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Research Article

Some Results of Conditionally Sequential Absorbing Maps in 2-Menger Space

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Abstract. This paper aims to present two results in probabilistic 2-metric space by using the concepts of conditionally sequential absorbing maps and reciprocally continuous mappings. These are generalizations of the theorem proved by Gupta *et al*. [5]. Further, these are justified by supporting examples.

Keywords. Self-mappings, Reciprocally continuous mappings, Conditionally sequential absorbing maps, Probabilistic 2-metric space

Mathematics Subject Classification (2020). 54H25

1. Introduction

The core area of present research in analysis is fixed point theory. These results will be applicable in science and engineering. One hotspot area in fixed point theory apart from metric and fuzzy space is Menger space, coined by Menger [6]. He interpreted the distance of two points by probabilistic notion through distribution function and introduced the theory of *probabilistic metric* (PM) space. Gahler [3] introduced the idea of 2-metric space as generalization of metric space. These concepts are developed in the formation of probabilistic 2-metric space introduced by Golet [4]. Further, these metric spaces improved through topological ideas by Schweizer and Sklar [1]. Mishra [7] introduced the compatible notion in Menger space, which channelizes to prove new results in probabilistic spaces. Singh and Jain [11] using the concept of weakly compatible mappings and obtained some results in menger space, which are attracted by many researchers. Recently, Gupta *et al*. [5] employed the weakly compatible property in 2-Menger space generated some results. Al-Thagafi and Shahzad [2] generalized the concept of weakly compatibility as occasionally weakly compatible mappings and produced some results in this area. Patel *et al*. [9] initiated the theme of conditionally sequential absorbing maps in metric space and obtained some results. Some more results can be witnessed by using the concepts like reciprocal continuous, sub sequentially continuous and semi compatible mappings in Menger space ([8], [10]).

2. Preliminaries

Definition 2.1. $\mathcal{F}:\mathcal{R}\to\mathcal{R}^+$ is distribution function [5] if it is

(i) non-decreasing

(ii) continuous from left

(iii) $\inf\{\mathcal{F}(\alpha):\alpha\in\mathcal{R}\}=0$

(iv) $\sup\{\mathcal{F}(\alpha):\alpha\in\mathcal{R}\}=1$.

The letter $\mathcal{L}$ is used to refer to a collection of all distribution functions.

Definition 2.2. A probabilistic 2-metric space (2-PM space) [5] ia a pair $(\Omega,\mathcal{F})$ with $\mathcal{F}:\Omega\times\Omega\times\Omega\to\mathcal{L}$ here $\mathcal{L}$ stands as the set of all distribution functions and the $\mathcal{F}$ value at $(e,f,g)\in\Omega\times\Omega\times\Omega$ is written as $\mathcal{F}_{e,f,g}$ and fulfill the following properties:

(a) $\mathcal{F}_{e,f,g}(0)=0$,

(b) $\exists\, g\in\Omega$ such that $\mathcal{F}_{e,f,g}(t_\varsigma)<1$, for all $e,f\in\Omega$, $e\neq f$, for some $t_\varsigma>0$,

(c) $\mathcal{F}_{e,f,g}(t_\varsigma)=1$, for all $t_\varsigma>0$ if $e=f=g$ or $e=f$ or $f=g$ or $e=g$,

(d) $\mathcal{F}_{e,f,g}(t_\varsigma)=\mathcal{F}_{f,g,e}(t_\varsigma)=\mathcal{F}_{g,f,e,}(t_\varsigma)$,

(e) $\mathcal{F}_{e,f,g}(t_x)=\mathcal{F}_{f,g,e}(t_y)=\mathcal{F}_{g,f,e,}(t_z)=1\implies\mathcal{F}_{e,f,g}(t_x+t_y+t_z)=1$.

Definition 2.3. The mapping $t_\varsigma:[0,1]^3\to[0,1]$ is a t- norm [5] it has the properties:

(i) $t_\varsigma(0,0,0)=0$,

(ii) $t_\varsigma(v,1,1)=v$,

(iii) $t_\varsigma(a_0,b_0,c_0)=t_\varsigma(b_0,c_0,a_0)=t_\varsigma(c_0,a_0,b_0)$,

(iv) $t_\varsigma(d,e,f)\geq t_\varsigma(d_1,e_1,f_1)$ for $d\geq d_1$, $e\geq e_1$, $f\geq f_1$,

(v) $t_\varsigma(t_\varsigma(a_0,b_0,c_0)),r,s)=t_\varsigma(a_0,t_\varsigma(b_0,c_0,r),s)=t_\varsigma(a_0,b_0,t_\varsigma(c_0,r,s))$.

