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Research Article

An Introduction to Multi Inner Product Spaces

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Abstract. In this paper, for the first time, notion of multi complex numbers and multi complex number valued inner product is introduced in multi linear (vector) space. Starting from the definition, some basic properties of multi inner product spaces are studied along with examples. Multi number valued parallelogram law and polarization identity are established in multi inner product space.

Keywords. Multi linear space, Multi complex number, Multi inner product, Schwarz inequality, Parallelogram law, Multi Hilbert space

Mathematics Subject Classification (2020). 46C05, 46C50

1. Introduction

Multiset has become an important concept and being widely used both in mathematics and in computer science ([7], [8], [22]). If we allow repeated occurrences of any object in an ordinary set, then the mathematical structure is called a multiset (or mset), ([20], [21]). Formally, a multiset is defined as a collection of objects with certain multiplicity to each element and is written as $\{m_1/a_1, m_2/a_2, \dots, m_n/a_n\}$ in which the element a_i occurs m_i times. We also observe that each multiplicity m_i is a positive integer.

Classical set theory assumes that all mathematical objects occur without repetition. But, the real physical world has enormous repetition. For example, many carbon atoms are there, many water molecules, many strands of RNA, etc.

Multiset theory, real valued multisets and negative membership of the elements of multisets were studied by Blizard ([1], [3], [2], [4]). Girish and John developed the concepts of multiset topologies, multiset relations, multiset functions ([12], [13], [14]). Multi set theory was further developed in various directions by many authors in recent past. For reference one can follow the references ([23], [16], [17], [5], [6], [15], [18]).

In our previous papers ([9], [11], [19]), [10]), we have introduced the notions of multi metric space, multi metric topology, multi linear (vector) space, multi normed linear space and operator on multi normed linear space along with their various properties and several examples and counter examples. In the present paper, we are going to introduce multi complex numbers and multi complex number valued multi inner product on multi linear (vector) space. Starting from the definition, some basic properties of multi inner product space are studied along with some examples. Parallelogram law and polarization identity are established in multi inner product space using multi numbers.

2. Preliminaries

Definition 2.1 ([12])**.** A *multi set* M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers.
Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We represent the mset M drawn from the set $X = \{x_1, x_2, \dots, x_n\}$ as $M = \{m_1/x_1, m_2/x_2, \dots, m_n/x_n\}$, where m_i is the number of occurrences of the element x_i in the mset M denoted by $x_i \in^{m_i} M$, $i = 1, 2, \dots, n$. However, those elements which are not included in the mset M have zero count.

Example 2.2 ([12])**.** Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Definition 2.3 ([12])**.** Let M and N be two msets drawn from a set X. Then, the following are defined:

(i) $M = N$ if $C_M(x) = C_N(x)$, for all $x \in X$.

(ii) $M \subset N$ if $C_M(x) \le C_N(x)$, for all $x \in X$.

(iii) $P = M \cup N$ if $C_P(x) = \max\{C_M(x), C_N(x)\}$, for all $x \in X$.

(iv) $P = M \cap N$ if $C_P(x) = \min\{C_M(x), C_N(x)\}$, for all $x \in X$.

(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$, for all $x \in X$.

(vi) $P = M \ominus N$ if $C_P(x) = \max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where $\oplus$ and $\ominus$ represents mset addition and mset subtraction, respectively.

Let M be an mset drawn from a set X. The *support set* of M, denoted by M^*, is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set. M^* is also called root set.
An mset M is said to be an *empty mset* if for all $x \in X$, $C_M(x) = 0$. The cardinality of an mset M drawn from a set X is denoted by Card(M) or $|M|$ and is given by Card$(M) = \sum_{x \in X} C_M(x)$.

Definition 2.4 (Multi point, [9])**.** Let M be a multi set over a universal set X. Then a multi point of M is defined by a mapping $P_x^k : X \to \mathbf{N}$ such that $P_x^k(x) = k$ where $k \le C_M(x)$.
x and k will be referred to as the *base* and the *multiplicity* of the multi point P_x^k, respectively.

Collection of all multi points of an mset M is denoted by M_{pt}.

Definition 2.5 ([9]). Let $m\mathbf{R}^+$ denotes the multi set over $\mathbf{R}^+$ (set of non-negative real numbers) having multiplicity of each element equal to w, $w \in \mathbf{N}$. The members of $(m\mathbf{R}^+)_{pt}$ will be called *non-negative multi real points*.

