(\bar{x}_0)$. □

Let $f_j := \frac{\partial f}{\partial x_j}$, and $f^i := g^{ij} f_j$ so that $\nabla f = f^i \frac{\partial}{\partial x} = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x}$, we have more propositions that will lead us to the main results of this work.

Proposition 3.5. If $X \in \Gamma(M)$ then $X = \operatorname{grad}\phi \iff \operatorname{curl} X = 0$ and $\operatorname{tr}(\operatorname{curl} X) = 0$.

Proof. Let $X = \operatorname{grad}\phi$, then $X^k = g^{kj} \frac{\partial \phi}{\partial x_j}$ and $X_i = \frac{\partial \phi}{\partial x_i}$. So, we have

$$(\operatorname{curl} X)_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0.$$

Conversely, let $X \in \Gamma(M)$ such that $\operatorname{curl} X = 0$. Then, $\frac{\partial X_i}{\partial x_j} = \frac{\partial X_j}{\partial x_i}$. Thus, the 1-form $\omega = \sum X_k dx_k$ is exact. This means there is a locally defined function such that $\omega = df = \sum \frac{\partial f}{\partial x_k} dx_k$. Thus, $X_k = X^j$. Besides, $\operatorname{tr}(\operatorname{curl} X) = g^{ij}(X_{i;j} - X_{j;i}) = X^j_{;j} - X^i_{;i} = 0$; which completes the proof. □

Let $\langle , \rangle$ be the Riemannian metric with associated Levi-Civita connection ∇. We prove global and invariant properties of the curl operator on M.

Proposition 3.6. If $A = \operatorname{curl} X$, then $A(U,V) = \langle \nabla_V X, U \rangle - \langle \nabla_U X, V \rangle$, for all $U, V \in \Gamma(M)$.

Proof. For every $U, V \in \Gamma(M)$, we have

$$\begin{aligned} A(U,V) &= A_{ij} U^i V^j = (X_{i;j} - X_{j;i}) U^i V^j \\ &= (\nabla_{\partial_j} X)_i U^i V^j - (\nabla_{\partial_i} X)_j U^i V^j \\ &= \langle \nabla_{\partial_j} X, U \rangle V^j - \langle \nabla_{\partial_i} X, U \rangle U^i \\ &= \langle \nabla_{V^j \partial_j} X, U \rangle - \langle \nabla_{U^i \partial_j} X, V \rangle \\ &= \langle \nabla_V X, U \rangle - \langle \nabla_U X, V \rangle. \end{aligned}$$

□

Proposition 3.7. Let $A = \operatorname{curl} X$, where $X \in \Gamma(M)$. Then

$$A(U,V) = V\langle X, U \rangle - U\langle X, V \rangle + \langle X, [U,V] \rangle.$$

Proof. Since ∇ is a metric connection

$$V\langle X, U \rangle = \langle \nabla_V X, U \rangle + \langle X, \nabla_V U \rangle \quad \text{and} \quad U\langle X, V \rangle = \langle \nabla_U X, V \rangle + \langle X, \nabla_U V \rangle.$$

Using the symmetry of ∇, subtracting we obtain

$$V\langle X, U \rangle - U\langle X, V \rangle = A(U,V) + \langle X, [V,U] \rangle,$$

which proves the claim. □

The following result shows the relation between the curl, Levi-Civita connection and the Lie derivative.

Proposition 3.8. If $A = \operatorname{curl} X$ and ∇ is the Levi-Civita connection on M, then

$$A(U,V) = 2\langle \nabla_V X, U \rangle - (L_X g)(U,V).$$

Proof. We have

$$2\langle\nabla_V X, U\rangle = V\langle X, U\rangle + X\langle U, V\rangle - U\langle V, X\rangle - \langle V, [X, U]\rangle + \langle X, [U, V]\rangle + \langle U, [V, X]\rangle.$$

That is,

$$\begin{aligned}2\langle\nabla_V X, U\rangle &= A(U, V) + X\langle U, V\rangle - \langle V, [X, U]\rangle + \langle U, [V, X]\rangle\\ &= A(U, V) + X\langle U, V\rangle - \langle V, L_X U\rangle - \langle U, L_X V\rangle.\end{aligned}$$