Definition 2.4. A Menger probabilistic 2-metric space [5] is a triplet $(\Omega,\mathrm{F},t_\varsigma)$, where $(\Omega,\mathcal{F})$ is a 2-PM space and t_ς is a t-norm having triangle inequality

$$\mathrm{F}_{u,v,w}(t_x+t_y+t_z)\geq t(\mathrm{F}_{u,v,p}(t_x),\mathrm{F}_{u,p,w}(t_y),\mathrm{F}_{p,v,w}(t_z)),\ \text{for all } w,p,v,u\in\Omega \text{ and } t_x,t_y,t_z\geq 0.$$

Definition 2.5. A sequence (p_n) in 2-Menger space $(\Omega,\mathcal{F},t_\varsigma)$ [5]

(i) *converges* to β if for each $\epsilon>0$, $t_\varsigma>0$, $\exists\ N(\epsilon)\in N \implies \mathcal{F}_{p_n,\beta,a}(\varsigma)>1-t_\varsigma$, for all $a\in\Omega$ and $n\geq N(\epsilon)$;

(ii) *Cauchy* if for each $\epsilon>0$, $t_\varsigma>0$, $\exists\ N(\epsilon)\in N \implies \mathcal{F}_{p_n,p_m,a}(\epsilon)>1-t_\varsigma$, for all $a\in\Omega$ and $n,m\geq N(\epsilon)$;

(iii) if each Cauchy sequence converges in Ω then it is mentioned as *complete* 2-Menger space.

Definition 2.6. Self-mappings P, S in 2-Menger space $(\Omega,\mathcal{F},t_\varsigma)$ are known as

(a) *Compatible* [5] if $\mathcal{F}_{PSx_n,SPx_n,a}(\beta)\to 1$, for all $a\in\Omega$ and $\beta>0$ whenever a sequence $(x_n)\in\Omega$ such that $Px_n,Sx_n\to\theta$ where θ is some element of Ω as $n\to\infty$.

(b) *Weakly compatible* [5], [11] if commute at their coincidence points.

(c) *Occasionally Weakly Compatible* (OWC) [2] if there is a coincidence point at which maps are commuting.

Example 2.1. Define $\forall\ t_\varsigma\in[0,1]$

$$\mathcal{F}_{\upsilon,\beta,\gamma}(t_\varsigma)=\begin{cases}\frac{t_\varsigma}{t_\varsigma+d(\upsilon,\beta)} & \text{if } t_\varsigma>0\\ 0, & \text{if } t_\varsigma=0\end{cases} \tag{2.1}$$

$\forall\ \upsilon$, β and fixed $\gamma=0$, $t_\varsigma>0$.

By considering $\Omega=(-\infty,\infty)$ and d is usual distance on Ω then by (2.1) $(\Omega,\mathcal{F},t_\varsigma)$ forms 2-Menger space.

The mappings $P,S:\Omega\to\Omega$ are defined as

$$P(a)=a+3 \quad \text{for all } a\in\Omega \tag{2.2}$$

$$S(a)=\frac{(a+3)^2}{2} \quad \text{for all } a\in\Omega. \tag{2.3}$$

From (2.2) and (2.3) $a=-3,-2$ are coincidence points for the mappings P, S.

At $a=-3$

$$P(-3)=S(-3)=0, \tag{2.4}$$

$$PS(-3)=P(0)=3, \tag{2.5}$$

$$SP(-3)=S(0)=9. \tag{2.6}$$

From (2.4), (2.5) and (2.6).

Resulting that the maps P, S are *owc* mappings however not weakly compatible.

Definition 2.7. Self-mappings P, S in 2-Menger space $(\Omega,\mathcal{F},t_\varsigma)$ are termed as Conditionally sequentially absorbing [9] if whenever the sequence (c_m) satisfying

$$\left\{(c_m):\lim_{m\to\infty}Pc_m=\lim_{m\to\infty}Sc_m\right\}\neq\phi$$

then there exists another sequence (e_m) in Ω with

$$\lim_{m\to\infty}Pe_m=\lim_{m\to\infty}Se_m=\eta,\ \text{ for some } \eta\in X$$

such that

$$\lim_{m\to\infty} \mathcal{F}_{Pe_n,PSe_n,a}(\beta)=1$$

and

$$\lim_{m\to\infty} \mathcal{F}_{Se_n,SPe_n,a}(\beta)=1, \quad \text{for all } a\in\Omega \text{ and } \beta>0.$$

Example 2.2. Let $(\Omega,\mathcal{F},t_\varsigma)$ be a 2-Menger space where $\mathcal{F},t_\varsigma$ be as in (2.1), choose $\Omega=[-3,3]$. The mappings $P,S:\Omega\to\Omega$ are defined as

$$P(a)=\begin{cases}-3 & \text{if } a\in[-3,0)\\ (7)^{-a^2} & \text{if } a\in[0,3]\end{cases} \tag{2.7}$$

$$S(a)=\begin{cases}\frac{a^3}{9} & \text{if } a\in[-3,0)\\ (7)^{-3a} & \text{if } a\in[0,3].\end{cases} \tag{2.8}$$