Definition 2.6 ([9]). Let P^i_a and P^j_b be two multi real points of $m\mathbf{R}^+$. We define $P^i_a > P^j_b$ if $a > b$ or $P^i_a > P^j_b$ if $i > j$ when $a = b$.

Definition 2.7 (Addition of multi real points, [9]). We define $P^i_a + P^j_b = P^k_{a+b}$, where $k = \max\{i, j\}$, $P^i_a, P^j_b \in (m\mathbf{R}^+)_{pt}$.

Definition 2.8 (Multiplication of multi real points, [9]). We define multiplication of two multi real points in $m\mathbf{R}^+$ as follows:

$$P^i_a \times P^j_b = \begin{cases} P^1_0, & \text{if either } P^i_a \text{ or } P^j_b \text{ equal to } P^1_0, \\ P^k_{ab}, & \text{otherwise,} \end{cases}$$

where $k = \max\{i, j\}$.

Definition 2.9 (Multi metric, [9]). Let $d : M_{pt} \times M_{pt} \to (m\mathbf{R}^+)_{pt}$($M$ being a multi set over a Universal set X having multiplicity of any element atmost equal to w) be a mapping which satisfy the following:

(M1): $d(P^l_x, P^m_y) \geq P^1_0$, for all $P^l_x, P^m_y \in M_{pt}$,

(M2): $d(P^l_x, P^m_y) = P^1_0$ iff $P^l_x = P^m_y$, for all $P^l_x, P^m_y \in M_{pt}$,

(M3): $d(P^l_x, P^m_y) = d(P^m_y, P^l_x)$, for all $P^l_x, P^m_y \in M_{pt}$,

(M4): $d(P^l_x, P^m_y) + d(P^m_y, P^n_z) \geq d(P^l_x, P^n_z)$, for all $P^l_x, P^m_y, P^n_z \in M_{pt}$,

(M5): for $l \neq m$, $d(P^l_x, P^m_y) = P^k_0$, $\Leftrightarrow x = y$ and $k = \max\{l, m\}$.

Then d is said to be a multi metric on M and (M, d) is called a Multi metric (or an M-metric) space.

Example 2.10 ([9]). Let M be a multi set over X having multiplicity of any element atmost equal to w. We define

$$d : M_{pt} \times M_{pt} \to (m\mathbf{R}^+)_{pt}$$

such that

$$d(P^l_x, P^m_y) = \begin{cases} P^1_0 & \text{if } P^l_x = P^m_y, \\ P^{\max\{l,m\}}_0 & \text{if } x = y \text{ and } l \neq m, \\ P^j_1 & \text{if } x \neq y, \text{ for all } P^l_x, P^m_y \in M_{pt}, \ (1 \leq j \leq w \text{ is some fixed positive integer}). \end{cases}$$

Then d is an M-metric on M.

Definition 2.11 ([11]). Let (M, d) be an M-metric space, $r > 0$ and $P^k_a \in M_{pt}$. Then the *open ball* with centre P^k_a and radius P^i_r ($r > 0$, $i \in \mathbf{N}$, $1 \leq i \leq w$), is denoted by $B(P^k_a, P^i_r)$ and is defined by

$$B(P^k_a, P^i_r) = \{P^l_x : d(P^l_x, P^k_a) < P^i_r\}$$

$MS[B(P^k_a, P^i_r)]$ will be called a *multi open ball* with centre P^k_a and radius $P^i_r > P^1_0$.

Definition 2.12 ([11]). Let (M,d) be an M-metric space and $P_a^k \in M_{pt}$. A collection $N(P_a^k)$ of multi points of M is said to be a *nbd* of the multi point P_a^k if $\exists\ r > 0$ such that $P_a^k \in B(P_a^k, P_r^1) \subset N(P_a^k)$.
$MS[N(P_a^k)]$ will be called a *multi nbd* of the multi point P_a^k.

Definition 2.13 ([11]). Let (M,d) be an M-metric space. Then a collection B of multi points of M is said to be *open* if every multi point of B is an interior point of B, i.e., for each $P_a^k \in B$, $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k and $r > 0$ such that $B(P_a^k, P_r^1) \subset B$.
ϕ is separately considered as an open set.

