Communications in Mathematics and Applications
Vol. 13, No. 1, pp. 9–17, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i1.1702

http://www.rgnpublications.com

Research Article

On Some Bicomplex Hartley Transformations

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Received: October 18, 2021 **Accepted:** January 22, 2022

Abstract. In this study, we have defined Hartley transforms for functions of bicomplex variables by using some known integral transformations, and then we examined the properties of these transformations.

Keywords. Bicomplex function, Integral transformations, Hartley transform

Mathematics Subject Classification (2020). 30G35, 44A15, 42A38

1. Introduction

Bicomplex analysis is a field that has been frequently used recently to develop function theory in the frequency domain. Bicomplex numbers are more advantageous than quaternions in solving some problems such as electromagnetism problems. Hamilton defined quaternions with all the properties of both real numbers and complex numbers, except for the commutative property, as physical rotations in 4-dimensional space. During the study of irrational equations, Cockle [5] defined a new imaginary unit using the concept of quaternion, and the author defined the bicomplex Z number as follows:

$$Z = x_1 + ix_2 + jx_3 + ijx_4. \tag{1.1}$$

Here i and j are the complex units, and ij is the hyperbolic unit, and between these units, the relations $ij = ji$, $i^2 = j^2 = -1$ are valid. According to the addition and multiplication operations defined on $\mathbb{BC}$, which is the set of numbers with this property, it is a commanding ring and is a Banach algebra with the help of scalar product and norm functions. In this algebra, apart from the obvious idempotent elements, there are two different idempotent elements, denoted e_1

and e_2. Each element of the bicomplex algebra can be represented in a unique way with the help of these elements:

$$Z = c_1 e_1 + c_2 e_2; \;\; e_1 + e_2 = 1, \; e_1 e_2 = 0. \tag{1.2}$$

Thus, a holomorphic function with a bicomplex variable is represented by a complex variable (f_1, f_2) holomorphic function pair:

$$f(Z) = f_1(z_1 - iz_2)e_1 + f_2(z_1 + iz_2)e_2; \;\; z_1, z_2 \in \mathbb{C}. \tag{1.3}$$

The principal aim of this study is to examine integral transformations on bicomplex algebra using idempotent basis elements and to define Hartley transformations, which will be defined for the first time on this space. Detailed information on these transformations can be found in [8, 10] studies.

The integral transformation of a function $f(x)$ defined in the interval $[a, b]$ is given by the following equation:

$$I\{f(x)\} = F(k) = \int_a^b K(x,k) f(x) dx. \tag{1.4}$$

Different types of transformations are obtained with different selections of the kernel function $K(x,k)$. The most commonly used transforms are Fourier, Laplace and Mellin transforms. The theory of integral transformations of multivariable functions can be developed using properties of Banach spaces. The importance of these transformations is that they transforms a difficult mathematical problem to a relatively easy problem can be easily be solved.

The Fourier transform plays an important role in solving differential equations and integral equations, reducing the number of independent variables of partial differential equations. Especially in mathematical statistics, statistical mechanics problems, diffusion, free vibration problems, geophysical engineering; Fourier transforms are used in measuring the resistance and strength of fault lines, in two-dimensional wave equation or Cauchy problems, in solving unknown $f(x)$ functions in integral equations [2–4, 8, 10].

Banarjee *et al*. [1] studied Fourier transforms in bicomplex space. These authors gave some basic properties for the bicomplex version of Fourier transforms in their work. Kumar and Kumar [9], and Banarjee *et al*. [1] studied the Laplace integral transform in bicomplex space. References [1, 6, 9] can be consulted for different studies in bicomplex space. The authors in [1, 9] studied the bicomplex Laplace and Fourier integral transforms.

2. Bicomplex Hartley Transformation

The Hartley transform is closely related to the Fourier transform. The Hartley transform is an integral transform that maps a real-valued temporal or spatial function to a real-valued frequency function through the kernel $\text{cas}(vx) = \cos(vx) + \sin(vx)$. This transformation was defined and studied by Ralph Vinton Lyon Hartley in 1942 by considering the $\text{cas}(vx)$ function instead of the kernel function in the Fourier transform [8].