From (2.7) and (2.8), the mappings P, S have coincidence points $-3,0$ and 3.
At $a=3$

$$P(3)=S(3)=7^{-9}, \tag{2.9}$$

$$PS(3)=p(7^{-9})=7^{-(7^{-9})^2}, \tag{2.10}$$

$$SP(3)=S(7^{-9})=7^{-3(7^{-9})}. \tag{2.11}$$

From (2.9), (2.10) and (2.11) the mappings are not weakly compatible.
Let $(a_m)=\frac{\sqrt{7}}{m}$, for all $m\ge 1$. Then from (2.7)

$$\lim_{m\to\infty} Pa_m=\lim_{m\to\infty} P\left(\frac{\sqrt{7}}{m}\right)=\lim_{m\to\infty} 7^{-\left(\frac{\sqrt{7}}{m}\right)^2}=1 \tag{2.12}$$

and

$$\lim_{m\to\infty} Sa_m=\lim_{m\to\infty} S\left(\frac{\sqrt{7}}{m}\right)=\lim_{m\to\infty} 7^{-3\left(\frac{\sqrt{7}}{m}\right)}=1. \tag{2.13}$$

From (2.12), (2.13), we get

$$\lim_{m\to\infty} Pa_m=\lim_{m\to\infty} Sa_m. \tag{2.14}$$

From (2.14) implies

$$\left\{(a_m):\lim_{m\to\infty} Pa_m=\lim_{m\to\infty} Sa_m\right\}\ne\phi.$$

Then there exists another sequence $(c_m)=-3+\frac{4}{m}$, for all $m\ge1$ and from (2.7)

$$\lim_{m\to\infty} Pc_m=\lim_{m\to\infty} P\left(-3+\frac{4}{m}\right)=\lim_{m\to\infty} -3=-3 \tag{2.15}$$

and from (2.8)

$$\lim_{m\to\infty} Sc_m=\lim_{m\to\infty} S\left(-3+\frac{4}{m}\right)=\lim_{m\to\infty} \frac{\left(-3+\frac{4}{m}\right)^3}{9}=-3. \tag{2.16}$$

From (2.15), (2.16)

$$\lim_{m\to\infty} Pc_m=\lim_{m\to\infty} Sc_m=-3. \tag{2.17}$$

Further from (2.7), (2.16)

$$\lim_{m\to\infty} PSc_m = \lim_{m\to\infty} P\left(\frac{\left(-3+\frac{4}{m}\right)^3}{9}\right) = \lim_{m\to\infty} -3 = -3, \tag{2.18}$$

from (2.8), (2.15)

$$\lim_{m\to\infty} SPc_m = \lim_{m\to\infty} S(-3) = \lim_{m\to\infty} -3 = -3. \tag{2.19}$$

Thus from (2.15), (2.18), (2.16) and (2.19)

$$\lim_{m\to\infty} \mathcal{F}_{Pc_n,PSc_n,a}(\beta) = 1 \text{ and } \lim_{m\to\infty} \mathcal{F}_{Sc_n,SPc_n,a}(\beta) = 1. \tag{2.20}$$

We can conclude that from (2.10), (2.11) and (2.20) the pair (P,S) is conditionally sequentially absorbing but not weakly compatible.

Definition 2.8. Self-mappings P, S in 2-Menger space $(\Omega,\mathcal{F},t_\varsigma)$ are said to be Reciprocally continuous [8], if whenever the sequence $\{c_m\}$ such that

$$\lim_{m\to\infty} Pc_m = \lim_{m\to\infty} Sc_m = \eta, \quad \text{for some } \eta \in \Omega$$

implies

$$\lim_{m\to\infty} \mathcal{F}_{P\eta,PSc_m,a}(\beta) = 1$$

and

$$\lim_{m\to\infty} \mathcal{F}_{S\eta,SPc_m,a}(\beta) = 1 \quad \text{for all } a \in \Omega$$

and for some $\beta > 0$.

Example 2.3. Let $(\Omega,\mathcal{F},t_\varsigma)$ be a 2-Menger space where $\mathcal{F},t_\varsigma$ be as in (2.1) and $\Omega = [0,\infty)$. The mappings $P,S:\Omega\to\Omega$ are defined as

$$P(a) = \begin{cases} \sin(a) & \text{if } a \in [0,\frac{\pi}{2}) \\ a^2 & \text{if } a \in \left[\frac{\pi}{2},\infty\right) \end{cases} \tag{2.21}$$

$$S(a) = \begin{cases} \tan(a) & \text{if } a \in \left[0,\frac{\pi}{2}\right) \\ \pi a & \text{if } a \in \left[\frac{\pi}{2},\infty\right). \end{cases} \tag{2.22}$$

From (2.21), (2.22) the mappings P, S have coincidence points 0, and π.
At $a = \pi$

$$P(\pi) = S(\pi) = (\pi)^2, \tag{2.23}$$

$$PS(\pi) = p((\pi)^2) = (\pi)^4, \tag{2.24}$$

$$SP(\pi) = S((\pi)^2) = (\pi)^3. \tag{2.25}$$

From (2.23), (2.24) and (2.25) the mappings are not weakly compatible.
Let $(a_m) = \pi - \frac{2}{m^2}$, for all $m \ge 1$. Then from (2.1)

$$\lim_{m\to\infty} Pa_m = \lim_{m\to\infty} P\left(\pi - \frac{2}{m^2}\right) = \lim_{m\to\infty}\left(\pi - \frac{2}{m^2}\right)^2 = (\pi)^2, \tag{2.26}$$

from (2.22)

$$\lim_{m\to\infty} Sa_m = \lim_{m\to\infty} S\left(\pi - \frac{2}{m^2}\right) = \lim_{m\to\infty} \pi\left(\pi - \frac{2}{m^2}\right) = (\pi)^2. \tag{2.27}$$