Using that $(L_X g)(U, V) = X\langle U, V\rangle - \langle L_X U, V\rangle - \langle U, L_X V\rangle$, we obtain

$$2\langle\nabla_V X, U\rangle = A(U, V) + (L_X g)(U, V).$$

□

We can now show a Helmholtz decomposition of vector fields on M. This is the theorem that follows. That is, we prove that a vector field X on a compact Riemannian manifold can be uniquely decomposed as a sum of two vectors Y and Z, where Y is the rotation component and Z the expansion component.

Theorem 3.9. If $X \in \Gamma(M)$, there are two vector fields Y and Z on M such that $X = Y + Z$, with $\operatorname{div} Y = 0$ and $\operatorname{curl} Z = 0$. Moreover, the decomposition is unique.

Proof. Let $\eta = \operatorname{div} X$ and let ϕ solve the elliptic equation $\nabla \circ \nabla\phi = \Delta\phi = \eta$. Take $Z = \nabla\phi$ and $Y = X - \nabla\phi$. Then $\operatorname{curl} Z = \operatorname{curl}\nabla\phi = 0$ and $\operatorname{div} Y = \eta - \nabla\phi = 0$. This proves the existence of Y and Z.

Now, suppose two decompositions of X so that $X = Y_1 + Z_1 = Y_2 + Z_2$. As $\operatorname{curl} Z_1 = 0$, it follows that there are two functions ϕ_i, $i = 1, 2$ such that $Z_i = \nabla\phi_i$, $i = 1, 2$.

So, $Y_2 - Y_1 = \nabla(\phi_2 - \phi_1)$.

Denote $W = Y_2 - Y_1$ and $\phi = \phi_2 - \phi_1$ then $\operatorname{div} W = \operatorname{div}\nabla\phi$. Since $\operatorname{div} Y_2 - \operatorname{div} Y_1 = 0$ we get $\Delta\phi = 0$, thus, $\phi_2 - \phi_1$ must be constant. Taking the gradient yields $Z_2 - Z_1 = 0$. Then, we have also that $Y_1 = Y_2$, hence the decomposition is unique. □

For example, let $X = (x_1 - x_2)\partial_{x_1} + (x_1 + x_2)\partial_{x_2}$. Then the Helmholtz decomposition is

$$X = Y + Z \quad \text{with} \quad Y = x_1\partial_{x_1} + x_2\partial_{x_2} \quad \text{and} \quad Z = -x_2\partial_{x_1} + x_1\partial_{x_2}.$$

To visualise this, we consider the 2-dimensional unit sphere with a local parametrisation

$$\Phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^3 \quad \text{with} \quad \Phi(\theta, \phi) = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ \sin\theta\end{pmatrix}.$$

The differentials $\frac{\partial\Phi}{\partial\theta}$ and $\frac{\partial\Phi}{\partial\phi}$ give the Gramian $g = \left(g_{ij}\right)_{ij} = \begin{pmatrix}1 & 0\\ 0 & \cos^2\theta\end{pmatrix}$ which has the associated vector field flow $X_{S^2} = (1, \cos^2\theta)$ and stream plot as Figure 1. Observe that X_{S^2} is divergence-free and that $\operatorname{curl} X_{S^2} = -2\cos\theta\sin\theta$.

By the Helmholtz decomposition,

$$X_{S^2} = Y_{S^2} + Z_{S^2} \quad \text{with} \quad Y_{S^2} = X_{S^2} \quad \text{and} \quad Z_{S^2} = (-\cos^2\theta, 1) \quad \text{with} \quad \operatorname{curl} Z_{S^2} = 0.$$

The stream plot of the Helmholtz decomposed X_{S^2} flow is Figure 2.