Definition 2.14 ([11]). Let (M,d) be an M-metric space. Then $N \subset M$ is said to be *multi open* in (M,d) iff $\exists$ a collection B of multi points of N such that B is open and $MS(B) = N$.
The null multiset Φ separately considered as multi open in (M,d).

Theorem 2.15 ([11]). *In an M-metric space (M,d),*
- (i) *The null sub mset $\emptyset$ is multi open.*
- (ii) *M is multi open.*
- (iii) *Arbitrary union of multi open sets is multi open.*
- (iv) *Intersection of two multi open sets is multi open.*

Example 2.16 ([11]). Arbitrary intersection of multi open sets may not be multi open.
For example consider $\mathbf{R}$ to be a multi set with multiplicity of each element 1.
Define $d : \mathbf{R}_{pt} \times \mathbf{R}_{pt} \to (m\mathbf{R}^+)_{pt}$ by $d(P_x^1, P_y^1) = P_{|x-y|}^1$, for all $P_x^1, P_y^1 \in \mathbf{R}_{pt}$.
Consider the collection $\{P_n : n \in \mathbf{N}\}$ of multi sets such that
$$P_n = \left\{\frac{1}{x} : -\frac{1}{n} < x < \frac{1}{n}\right\}.$$
Then P_n, $n \in \mathbf{N}$ are multi open sets as $(P_n)_{pt} = \{P_x^1 : -\frac{1}{n} < x < \frac{1}{n}\}$, $n \in \mathbf{N}$ are open sets of multi points in $(\mathbf{R},d)$ and $P_n = MS((P_n)_{pt})$.
But $\bigcap_{n\in\mathbf{N}} P_n = \{1/0\}$ which is not multi open in $(\mathbf{R},d)$.

Definition 2.17 (Multi vector space, [19]). Let V be vector space over a field K. A multiset X over V is said to be a multi vector space or a multi linear space or Mvector space of V over K if every element of X has the same multiplicity and the support X^* of X is a subspace of V.
The multiplicity of every element of X will be denoted by w_X.

Example 2.18 ([19]). Let $\mathbf{R}^3$ be the Euclidean 3-dimensional pace over $\mathbf{R}$. Let $X = \{5/(a,b,0) : a,b \in \mathbf{R}\}$. Then X is a multi vector space of $\mathbf{R}^3$ over $\mathbf{R}$.

Definition 2.19 (Multivectors, [19]). Let X be an Mvector space over a vector space V_k. Then every multi point of X, i.e., every element of X_{pt} will be called a multivector of X.

Definition 2.20 (Multi scalar field, [19]). Let K be a field. Then a multi set L over K is called a multi scalar field or Mscalar field if every element of K has the same multiplicity and the support L^* of L is a subfield of K.
Multi points of L will be referred to as *multi scalars* or *Mscalars* of L.
Multiplicity of each element of L will be denoted by w_L.

Example 2.21 ([19])**.** In Example 2.18, $P^1_{(1,1,0)}$, $P^2_{(1,1,0)}$, $P^4_{(1,5,0)}$ etc. are Mvectors of the given Mvector space.

Definition 2.22 ([19])**.** Let X be an Mvector space over V_K. Then an Mvector P^k_x of X will be called a null Mvector if its base $x = \theta$ (θ being the null vector of X^* ie V_K).
It will be denoted by Θ^k. An Mvector P^k_x will be called non null if $x \neq \theta$.

Definition 2.23 ([19])**.** Let X be an Mvector space over a vector space V_K, L be an Mscalar field over K such that $w_L \le w_X$, $P^l_x, P^m_y \in X_{pt}$ and $P^i_a \in L_{pt}$.
Then, we define

$$P^l_x + P^m_y = \begin{cases} P^1_\theta & \text{iff } x = -y \text{ and } l = m, \\ P^{l\vee m}_{x+y} & \text{otherwise} \end{cases}$$

and

$$P^i_a \cdot P^l_x = \begin{cases} P^1_\theta & \text{iff } P^i_a = P^1_0 \text{ or } P^l_x = P^1_\theta, \\ P^{i\vee l}_{ax} & \text{otherwise,} \end{cases}$$

where 0 is the null element of K.