Again from (2.21), (2.27) we get

$$\lim_{m\to\infty} PSa_m = \lim_{m\to\infty} P\left((\pi)^2 - \pi\left(\frac{2}{m^2}\right)\right) = \lim_{m\to\infty}\left((\pi)^2 - \pi\left(\frac{2}{m^2}\right)\right)^2 = (\pi)^4. \tag{2.28}$$

From (2.22), (2.26)

$$\lim_{m\to\infty} SPa_m = \lim_{m\to\infty} S\left(\left(\pi - \frac{2}{m^2}\right)^2\right) = \lim_{m\to\infty} \pi\left(\left(\pi - \frac{2}{m^2}\right)^2\right) = (\pi)^3. \tag{2.29}$$

From (2.21), (2.22), (2.28), (2.29)

$$\lim_{m\to\infty} \mathcal{F}_{P(\pi)^2, PSc_m, a}(\beta) = 1 \text{ and } \lim_{m\to\infty} \mathcal{F}_{S(\pi)^2, SPc_m, a}(\beta) = 1 \tag{2.30}$$

for all $a \in \Omega$ and for some $\beta > 0$ and also if $(a_m) \to 0$ as $m \to \infty$ implies

$$\lim_{m\to\infty} \mathcal{F}_{P(0), PSc_m, a}(\beta) = 1 \text{and } \lim_{m\to\infty} \mathcal{F}_{S(0), SPc_m, a}(\beta) = 1 \tag{2.31}$$

for all $a \in \Omega$ and for some $\beta > 0$.

Hence from (2.30) and (2.31) the mappings P, S are reciprocally continuous but not weakly compatible.

The following theorem was proved by Gupta *et al*. [5].

Theorem 2.1. *Let A, B, S and T be self-mappings on a complete probabilistic 2-metric space $(\mathtt{X}, \mathcal{F}, t_\varsigma)$ satisfying*

(i) $A(\mathtt{X}) \subseteq T(\mathtt{X}), B(\mathtt{X}) \subseteq S(\mathtt{X})$,

(ii) *one of $A(\mathtt{X}), B(\mathtt{X}), T(\mathtt{X})$ or $S(\mathtt{X})$ is complete,*

(iii) *pairs (A, S) and (B, T) are weakly compatible,*

(iv) $\mathcal{F}_{Ax,By,\gamma}(t_\epsilon) \geq r\mathcal{F}_{Sx,Ty,\gamma}(t_\varsigma)$ *for all x, y in $\mathtt{X}$ and $t_\varsigma > 0$,*

where $r : [0,1] \to [0,1]$ is some continuous function such that $r(t_\epsilon) > t_\epsilon$ for each $o < t_\varsigma < 1$.

Then the mappings A, B, S and T have unique common fixed point in $\mathtt{X}$.

Now we give generalization of Theorem 2.1 as under.

3. Main Results

Theorem 3.1. *Let A, B, S and T be mappings on a complete probabilistic 2-metric space $(\Omega, \mathcal{F}, t_\varsigma)$ to itself satisfying*

$$A(\Omega) \subseteq T(\Omega), \quad B(\Omega) \subseteq S(\Omega) \tag{3.1}$$

the pair of mappings (A, S) reciprocally continuous and conditionally sequentially absorbing and (B, T) is occasionally weakly compatible

$$\mathcal{F}_{Ax,By,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{Sx,Ty,\gamma}(t_\varsigma)) \tag{3.2}$$

whenever $x,y \in \Omega$ *and* $t_\varsigma > 0$ *for some continuous self-map on* $[0,1]$ *such that* $r(t_\varsigma) > t_\varsigma$ *for each* $o < t_\varsigma < 1$.
Then A, B, S and T have unique common fixed point in Ω.

Proof. By using (3.1) the sequence (y_n) derived as

$$\langle y_{2n}\rangle = Ax_{2n} = Tx_{2n+1}, \tag{3.3}$$

$$\langle y_{2n+1}\rangle = Bx_{2n+1} = Sx_{2n+2}. \tag{3.4}$$

Now, our claim is to show y_n is a Cauchy sequence.
By taking the values $x = x_{2n}$, $y = x_{2n+1}$ in (3.2), we get

$$\mathcal{F}_{Ax_{2n},Bx_{2n+1},\gamma}(t_\varsigma) \geq r(\mathcal{F}_{Sx_{2n},Tx_{2n+1},\gamma}(t_\varsigma))$$

$$\mathcal{F}_{y_{2n},y_{2n+1},\gamma}(t_\varsigma) \geq r(\mathcal{F}_{y_{2n-1},y_{2n},\gamma}(t_\varsigma)) > \mathcal{F}_{y_{2n-1},y_{2n},\gamma}(t_\varsigma).$$