Unlike the Fourier transform, the Hartley transform is more advantageous because it works with real functions. This type of conversion is widely used in electrical engineering. *Discrete Fourier Transform* (DFT) plays an important role in the analysis, design, and implementation of discrete-time signal processing algorithms and systems, and in signal processing applications such as linear filtering, correlation analysis, and spectrum analysis. *Fast Fourier Transform* (FFT) is an algorithm that calculates the discrete Fourier transform of a sequence.

FFT algorithms are frequently used in engineering, mathematics and music. Fourier analysis was used in the discovery of double-stranded DNA structure [12]. Bracewell [4] also examined the effectiveness of the Hartley transform in the use of optics and microwaves. This author also studied the Discrete Hartley transform and its properties in 1983 [2], and the same author, in his 1984 study, examined the Fast Hartley transform with the help of the Discrete Hartley transform [3]. The Hartley transformation of the function $V(t):\mathbb{R}\to\mathbb{R}$, with t time, is given by the following equation, and is denoted by $H(f)$:

$$H(f)=\int_{-\infty}^{\infty}\operatorname{cas} wt V(t)dt=\int_{-\infty}^{\infty}\operatorname{cas} 2\pi f t V(t)dt. \tag{2.1}$$

Here, f is the frequency, w is the angular frequency, and $w=2\pi f$. The $V(t)$ function, that is, the inverse of the Hartley transform, can be found as follows [8]:

$$V(t)=\int_{-\infty}^{\infty}\operatorname{cas} wt H(f)dt=\int_{-\infty}^{\infty}\operatorname{cas} 2\pi f t H(f)dt. \tag{2.2}$$

As an example, the Hartley transformation of the following function $V(t)$ can be taken:

$$V(t)=\begin{cases} e^{-t}, & t>0 \\ 0, & t<0. \end{cases} \tag{2.3}$$

From the equation (2.1), we write

$$H(f)=\int_{0}^{\infty}\operatorname{cas} 2\pi f t e^{-t}=dt=\int_{0}^{\infty}\cos 2\pi f t e^{-t}dt+\int_{0}^{\infty}\sin 2\pi f t e^{-t}dt. \tag{2.4}$$

Since the above equation consists of Fourier cosine and Fourier sine integrals, the following result is easy to find.

$$H(f)=\frac{1}{v^2+1}+\frac{v}{v^2+1}=\frac{1+v}{v^2+1}. \tag{2.5}$$

We described Hartley transforms for bicomplex variables in our study in [7]. In this study, it is to study some important properties of these transformations, which were defined for the first time.

Definition 2.1. *Provided that* $|H_1(w_1)|<\infty$, $|H_2(w_2)|<\infty$ *the Hartley transform of the* $V(t)$ *function,* $H(w)$, *is obtained with the help of* $H_1(w_1)$ *and* $H_2(w_2)$ *transformations:*

$$H(w)=\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w_1 t V(t)dt e_1+\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w_2 t V(t)dt e_2,$$

$$H(w)=\int_{-\infty}^{\infty}(\cos 2\pi w_1 t+\sin 2\pi w_1 t)V(t)e_1 dt+\int_{-\infty}^{\infty}(\cos 2\pi w_2 t+\sin 2\pi w_2 t)V(t)e_2 dt,$$

$$H(w)=\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w t V(t)dt. \tag{2.6}$$

Here, w_1, w_2 *are the complex frequency, and* $H_1(w_1)$ *and* $H_2(w_2)$ *are holomorphic in regions* Ω_1 *and* Ω_2, *respectively:*

$$H_1(w_1)=\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w_1 t V(t)dt, \qquad H_2(w_2)=\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w_2 t V(t)dt,$$

$$\Omega_1=\{w_1\in\mathbb{C}:-\infty<\operatorname{Re}(w_1)<\infty\}, \qquad \Omega_2=\{w_2\in\mathbb{C}:-\infty<\operatorname{Re}(w_2)<\infty\}.$$

With the help of the properties of the $H_1(w_1)$ and $H_2(w_2)$ functions, the $H(w)$ function becomes holomorphic in the following region Ω:

$$\Omega = \{w \in \mathbb{BC} : w = w_1 e_1 + w_2 e_2, w_1 \in \Omega_1 \text{ and } w_2 \in \Omega_2\}. \tag{2.7}$$

In the following theorem, we give the existence theorem for the Hartley transform with bicomplex variables.