Definition 2.24 ([19])**.** An Mvector space X over V_K is said to be finite dimensional if there is a finite set of ML.Id Mvectors in X that also generates M i.e., there exists a finite set $S = \{P^{l_1}_{x_1}, P^{l_2}_{x_2}, \dots, P^{l_n}_{x_n}\}$ of Mvectors of X which is ML.Id and $MS[LS(S)] = X$.
The number of elements of such a set S is called the dimension of X and is denoted by $\dim(X)$.

Notation. Through out this paper we shall consider **V** as a vector space over **R**/**C**, **X** as an Mvector space over V_K with $w_X \le w$ (w being the multiplicity of every element of $m\mathbf{R}^+$) and **L** as an Mscalar field over K with support $L^* = K$ and $w_l \le w_X$.

Definition 2.25 ([19])**.** A mapping $\|\ \| : X_{pt} \to (m\mathbf{R}^+)_{pt}$ will be called a multi norm or mnorm on X if it satisfies the following:
(N1): $\|P^l_x\| \ge P^1_0$, for all $P^l_x \in X_{pt}$,
(N2): $\|P^l_x\| = P^k_0$ iff $x = \theta$ and $l = k$,
(N3): $\|P^i_a P^l_x\| = P^i_{|a|}\|P^l_x\|$, for all $P^i_a \in L_{pt}$, $P^l_x \in X_{pt}$,
(N4): $\|P^l_x + P^m_y\| \le \|P^l_x\| + \|P^m_y\|$, for all $P^l_x, P^m_y \in X_{pt}$.
An Mvector space X with an Mnorm $\|\ \|$ on X is called a multi normed linear space or Mnormed linear space and is denoted by $(X, \|\ \|)$. (N1), (N2), (N3) and (N4) are called norms or axioms.

Definition 2.26 (Completeness, [19])**.** An Mnormed linear space $(X, \|\ \|)$ is said to be complete if every Cauchy sequence of Mvectors in $(X, \|\ \|)$ converges to an Mvector of X.

Theorem 2.27 ([19])**.** *In an Mnormed linear space $(X, \|\ \|)$, if $P^{l_n}_{x_n} \to P^l_x$ and $P^{k_n}_{y_n} \to P^k_y$, then $P^{l_n}_{x_n} + P^{k_n}_{y_n} \to P^l_x + P^k_y$.*

Theorem 2.28 ([19]). *In an Mnormed linear space $(X, \| \, \|)$ over a vector space V_K, if $\{P_{x_n}^{l_n}\}$ be a sequence of Mvectors such that $P_{x_n}^{l_n} \to P_x^l$ and $\{P_{a_n}^{k_n}\}$ be a sequence of Mscalars such that $P_{a_n}^{k_n} \to P_a^k$, then $P_{a_n}^{k_n} \cdot P_{x_n}^{l_n} \to P_a^k \cdot P_x^l$.*

Theorem 2.29 ([19]). *In an Mnormed linear space $(X, \| \, \|)$ over a vector space V_K, if $\{P_{x_n}^{l_n}\}$, $\{P_{y_n}^{m_n}\}$ are Cauchy sequences of Mvectors and $\{P_{a_n}^{k_n}\}$ is a Cauchy sequence of Mscalars, then $\{P_{x_n}^{l_n} + P_{y_n}^{m_n}\}$, $\{P_{a_n}^{k_n} \cdot P_{x_n}^{l_n}\}$ are Cauchy sequences of Mvectors.*

3. Multi Complex Numbers

Definition 3.1. Let $m\mathbf{C}$ denotes the multi set over $\mathbf{C}$ (set of complex numbers) having multiplicity of each element equal to w, $w \in \mathbf{N}$. The members of $(m\mathbf{C})_{pt}$ will be called *multi complex numbers*.

Definition 3.2 (Addition of multi complex numbers). Let P_z^i and P_w^j be two multi complex numbers. We define $P_z^i + P_w^j = P_{z+w}^k$ where $k = \max\{i, j\}$, $P_z^i, P_w^j \in (m\mathbf{C})_{pt}$, where $z + w$ denotes the standard addition of complex numbers.

Definition 3.3 (Multiplication of multi complex numbers). Let P_z^i and P_w^j be two multi complex numbers. We define multiplication of two multi complex numbers as the following:

$$P_z^i \times P_w^j = \begin{cases} P_0^1, & \text{if either } P_z^i \text{ or } P_w^j \text{ equal to } P_0^1, \\ P_{zw}^k, & \text{otherwise,} \end{cases}$$

where $k = \max\{i, j\}$. zw denotes the standard multiplication of complex numbers.