In general, we have

$$\mathcal{F}_{y_{n+1},y_n,\gamma}(t_\varsigma) > \mathcal{F}_{y_n,y_{n-1},\gamma}(t_\varsigma)$$

for all $n \geq 1$.
Then we have $\{\mathcal{F}_{y_{n+1},y_n,\gamma}(t_\varsigma)$, for all $n \geq 1\}$ is an increasing sequence of positive real numbers bounded above by 1 therefore it must be converge to a limit say $L \leq 1$.
If $L < 1$ then $\mathcal{F}_{y_{n+1},y_n,\gamma}(t_\varsigma) = L > r(1) > 1$ which is a conflict. Hence $L = 1$.
Therefore, for all n and p, $\mathcal{F}_{y_{n+p},y_n,\gamma}(t_\varsigma) = 1$.
Thus (y_n) being Cauchy sequence in complete space Ω so it has limit $z \in \mathbb{X}$ resulting each sub sequence has the same limit z.
That is from (3.3) and (3.4)

$$Ax_{2n}, Sx_{2n} \to z, \tag{3.5}$$

$$Tx_{2n+1}, Bx_{2n+1} \to z$$

as $n \to \infty$.
Use the notion

$$L\{A,S\} = \{(x_n) : \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n\}.$$

Since the pair (A,S) is conditionally sequential absorbing from (3.5)

$$L\{A,S\} \neq \phi \implies \exists\, \langle y_n\rangle$$

such that

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \theta \text{ (say)} \tag{3.6}$$

$$\implies \lim_{n\to\infty} \mathcal{F}_{Ay_n,ASy_n,\gamma}(t_\varsigma) = 1 \text{ and } \mathcal{F}_{Sy_n,SAy_n,\gamma}(t_\varsigma) = 1 \tag{3.7}$$

for all $\gamma \in \mathbb{X}$, $t_\varsigma > 0$.
Also, the pair (A,S) is reciprocally continuous implies whenever

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \theta \;\text{ (say)} \tag{3.8}$$

$$\implies \lim_{n\to\infty} \mathcal{F}_{A\theta,ASy_n,\gamma}(t_\varsigma) = 1 \text{ and } \lim_{n\to\infty} \mathcal{F}_{S\theta,SAy_n,\gamma}(t_\varsigma) = 1. \tag{3.9}$$

Using (3.6) and (3.9) in (3.7), we get

$$A\theta = S\theta = \theta.$$

But $A\theta$ is element in $A(\Omega)$ by (3.1) there exists η such that

$$\theta = S\theta = A\theta = T\eta. \tag{3.10}$$

Claim $B\eta = T\eta$.
By putting $x = \theta, y = \eta$ in (3.2)

$$\mathcal{F}_{A\theta,B\eta,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{S\theta,T\eta,\gamma}(t_\varsigma)).$$

From (3.10)

$$\mathcal{F}_{A\theta,B\eta,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{S\theta,S\theta,\gamma}(t_\varsigma)) = r(1) = 1. \tag{3.11}$$

From (3.11) $\implies A\theta = B\eta$.
This gives

$$\theta = S\theta = A\theta = T\eta = B\eta. \tag{3.12}$$

The pair (B,T) is occasionally weakly compatible gives $BT\eta = TB\eta \implies B\theta = T\theta$ from (3.12).
Claim $\theta = B\theta$.
By taking $x = y = \theta$ in (3.2)

$$\mathcal{F}_{A\theta,B\theta,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{S\theta,T\theta,\gamma}(t_\varsigma))$$

using (3.12) and $B\theta = T\theta$

$$\mathcal{F}_{\theta,B\theta,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{\theta,B\theta,\gamma}(t_\varsigma)) > \mathcal{F}_{\theta,B\theta,\gamma}(t_\varsigma)$$

$$\mathcal{F}_{\theta,B\theta,\gamma}(t_\varsigma) > \mathcal{F}_{\theta,B\theta,\gamma}(t_\varsigma)$$

which is absurd. Hence $\theta = B\theta$.
Resulting

$$\theta = B\theta = T\theta = A\theta = S\theta. \tag{3.13}$$

Therefore, θ is the required common fixed point.
Uniqueness can be easily obtained from the contraction condition. □

Now, we provide a supporting illustration to justify Theorem 3.1.