Theorem 2.1 ([7]). *Let $V(t)$ be a continuous, real-valued function in the range $-\infty < t < \infty$. Then there exists the Hartley transform $H(w)$ and it is defined in the region below:*

$$\Omega = \{w \in \mathbb{BC} : -\infty < a_0 < \infty, -\infty < a_3 < \infty\}. \tag{2.8}$$

Proof. For the proof, refer to reference [7]. □

In the following theorem, we give the uniqueness of the bicomplex Hartley transform.

Theorem 2.2. *Let the bicomplex Hartley transforms of functions $V_1(t)$ and $V_2(t)$ be $H(w)$ and $G(w)$, respectively. If $H(w) = G(w)$, then the following equality is true:*

$$V_1(t) = V_2(t). \tag{2.9}$$

Proof. The transformations $H(w)$ and $G(w)$ can be written as follows:

$$H(w) = H_1(w_1)e_1 + H_2(w_2)e_2, \quad G(w) = G_1(w_1)e_1 + G_2(w_2)e_2.$$

From this fact,

$$H(w) = G(w) \Leftrightarrow H_1(w_1) = G_1(w_1), H_2(w_2) = G_2(w_2)$$

is obtained. Thus, we write

$$\int_{-\infty}^{\infty} \operatorname{cas} 2\pi w_1 t V_1(t) dt = \int_{-\infty}^{\infty} \operatorname{cas} 2\pi w_1 t V_2(t) dt$$

and

$$\int_{-\infty}^{\infty} \operatorname{cas} 2\pi w_2 t V_1(t) dt = \int_{-\infty}^{\infty} \operatorname{cas} 2\pi w_2 t V_2(t) dt.$$

The above last equations can only be valid when $V_1(t) = V_2(t)$. Then, the theorem is proven. □

Theorem 2.3. *Let the functions $V_1(t), V_2(t)$ have transformations of $H_1(w), H_2(w)$, respectively. Then,*

$$H\{aV_1(t) + bV_2(t)\} = aH_1(w) + bH_2(w) \tag{2.10}$$

equality is true. Here, a and b in Ω.

Proof. If we use definitions for proof:

$$H\{aV_1 f(t) + bV_2 g(t)\} = \int_{-\infty}^{\infty} \{aV_1(t) + bV_2(t)\} \operatorname{cas} 2\pi w t\, dt,$$

$$H\{aV_1 f(t) + bV_2 g(t)\} = \int_{-\infty}^{\infty} aV_1(t) \operatorname{cas} 2\pi w t\, dt + \int_{-\infty}^{\infty} bV_2(t) \operatorname{cas} 2\pi w t\, dt,$$

$$H\{aV_1 f(t) + bV_2 g(t)\} = a\int_{-\infty}^{\infty} V_1(t) \operatorname{cas} 2\pi w t\, dt + b\int_{-\infty}^{\infty} V_2(t) \operatorname{cas} 2\pi w t\, dt.$$

Then, we get

$$H\{aV_1(t) + bV_2(t)\} = aH_1(w) + bH_2(w).$$

Thus, the proof is completed. □

Theorem 2.4. *Let the bicomplex Hartley transform of $V(t)$ be $H(w)$. Then, with $t-a\in\Omega$, the following equation is true:*

$$H\{V(t-a)\}=(\cos 2\pi wa)H(w)+(\sin 2\pi wa)H(-w). \tag{2.11}$$

Proof. If $t-a=u$, then the following equations can be written for the Hartley transform of $V(t-a)$, i.e.,

$$\int_{-\infty}^{\infty} V(t-a)\operatorname{cas}2\pi wt\,dt=\int_{-\infty}^{\infty} V(u)\operatorname{cas}2\pi w(u+a)dt,$$

$$\int_{-\infty}^{\infty} V(t-a)\operatorname{cas}2\pi wt\,dt=\int_{-\infty}^{\infty} V(u)(\cos 2\pi w(u+a)+\sin 2\pi w(u+a))dt.$$

If using the trigonometric transformation formulas for the second side of the last equation, then we get

$$\cos 2\pi wa\int_{-\infty}^{\infty} V(u)\operatorname{cas}2\pi wu\,dt\sin 2\pi wa\int_{-\infty}^{\infty} V(u)\operatorname{cas}(-2\pi wu)dt.$$

Then, we have

$$\int_{-\infty}^{\infty} V(t-a)\operatorname{cas}2\pi wt\,dt=\cos 2\pi waH(w)+\sin 2\pi waH(-w).$$

Thus, the proof is completed. □

Obtaining meaningful data with the operations performed on signals is called signal processing. Convolution operation is a signal processing technique. Some areas where convolution and its properties are used are electrical engineering, probability and image processing.