Proposition 3.4 (Properties of multiplication). *Multiplication of multi complex numbers satisfies the following properties:*

(i) *Multiplication is Commutative.*

(ii) *Multiplication is Associative.*

(iii) *Multiplication is distributive over addition.*

Proof. (i) holds from the definition of multiplication of multi complex numbers.

(ii) If none of P_z^i, P_w^j, P_v^k is equal to P_0^1,

$$P_z^i \times (P_w^j \times P_v^k) = P_z^i \times P_{wv}^{\max\{j,k\}} = P_{zwv}^{\max\{i,j,k\}} = (P_z^i \times P_w^j) \times P_v^k.$$

If any of P_z^i, P_w^j, P_v^k is equal to P_0^1, then clearly each of the product is P_0^1.

(iii) Let P_z^i, P_w^j, $P_v^k \in (m\mathbf{C})_{pt}$. To show that $P_z^i \times (P_w^j + P_v^k) = P_z^i \times P_w^j + P_z^i \times P_v^k$.

If none of P_z^i, P_w^j, P_v^k is equal to P_0^1,

$$P_z^i \times (P_w^j + P_v^k) = P_z^i \times P_{w+v}^{\max\{j,k\}} = P_{z(w+v)}^{\max[i,\max\{j,k\}]} = P_{zw+zv}^{\max\{i,j,k\}}$$

and

$$P_z^i \times P_w^j + P_z^i \times P_v^k = P_{zw}^{\max\{i,j\}} + P_{zv}^{\max\{i,k\}} = P_{zw+zv}^{\max[\max\{i,j\},\max\{i,k\}]} = P_{zw+zv}^{\max\{i,j,k\}}.$$

So, $P_z^i \times (P_w^j + P_v^k) = P_z^i \times P_w^j + P_z^i \times P_v^k$.

If $P_z^i = P_0^1$ then $P_z^i \times (P_w^j + P_v^k) = P_0^1$ and $P_z^i \times P_w^j + P_z^i \times P_v^k = P_0^1 + P_0^1 = P_0^1$, for all $P_w^j, P_v^k \in (m\mathbf{C})_{pt}$ which gives the desired result.

If $P_w^j = P_0^1$ or $P_v^k = P_0^1$ (for definiteness say $P_w^j = P_0^1$), $P_z^i \times (P_w^j + P_v^k) = P_z^i \times (P_0^1 + P_v^k) = P_z^i \times P_v^k$ and $P_z^i \times P_w^j + P_z^i \times P_v^k = P_0^1 + P_z^i \times P_v^k = P_z^i \times P_v^k$ which also gives the desired result.

Thus the multiplication is distributive over addition. □

Definition 3.5 (Complex conjugate of multi complex number)**.** Let P_z^i be a multi complex number. The complex conjugate of P_z^i, denoted by $\overline{P}_z^i$, is defined by $\overline{P}_z^i = P_{\overline{z}}^i$, where $\overline{z}$ denotes the standard complex conjugate of the complex number z.

Definition 3.6 (Modulus of multi complex number)**.** Let P_z^i be a multi complex number. The modulus of P_z^i, denoted by $|P_z^i|$, is defined by $|P_z^i| = P_{|z|}^i$, where $|z|$ denotes the standard modulus of the complex number z. Clearly, modulus of a multi complex number is a non-negative multi real number (point).

Definition 3.7 (Real and Imaginary part of multi complex number)**.** Let P_z^i be a multi complex number. The real part of P_z^i, denoted by $Re(P_z^i)$, is defined by $Re(P_z^i) = P_{Re(z)}^i$, where $Re(z)$ denotes the standard real part of the complex number z.
The imaginary part of P_z^i, denoted by $Im(P_z^i)$, is defined by $Im(P_z^i) = P_{Im(z)}^i$, where $Im(z)$ denotes the standard imaginary part of the complex number z.

Proposition 3.8. *For multi complex numbers P_z^i, P_w^j the followings hold:*

(i) $\overline{P_z^i + P_w^j} = \overline{P}_z^i + \overline{P}_w^j$,

(ii) $|P_z^i| = |\overline{P}_z^i|$,

(iii) $|P_z^i| > Re(P_z^i)$,

(iv) $|P_z^i| > Im(P_z^i)$,

(v) $P_z^i + \overline{P}_z^i$ *is a multi real number,*

(vi) $\overline{P_z^i \cdot P_w^j} = \overline{P}_z^i \cdot \overline{P}_w^j$,

(vii) $P_z^i \cdot \overline{P_z^i} = |P_z^i|^2$.