Example 3.1. Let $(\Omega, \mathcal{F}, t_\varsigma)$ be a 2-Menger space where $\mathcal{F}, t_\varsigma$ be as in (2.1) and $\Omega = [-1,2]$. The mappings $A,S,B,T : \Omega \to \Omega$ are defined as

$$A(a) = B(a) = \begin{cases} -1 & \text{if } a \in [-1,0) \\ e^{-a^2} & \text{if } a \in [0,2] \end{cases} \tag{3.14}$$

$$S(a) = T(a) = \begin{cases} a^3 & \text{if } a \in [-1,0) \\ e^{-2a} & \text{if } a \in [0,2]. \end{cases} \tag{3.15}$$

From (3.14) and (3.15), we have

$$A(\Omega)=\{-1\}\cup[e^{-4},1],\quad S(\Omega)=[-1,0)\cup[e^{-4},1].$$

This gives

$$A(\Omega)\subset T(\Omega),\quad B(\Omega)\subset T(\Omega).$$

From (3.14) and (3.15) $a=-1,0$ are coincidence points for mappings A, S.
At $a=0$, $S(0)=A(0)=1$ and

$$AS(0)=A(1)=e^{-1}, \tag{3.16}$$

$$SA(0)=S(1)=e^{-2}. \tag{3.17}$$

From (3.16) and (3.17)

$$AS(0)\neq SA(0). \tag{3.18}$$

Hence from (3.18) the mappings are not weakly compatible.
For a sequence $(a_m)=\frac{4}{3m}$, for all $m\geq 1$. Then (3.14)

$$\lim_{m\to\infty} Aa_m=\lim_{m\to\infty} A\left(\frac{4}{3m}\right)=\lim_{m\to\infty} e^{-(\frac{4}{3m})^2}=1, \tag{3.19}$$

from (3.15)

$$\lim_{m\to\infty} Sa_m=\lim_{m\to\infty} S\left(\frac{4}{3m}\right)=\lim_{m\to\infty} e^{-2(\frac{4}{3m})}=1 \tag{3.20}$$

and from (3.14), (3.20), we get

$$\lim_{m\to\infty} ASa_m=\lim_{m\to\infty} A(e^{\frac{-8}{3m}})=\lim_{m\to\infty} e^{-(e^{\frac{-8}{3m}})^2}=e^{-1}, \tag{3.21}$$

from (3.15) and (3.19)

$$\lim_{m\to\infty} SAa_m=\lim_{m\to\infty} S(e^{-(\frac{4}{3m})^2})=\lim_{m\to\infty} e^{-2(e^{-(\frac{4}{3m})^2})}=e^{-2}. \tag{3.22}$$

From (3.21) and (3.22) the pair (A,S) is non-compatible so that there exists another sequence $(c_m)=-1+\frac{3}{4m}$, for all $m\geq 1$. Then from (3.14)

$$\lim_{m\to\infty} Ac_m=\lim_{m\to\infty} A\left(-1+\frac{3}{4m}\right)=\lim_{m\to\infty}(-1)=-1, \tag{3.23}$$

and from (3.15)

$$\lim_{m\to\infty} Sc_m=\lim_{m\to\infty} S\left(-1+\frac{3}{4m}\right)=\lim_{m\to\infty}\left(-1+\frac{3}{4m}\right)^3=-1. \tag{3.24}$$

Now from (3.14), (3.24)

$$\lim_{m\to\infty} ASc_m=\lim_{m\to\infty} A\left(\left(-1+\frac{3}{4m}\right)^3\right)=\lim_{m\to\infty}(-1)=-1, \tag{3.25}$$

from (3.15), (3.23)

$$\lim_{m\to\infty} SAc_m=\lim_{m\to\infty} S(-1)=-1. \tag{3.26}$$

Thus from (3.23), (3.25), (3.24) and (3.26)

$$\lim_{m\to\infty} \mathcal{F}_{Ac_m,ASc_m,a}(\beta)=1 \ \text{ and } \ \lim_{m\to\infty} \mathcal{F}_{Sc_m,SAc_m,a}(\beta)=1. \tag{3.27}$$

Further from (3.14), (3.15), (3.25) and (3.26)

$$\lim_{m\to\infty} \mathrm{F}_{ASc_m,A(-1),a}(\beta)=1 \text{ and } \lim_{m\to\infty} \mathrm{F}_{SAc_m,S(-1),a}(\beta)=1. \tag{3.28}$$

From (3.21), (3.22) (3.27) and (3.28) we can conclude that the pairs of (A,R), (C,T) are non-compatible reciprocally continuous and conditionally sequential absorbing, having unique fixed point $a=-1$. Further, the pairs of (A,S), (B,T) are not weakly compatible and hence satisfied all the conditions of Theorem 3.1.

Now, we give another generalization of Theorem 2.1 as under.

Theorem 3.2. *Let A, B, S and T be mappings on a 2-menger space $(\Omega,\mathrm{F},t_\varsigma$ to itself satisfying the pairs of mappings (A,S) and (B,T) non-compatible reciprocally continuous and conditionally sequential absorbing*

$$\mathcal{F}_{Ax,By,\gamma}(t_\varsigma)\ge r(\mathcal{F}_{Sx,Ty,\gamma}(t_\varsigma))$$

whenever $x,y\in\Omega$ and $t_\varsigma>0$ for some continuous self-map on $[0,1]$ such that $r(t_\varsigma)>t_\varsigma$ for each $o<t_\varsigma<1$.
Then A, B, S and T have unique common fixed point in Ω. Moreover, all these mappings are discontinuous at their fixed point.