In the following theorem, the relationship between the Hartley transform and the convolution property is given.

Theorem 2.5. *Let the functions $V_1(t),V_2(t)$ have transformations of $H_1(w),H_2(w)$, respectively. Then, the bicomplex Hartley transformation of $V_1(t)*V_2(t)$ is as follows:*

$$H\{V_1(t)*V_2(t)\}=\frac{1}{2}(K_1e_1+K_2e_2), \tag{2.12}$$

where

$$K_1=H_2(w_1)[H_1(w_1)+H_1(-w_1)]+H_2(-w_1)[H_1(w_1)-H_1(-w_1)] \tag{2.13}$$

and

$$K_2=H_2(w_2)[H_2(w_2)+H_2(-w_2)]+H_2(-w_2)[H_2(w_2)-H_2(-w_2)]. \tag{2.14}$$

Proof. Let's see the proof using the definition of the bicomplex Hartley transform. We can write

$$H\{V_1(t)*V_2(t)\}=\int_{-\infty}^{\infty}\operatorname{cas}2\pi wt\left[\int_{-\infty}^{\infty}V_1(u)V_2(t-u)du\right]dt.$$

If we do the necessary editing and calculations, then the second side of the last equation above is as follows.

$$\int_{-\infty}^{\infty}\operatorname{cas}2\pi w_1t\left[\int_{-\infty}^{\infty}V_1(u)V_2(t-u)du\right]dte_1+\int_{-\infty}^{\infty}\operatorname{cas}2\pi w_2t\left[\int_{-\infty}^{\infty}V_1(u)V_2(t-u)du\right]dte_2$$

and

$$\int_{-\infty}^{\infty}V_1(u)\left[\int_{-\infty}^{\infty}V_2(t-u)\operatorname{cas}2\pi w_1tdt\right]due_1+\int_{-\infty}^{\infty}V_1(u)\left[\int_{-\infty}^{\infty}V_2(t-u)\operatorname{cas}2\pi w_2tdt\right]due_2.$$

If we use Theorem 2.4, then $H\{V_1(t)*V_2(t)\}$ is as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} V_1(u)[\cos 2\pi w_1 u H_2(w_1)+\sin 2\pi w_1 u H_2(-w_1)]du e_1\\
&+\int_{-\infty}^{\infty} V_1(u)[\cos 2\pi w_2 u H_2(w_2)+\sin 2\pi w_2 u H_2(-w_2)]du e_2\\
&\quad = H_2(w_1)\frac{H_1(w_1)+H_1(-w_1)}{2}e_1+H_2(-w_1)\frac{H_1(w_1)-H_1(-w_1)}{2}e_1\\
&\qquad +H_2(w_2)\frac{H_2(w_2)+H_2(-w_2)}{2}e_2+H_2(-w_2)\frac{H_2(w_2)-H_2(-w_2)}{2}e_2\\
&\quad =\frac{1}{2}\{[H_2(w_1)(H_1(w_1)+H_1(-w_1))+H_2(-w_1)(H_1(w_1)-H_1(-w_1))]e_1\\
&\qquad\quad +[H_2(w_2)(H_2(w_2)+H_2(-w_2))+H_2(-w_2)(H_2(w_2)-H_2(-w_2))]e_2\}.
\end{aligned}$$

When the necessary operations are performed here, the correctness of the desired equality is seen, that is

$$H\{V_1(t)*V_2(t)\}=\frac{1}{2}(K_1e_1+K_2e_2).$$

Thus, the proof is completed. □

The autocorrelation of $V(t)$ described by the equation below:

$$V(t)\cdot V(t)=\int_{-\infty}^{\infty} V(u)V(t+u)du. \tag{2.15}$$

Autocorrelation is a temporal function of similarity between observed values. We have given the autocorrelation property in the following theorem.