Definition 3.9 (Subtraction of multi complex numbers)**.** We define subtraction of two multi complex numbers as follows:

$$P_z^i - P_w^j = \begin{cases} P_{z-w}^{\max\{i,j\}}, & \text{if } P_z^i \neq P_w^j, \\ P_0^1, & \text{if } P_z^i = P_w^j. \end{cases}$$

Definition 3.10 (Division of multi complex numbers)**.** We define division of two multi complex numbers as follows:
for $w \neq 0$,

$$P_z^i / P_w^j = \begin{cases} P_{z/w}^{\max\{i,j\}}, & \text{if } P_z^i \neq P_w^j, \\ P_1^1, & \text{if } P_z^i = P_w^j. \end{cases}$$

Definition 3.11 (Integral power of multi real numbers)**.** The nth integral power of a multi real number is defined by

$$(P^i_a)^n = P^i_a \cdot P^i_a \dots (n\text{-times}) = P^i_{a^n} \qquad \text{(by multiplication of multi real numbers).}$$

Definition 3.12 (Square root of multi real numbers)**.** Square root of a multi real number is defined by $\sqrt{P^i_a} = P^i_{\sqrt{a}}$.

We observe that, $\left(\sqrt{P^i_a}\right)^2 = \left(P^i_{\sqrt{a}}\right)^2 = P^i_{\sqrt{a}} \cdot P^i_{\sqrt{a}} = P^i_a$, (by multiplication of multi real numbers).

4. Multi Inner Product Space

Notation. Throughout this section we shall consider $\mathbf{V}$ as a vector space over $K = \mathbf{R}/\mathbf{C}$, $\mathbf{X}$ as an Mvector space over V_K with $w_X \le w$ (w being the multiplicity of every element of $m\mathbf{R}^+/m\mathbf{C}$ and $\mathbf{L}$ as an Mscalar field over K with support $L^* = K$ and $w_L \le w_X$.

Definition 4.1. A mapping $\langle\cdot\rangle : X_{pt} \times X_{pt} \to m\mathbf{C}_{pt}$ is said to be a multi inner product on the multi vector space X if $\langle\cdot\rangle$ satisfies the following conditions:

(I1): $\langle P^l_x, P^l_x\rangle \ge P^1_0$, for all $P^l_x \in X_{pt}$ and $\langle P^l_x, P^l_x\rangle = P^1_0$ if and only if $x = \theta$ and $l = 1$,

(I2): $\langle P^l_x, P^m_y\rangle = \overline{\langle P^m_y, P^l_x\rangle}$, where bar denote the complex conjugate of multi complex numbers,

(I3): $\langle P^i_a . P^l_x, P^m_y\rangle = P^i_a \cdot \langle P^l_x, P^m_y\rangle$ for all $P^l_x, P^m_y \in X_{pt}$ and for every multi scalar $P^i_a \in L_{pt}$,

(I4): for all $P^l_x, P^m_y, P^n_z \in X_{pt}$, $\langle P^l_x + P^m_y, P^n_z\rangle = \langle P^l_x, P^n_z\rangle + \langle P^m_y, P^n_z\rangle$.

The multi vector space X with a multi inner product $\langle\cdot\rangle$ on X is said to be a multi inner product space and is denoted by $(X, \langle\cdot\rangle)$. (I1), (I2), (I3) and (I4) are said to be multi inner product axioms.

Example 4.2. Let $X = l_2$. Then X is an inner product space with respect to the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$, for $x = \{\xi_i\}$, $y = \{\eta_i\}$ of l_2. Let X be an Mvector space over V_K with $w_X \le w$ (w being the multiplicity of every element of $m\mathbf{R}^+/m\mathbf{C}$ and $\mathbf{L}$ as an Mscalar field over K with support $L^* = K$ and $w_L \le w_X$. Let P^l_x, P^m_y be multi points of the multi vector space X. Then x, y are elements of l_2. The mapping $\langle\cdot\rangle : X_{pt} \times X_{pt} \to m\mathbf{C}_{pt}$ defined by $\langle P^l_x, P^m_y\rangle = P^{\max\{l,m\}}_{\langle x,y\rangle}$ is a multi inner product on the multi vector space X.