Proof. Since the pair (A,S) is non-compatible implies some sequence (x_n) with

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=\theta \text{ (say)} \tag{3.29}$$

for some $\theta\in\Omega$

$$\implies \lim_{n\to\infty}\mathcal{F}_{ASy_n,ASy_n,\gamma}(\beta) \text{ not exist or } \lim_{n\to\infty}\mathcal{F}_{Sy_n,SAy_n,\gamma}(\beta)\neq 1.$$

Since the pair (A,S) is conditionally sequential absorbing from (3.29)

$$L\{A,S\}\neq\phi \implies \exists\,\langle y_n\rangle$$

such that

$$\lim_{n\to\infty} Ay_n=\lim_{n\to\infty} Sy_n=\theta \text{ (say)}$$

$$\implies \lim_{n\to\infty}\mathcal{F}_{Ay_n,ASy_n,\gamma}(t_\epsilon)=1 \text{ and } \lim_{n\to\infty}\mathcal{F}_{Sy_n,SAy_n,\gamma}(t_\varsigma)=1 \tag{3.30}$$

for all $\gamma\in\Omega$, $t_\varsigma>0$.
Also, the pair (A,S) is reciprocally continuous implies whenever

$$\lim_{n\to\infty} Ay_n=\lim_{n\to\infty} Sy_n=\theta \text{ (say)} \tag{3.31}$$

$$\implies \lim_{n\to\infty}\mathcal{F}_{A\theta,ASy_n,\gamma}(t_\varsigma)=1 \text{ and } \lim_{n\to\infty}\mathcal{F}_{S\theta,SAy_n,\gamma}(t_\varsigma)=1. \tag{3.32}$$

Using from (3.31), (3.32) in (3.30), we get

$$A\theta=S\theta=\theta. \tag{3.33}$$

Since the pair (B,T) ia non-compatible implies some sequence (x_n) with

$$\lim_{n\to\infty} Bx_n=\lim_{n\to\infty} Tx_n=\eta(say) \tag{3.34}$$

for some $\eta \in X$

$$\implies \lim_{n\to\infty} \mathcal{F}_{BTx_n,TBx_n,\gamma}(t_\varsigma) \text{ not exist or } \lim_{n\to\infty} \mathcal{F}_{BTx_n,TBx_n,\gamma}(t_\varsigma) \neq 1.$$

Since the pair (B,T) is conditionally sequential absorbing from (3.34)

$$L\{B,T\} \neq \phi \implies \exists \langle y_n \rangle$$

such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w \text{ (say)}$$

$$\implies \lim_{n\to\infty} \mathcal{F}_{By_n,BTy_n,\gamma}(t_\varsigma) = 1 \text{ and } \lim_{n\to\infty} \mathcal{F}_{Ty_n,TBy_n,\gamma}(t_\varsigma) = 1 \tag{3.35}$$

for all $\gamma \in \Omega$, $\beta > 0$.

Also, the pair (B,T) is reciprocally continuous implies whenever

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} STy_n = w \text{ (say)} \tag{3.36}$$

$$\implies \lim_{n\to\infty} \mathcal{F}_{Bw,BTy_n,\gamma}(t_\varsigma) = 1 \text{ and } \lim_{n\to\infty} \mathcal{F}_{Tw,TBy_n,\gamma}(t_\varsigma) = 1. \tag{3.37}$$

Using (3.36) and (3.37) in (3.35), we get

$$Bw = Tw = w. \tag{3.38}$$

Claim $w = \theta$.

On contrary if $w \neq \theta$.

Put $x = \theta$ and $y = w$ in (3.2)

$$\mathcal{F}_{A\theta,Bw,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{S\theta,Tw,\gamma}(t_\varsigma))$$

$$\implies \mathcal{F}_{\theta,w,\gamma}(t_\varsigma) \geq r(\mathcal{F}_{\theta,w,\gamma}(t_\varsigma)) > \mathcal{F}_{\theta,w,\gamma}(t_\varsigma)$$

from (3.33), (3.38)

$$\implies \mathcal{F}_{\theta,w,\gamma}(t_\varsigma) > \mathcal{F}_{\theta,w,\gamma}(t_\varsigma)$$

which is contradiction hence $\theta = w$.

Uniqueness follows easily comes from contraction condition.

Suppose A is continuous at w from (3.31) then

$$\lim_{n\to\infty} Sy_n = \theta \implies \lim_{n\to\infty} ASy_n = A\theta \text{ (say)}.$$

From (3.32)

$$\lim_{n\to\infty} SAy_n = S\theta$$

but $A\theta = S\theta = \theta$

$$\implies \lim_{n\to\infty} ASy_n = \lim_{n\to\infty} SAy_n. \tag{3.39}$$

Eq. (3.39) demonstrates that (A,S) is compatible pair, despite the fact that it is non-compatible.

Therefore, A should be discontinuous at w. Similarly, the other mappings are also discontinuous at w. □

Now, we justified our theorem with proper illustration.