Theorem 2.6. *Let the bicomplex Hartley transform of $V(t)$ be $H(w)$. Then, the Hartley transform for $V(t)\cdot V(t)$ is as follows:*

$$H\{V(t)\cdot V(t)\}=\frac{1}{2}\{[H^2(w_1)+H^2(-w_1)]e_1+[H^2(w_2)+H^2(-w_2)]e_2\}. \tag{2.16}$$

Proof. Let's use the definition and properties of the bicomplex Hartley transform. Then, from the definition of $V(t)\cdot V(t)$ we write

$$\int_{-\infty}^{\infty}\{V(t)\cdot V(t)\}\operatorname{cas} 2\pi wt\,dt=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} V(u)V(t+u)du\right]\operatorname{cas} 2\pi wt\,dt.$$

The second side of this equation is as follows.

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} V(u)V(t+u)du\right)\operatorname{cas} 2\pi w_1 t dt e_1+\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} V(u)V(t+u)du\right)\operatorname{cas} 2\pi w_2 t dt e_2.$$

If we rearrange this integral, then we get

$$\int_{-\infty}^{\infty} V(u)\left(\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w_1 t V(t+u)dt\right)du e_1+\int_{-\infty}^{\infty} V(u)\left(\int_{-\infty}^{\infty}\operatorname{cas} 2\pi w_2 t V(t+u)dt\right)du e_2.$$

By the aid of translation property of the Hartley transform we obtain that

$$\int_{-\infty}^{\infty}\{V(t)\cdot V(t)\}\operatorname{cas} 2\pi wt\,dt=W_1e_1+W_2e_2,$$

where

$$W_1=\int_{-\infty}^{\infty} V(u)\cos 2\pi w_1 u H(w_1)du-\int_{-\infty}^{\infty} V(u)\sin 2\pi w_1 u H(-w_1)du,$$

$$W_2=\int_{-\infty}^{\infty}V(u)\cos 2\pi w_2 uH(w_2)du-\int_{-\infty}^{\infty}V(u\sin 2\pi w_2 uH(-w_2)du.$$

If we calculate the two integrals above, then we get

$$\int_{-\infty}^{\infty}\{V(t)\cdot V(t)\}\operatorname{cas}2\pi wt\,dt=\frac{1}{2}\{[H^2(w_1)+H^2(-w_1)]e_1+[H^2(w_2)+H^2(-w_2)]e_2\}.$$

Thus, the proof is completed. □

In the following theorem, we give the bicomplex Hartley transform of product of the functions $V_1(t)$ and $V_2(t)$ functions.

Theorem 2.7. *Let the bicomplex Hartley transforms of the functions $V_1(t)$ and $V_2(t)$ be $H(w)$ and $G(w)$, respectively. Then, Hartley transformation of the function $V_1(t)V_2(t)$ is as follows.*

$$H\{V_1(t)V_2(t)\}=\frac{1}{2}[H(w)*G(w)+H(-w)*G(w)+H(w)*G(-w)-H(-w)*G(-w)]. \tag{2.17}$$

Proof. For proof, it will be sufficient to use definitions and properties. Then,

$$\int_{-\infty}^{\infty}V_1(t)V_2(t)\operatorname{cas}2\pi wt\,dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(w)G(w)\operatorname{cas}2\pi wt\,dt\operatorname{cas}2\pi wt\,dt$$

is written. The right-hand side of the last equation is as follows.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(w)G(w)(\operatorname{cas}2\pi w_1 te_1+\operatorname{cas}2\pi w_2 te_2)^2d^2t.$$