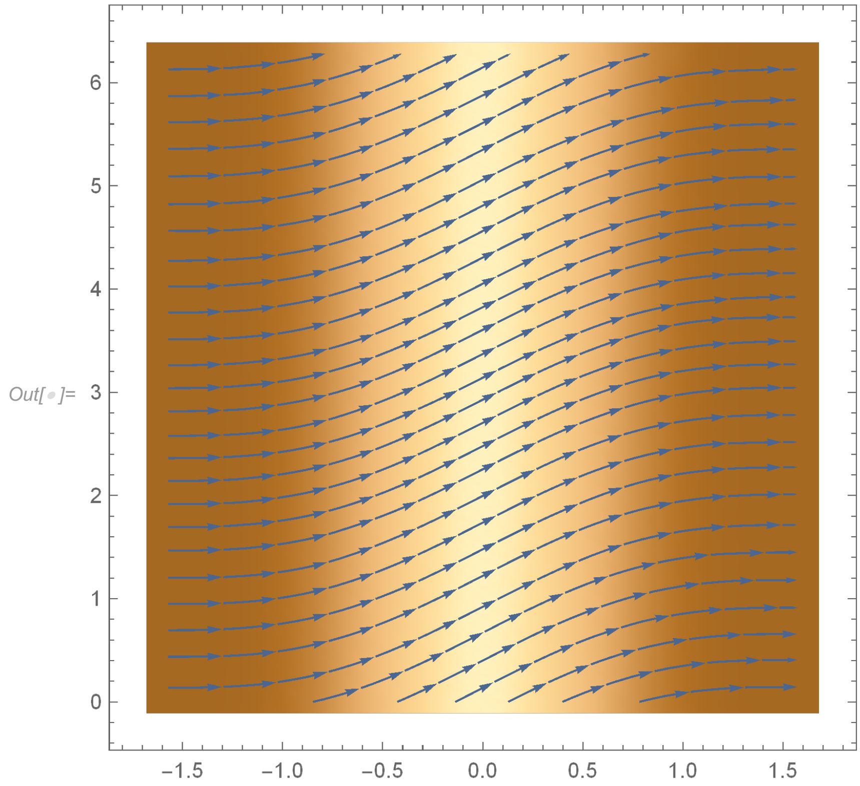

Figure 1. Stream plot of the vector field X_{S^2} on S^2

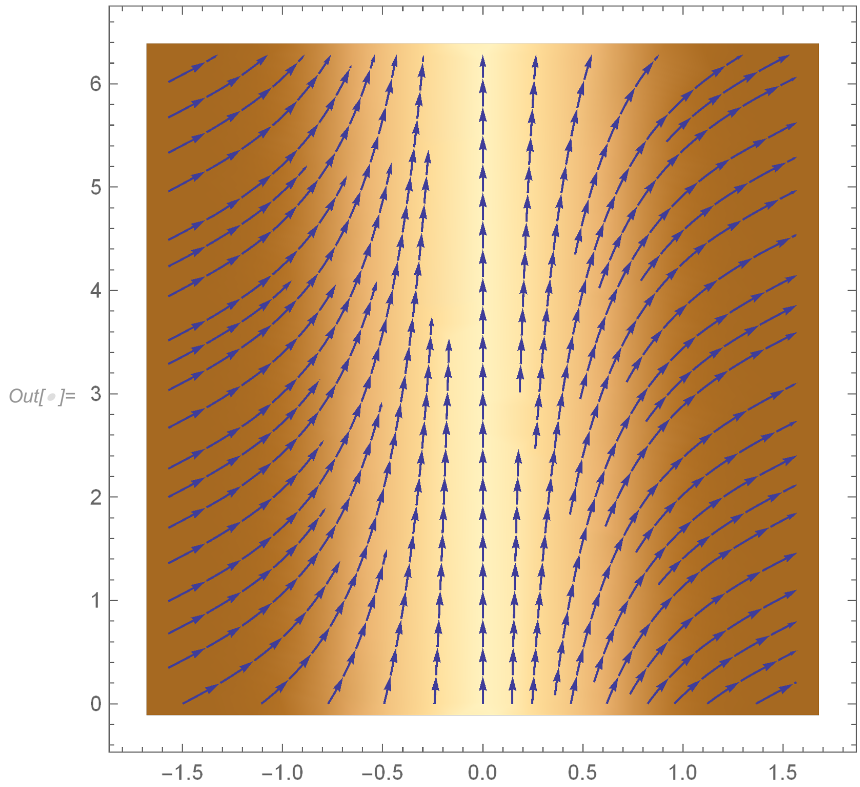

Figure 2. Stream plot of the decomposed X_{S^2}

Finally, another main result of this work is the Stokes' type result:

Theorem 3.10. Let X be smooth vector field on an open neighbourhood U of $Z \subseteq M$ with $X\big|_Z$ compactly supported and let dA be the area form on Z. Then the smooth function $\langle \mathrm{curl}(X)\big|_Z, \hat{N}\rangle$ on Z, where $\hat{N}$ is the outward unit normal field along Z in M, is compactly supported and satisfies the Stokes' type identity

$$\int_Z \langle \mathrm{curl}(X)\big|_Z, \hat{N}\rangle dA = \int_{\partial Z} \langle X\big|_{\partial Z}, T\rangle dl. \tag{3.2}$$

Proof. Let $\eta = \omega_X \in \Omega^1_M(U)$ be dual to X. Thus $\eta\big|_Z \in \Omega^1_Z(Z)$ is compactly supported as it vanishes at points where $X\big|_Z$ vanishes. By the definition of the curl, we have $\omega_{\mathrm{curl}(X)} = *(d\eta)$. Since $*\circ * = (-1)^{r(n-r)}$ on r-forms on an n- dimensional manifold, for $n = 3$ and $r = 1$, we have $*\circ * = 1$ on 1-forms on U, so $d\eta = *(\omega_{\mathrm{curl}(X)})$. Similarly,

$$\langle \mathrm{curl}(X)\big|_Z, \hat{N}\rangle dA = (d\eta)\big|_Z$$

inside of $\Omega^2_Z(Z)$. But $(d\eta)\big|_Z = d(\eta\big|_Z)$ since d and pullback commute along the closed embedding of Z into M. This is compactly supported on Z. For compactly supported 1-form $\eta\big|_Z$ on the 2-dimensional manifold Z with boundary, we get

$$\int_Z \langle \mathrm{curl}(X)\big|_Z, \hat{N}\rangle dA = \int_Z (d\eta)\big|_Z = \int_Z d(\eta\big|_Z) = \int_{\partial Z} \eta\big|_{\partial Z}$$

showing (3.2).

Thus our problem comes to proving the identity $\eta\big|_{\partial Z} = \langle X\big|_{\partial Z}, T\rangle dl \in \Omega^1_{\partial Z}(\partial Z)$. But since $\eta|_Z = \omega_{X|_Z} \in \Omega^1_Z(Z)$ it follows that $\omega_{X|_Z} = \omega_X Z$ pointwise.

Let $X|_Z = G$ and $G|_{\partial Z} = H$, we want to prove that for any smooth vector field H along ∂Z,

$$\langle H, T\rangle dl = \omega_H \in \Omega^1_{\partial Z}(\partial Z) \tag{3.3}$$

generally for any 1-dimensional Riemannian manifold C with length form dl dual to the tangent field T. Evaluating both sides of (3.3) at a point $p \in C$, we obtain a 1-dimensional real vector space V endowed with an inner product. So, let $V = T_pC$, if $v \in V$ with length for ϕ, and for $t \in V$ the vector dual to ϕ we have $\langle v, t\rangle \phi = \langle v, \cdot\rangle$ is in the dual space. It suffices to check this equality when evaluating both sides on the basis $\{t\}$. But since $\phi(t) = 1$ by the definition of t, the result follows. □

4. Conclusion

We have constructed vorticity of vector field flows on compact smooth Riemannian manifolds through differential operators mainly the curl and the divergence operators. Many properties of vorticity on the manifolds were established and we proved that a vector field on a compact Riemannian manifold admits a unique Helmholtz decomposition. We also proved a Stokes' type identity for the curl operator on smooth tensor fields on M.

This study can be extended to applications of the central ideas of the paper to physical flow problems in engineering and industry. One can as well study the vorticity of climate variabilities on the earth surface using the machinery developed in this paper. This will throw more light on the mathematical analysis of climate change.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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