Example 3.2. Let $(\Omega,\mathcal{F},t_\varsigma)$ be a 2-Menger space where $\mathcal{F},t_\varsigma$ be as in (2.1) and $\Omega = R$. The mappings $A,S,B,T:\Omega\to\Omega$ are defined as

$$A(a)=B(a)=\begin{cases} a^2 & \text{if } a\in(0,1] \\ 2 & \text{if } a\in(1,8) \end{cases} \tag{3.40}$$

$$S(a)=T(a)=\begin{cases} 1 & \text{if } a\in(0,1] \\ \log a & \text{if } a\in(1,8). \end{cases} \tag{3.41}$$

From (3.40) and (3.41) $a=e^2$ and 1 are coincidence points for the mappings A, S.
At $a=e^2$, $S(e^2)=A(e^2)=2$ and

$$AS(e^2)=A(2)=4, \tag{3.42}$$

$$SA(e^2)=S(2)=\log 2. \tag{3.43}$$

From (3.42) and (3.43)

$$AS(2)\neq SA(2). \tag{3.44}$$

Hence from (3.44) the mappings A, S are not weakly compatible.
For a sequence $(p_m)=e^2+\frac{2}{3m}$, for all $m\geq 1$. Then (3.40)

$$\lim_{m\to\infty} Aa_m = \lim_{m\to\infty} A\left(e^2+\frac{2}{3m}\right) = \lim_{m\to\infty} 2 = 2, \tag{3.45}$$

from (3.41)

$$\lim_{m\to\infty} Sa_m = \lim_{m\to\infty} S\left(e^2+\frac{2}{3m}\right) = \lim_{m\to\infty} \log\left(e^2+\frac{2}{3m}\right) = 2. \tag{3.46}$$

From (3.40), (3.46)

$$\lim_{m\to\infty} ASa_m = \lim_{m\to\infty} A\left(\log\left(e^2+\frac{2}{3m}\right)\right) = \lim_{m\to\infty} 2 = 2, \tag{3.47}$$

from (3.41), (3.45)

$$\lim_{m\to\infty} SAa_m = \lim_{m\to\infty} S(2) = \lim_{m\to\infty} \log 2 = \log 2. \tag{3.48}$$

From (3.47), (3.48) we get

$$\lim_{m\to\infty} ASa_m \neq \lim_{m\to\infty} SAa_m. \tag{3.49}$$

From (3.49) the pair (A,R) is non-compatible.
Further from (3.45), (3.46)

$$\{(p_m): \lim_{m\to\infty} Ap_m = \lim_{m\to\infty} Sp_m\} \neq \phi.$$

There exists another sequence $(q_m)=1-\frac{4}{3m}$, for all $m\geq 1$. Then from (3.40)

$$\lim_{m\to\infty} Aq_m = \lim_{m\to\infty} A\left(1-\frac{4}{3m}\right) = \lim_{m\to\infty} \left(1-\frac{4}{3m}\right)^2 = 1, \tag{3.50}$$

and from (3.41)

$$\lim_{m\to\infty} Sq_m = \lim_{m\to\infty} S\left(1-\frac{4}{3m}\right) = \lim_{m\to\infty} 1 = 1. \tag{3.51}$$

From (3.40) from (3.51)

$$\lim_{m\to\infty} ASq_m = \lim_{m\to\infty} A(1) = \lim_{m\to\infty} (1) = 1, \tag{3.52}$$

from (3.41), (3.50)

$$\lim_{m\to\infty} SAq_m = \lim_{m\to\infty} S\left(1-\frac{4}{3m}\right)^2 = \lim_{m\to\infty} 1 = 1. \tag{3.53}$$

Thus from (3.50), (3.51),from (3.52) and (3.53)

$$\lim_{m\to\infty} \mathcal{F}_{ASq_m,Aq_m,a}(\beta) = 1 \text{and} \lim_{m\to\infty} \mathcal{F}_{SAq_m,Sq_m,a}(\beta) = 1. \tag{3.54}$$

Further from (3.40), (3.52) and from (3.41), (3.53)

$$\lim_{m\to\infty} \mathcal{F}_{ASq_m,A(1),a}(\beta) = 1 \text{ and } \lim_{m\to\infty} \mathcal{F}_{SAq_m,S(1),a}(\beta) = 1. \tag{3.55}$$

From (3.54), (3.55) and (3.49) the pairs of (A,R), (C,T) are non-compatible reciprocally continuous and conditionally sequentially absorbing, having unique fixed point $a = 1$. Further, maps A, R, C and T are discontinuity at $a = 1$. Moreover, the pairs of (A,R), (C,T) are not weakly compatible and hence satisfying all the conditions of Theorem 3.2.

4. Conclusion

In this result we improvised Theorem 2.1 in two ways by using

(i) reciprocally continuous and conditionally sequential absorbing in the place of weakly compatible mappings in first pair and use the owc in place of weakly compatible mappings in second pair assumed in Theorem 2.1.

(ii) non-compatible reciprocally continuous and conditionally sequential absorbing maps in the place of weakly compatible mappings of both pairs in Theorem 3.2.

Moreover these two results are justified with suitable examples, removing closed sub spaces in Theorem 2.1 and removing completeness, closed sub spaces discontinuous of mappings at fixed point in Theorem 2.1.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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