If necessary calculations are taken, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(w)G(w)(\cos 2\pi w_1t+\sin 2\pi w_1t)^2e_1d^2t$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(w)G(w)(\cos 2\pi w_2t+\sin 2\pi w_2t)^2e_2d^2t$$

can be written. After the calculations

$$\int_{-\infty}^{\infty}V_1(t)V_2(t)\operatorname{cas}2\pi wt\,dt=[A_1B_1+A_1D_1+C_1B_1+C_1D_1]e_1+[A_2B_2+A_2D_2+C_2B_2+C_2D_2]e_2.$$

is obtained. Where,

$$A_1=\int_{-\infty}^{\infty}H(w)\cos 2\pi w_1tdt,\quad B_1=\int_{-\infty}^{\infty}G(w)\cos 2\pi w_1tdt,\quad C_1=\int_{-\infty}^{\infty}H(w)\sin 2\pi w_1tdt,$$
$$D_1=\int_{-\infty}^{\infty}G(w)\sin 2\pi w_1tdt,\quad A_2=\int_{-\infty}^{\infty}H(w)\cos 2\pi w_2tdt,\quad B_2=\int_{-\infty}^{\infty}G(w)\cos 2\pi w_2tdt,$$
$$C_2=\int_{-\infty}^{\infty}H(w)\sin 2\pi w_2tdt,\quad D_2=\int_{-\infty}^{\infty}G(w)\sin 2\pi w_2tdt. \tag{2.18}$$

and then

$$[A_1B_1+A_1D_1+C_1B_1+C_1D_1]e_1+[A_2B_2+A_2D_2+C_2B_2+C_2D_2]e_2$$

is as follows.

$$\left\{\left(\frac{V_1(t)+V_1(-t)}{2}\right)\left(\frac{V_2(t)+V_2(-t)}{2}\right)+\left(\frac{V_1(t)+V_1(-t)}{2}\right)\left(\frac{V_2(t)-V_2(-t)}{2}\right)\right\}e_1$$
$$+\left\{\left(\frac{V_1(t)-V_1(-t)}{2}\right)\left(\frac{V_2(t)+V_2(-t)}{2}\right)+\left(\frac{V_1(t)-V_1(-t)}{2}\right)\left(\frac{V_2(t)-V_2(-t)}{2}\right)\right\}e_1$$
$$+\left\{\left(\frac{V_1(t)+V_1(-t)}{2}\right)\left(\frac{V_2(t)+V_2(-t)}{2}\right)+\left(\frac{V_1(t)+V_1(-t)}{2}\right)\left(\frac{V_2(t)-V_2(-t)}{2}\right)\right\}e_2$$

$$+\left\{\left(\frac{V_1(t)-V_1(-t)}{2}\right)\left(\frac{V_2(t)+V_2(-t)}{2}\right)+\left(\frac{V_1(t)-V_1(-t)}{2}\right)\left(\frac{V_2(t)-V_2(-t)}{2}\right)\right\}e_2.$$

From these facts,

$$\begin{aligned}&\frac{1}{2}\left(\frac{V_1(t)V_2(t)+V_1(t)V_2(-t)+V_1(-t)V_2(t)+V_1(-t)V_2(-t)}{2}\right)e_1\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)-V_1(t)V_2(-t)-V_1(-t)V_2(t)-V_1(-t)V_2(-t)}{2}\right)e_1\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)+V_1(t)V_2(-t)-V_1(-t)V_2(t)+V_1(-t)V_2(-t)}{2}\right)e_1\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)-V_1(t)V_2(-t)-V_1(-t)V_2(t)-V_1(-t)V_2(-t)}{2}\right)e_1\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)+V_1(t)V_2(-t)+V_1(-t)V_2(t)+V_1(-t)V_2(-t)}{2}\right)e_2\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)-V_1(t)V_2(-t)+V_1(-t)V_2(t)-V_1(-t)V_2(-t)}{2}\right)e_2\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)+V_1(t)V_2(-t)-V_1(-t)V_2(t)+V_1(-t)V_2(-t)}{2}\right)e_2\\
&\quad+\frac{1}{2}\left(\frac{V_1(t)V_2(t)-V_1(t)V_2(-t)-V_1(-t)V_2(t)-V_1(-t)V_2(-t)}{2}\right)e_2\end{aligned}$$

is written. If the above equation is arranged in parentheses e_1 and e_2, then the following equality is obtained:

$$H\{V_1(t)V_2(t)\}=\frac{1}{2}[H(w)*G(w)+H(-w)*G(w)+H(w)*G(-w)-H(-w)*G(-w)].$$

Thus, the proof is completed. □

3. Conclusion

In this study, we investigated the existence of the Hartley transform for functions with bicomplex variables and some basic properties. By benefiting from this study, similarities and differences between bicomplex Fourier transform and bicomplex Hartley transform can be studied.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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