Let us verify (I1), (I2), (I3) and (I4) for multi inner product.

(I1): We have $\langle P^l_x, P^l_x\rangle = P^{\max\{l,l\}}_{\langle x,x\rangle} \ge P^l_0$, since, $\langle x, x\rangle \ge 0$ and $l \ge 1$.

Also, $\langle P^l_x, P^l_x\rangle = P^l_0 \iff P^{\max\{l,l\}}_{\langle x,x\rangle} = P^l_0 \iff \langle x, x\rangle = 0$ and $\max\{l,l\} = 1 \iff \tilde{x} = \theta$ and $l = 1$.

Thus, (I1) is satisfied.

(I2): We have, $\langle P^l_x, P^m_y\rangle = P^{max\{l,m\}}_{\langle x,y\rangle} = P^{\max\{l,m\}}_{\overline{\langle y,x\rangle}} = \overline{P}^{\max\{l,m\}}_{\langle y,x\rangle}$.

(I3): We have, $\langle P^i_a . P^l_x, P^m_y\rangle = \langle P^{\max\{i,l\}}_{ax}, P^m_y\rangle = P^{\max\{i,l,m\}}_{\langle ax,y\rangle} = P^{\max\{i,l,m\}}_{a\langle x,y\rangle} = P^i_a \cdot P^{\max\{l,m\}}_{\langle x,y\rangle} = P^i_a \cdot \langle P^l_x, P^m_y\rangle$.

(I4): For all $P^l_x, P^m_y, P^n_z \in X$, $\langle P^l_x + P^m_y, P^n_z\rangle = \langle P^{\max\{l,m\}}_{x+y}, P^n_z\rangle = P^{\max\{l,m,n\}}_{\langle x+y,z\rangle} = P^{\max\{l,m,n\}}_{\langle x,z\rangle + \langle y,z\rangle} = P^{\max\{l,n\}}_{\langle x,z\rangle} + P^{\max\{m,n\}}_{\langle y,z\rangle} = \langle P^l_x, P^n_z\rangle + \langle P^m_y, P^n_z\rangle$.

Thus (I4) is satisfied.

Therefore, $\langle\cdot\rangle$ is a multi inner product on X and consequently $(X,\langle\cdot\rangle)$ is a multi inner product space.

Corollary 4.3. *Every crisp inner product $\langle\cdot\rangle$ on a crisp vector space X can be extended to a multi inner product on the multi vector space X.*

Proposition 4.4. *Let $(X,\langle\cdot\rangle)$ be a multi inner product space, $P^l_x, P^m_y, P^n_z \in X$ and P^i_a, P^j_b etc. be multi scalars. Then*

(i) $\langle P^i_a \cdot P^l_x + P^j_b \cdot P^m_y, P^n_z\rangle = P^i_a \cdot \langle P^l_x, P^n_z\rangle + P^j_b \cdot \langle P^m_y, P^n_z\rangle$,

(ii) $\langle P^l_x, P^i_a \cdot P^m_y\rangle = \overline{P^i_a} \cdot \langle P^l_x, P^m_y\rangle$,

(iii) $\langle P^l_x, P^i_a \cdot P^m_y + P^j_b \cdot P^n_z\rangle = \overline{P^i_a} \cdot \langle P^l_x, P^m_y\rangle + \overline{P^j_b} \cdot \langle P^l_x, P^n_z\rangle$.

Proof. (i) We have, $\langle P^i_a \cdot P^l_x + P^j_b \cdot P^m_y, P^n_z\rangle = \langle P^i_a \cdot P^l_x, P^n_z\rangle + \langle P^j_b \cdot P^m_y, P^n_z\rangle = P^i_a \cdot \langle P^l_x, P^n_z\rangle + P^j_b \cdot \langle P^m_y, P^n_z\rangle$,

(ii) $\langle P^l_x, P^i_a \cdot P^m_y\rangle = \overline{\langle P^i_a \cdot P^m_y, P^l_x\rangle} = \overline{P^i_a \cdot \langle P^m_y, P^l_x\rangle} = \overline{P^i_a} \cdot \langle P^l_x, P^m_y\rangle$,

(iii) $\langle P^l_x, P^i_a \cdot P^m_y + P^j_b \cdot P^n_z\rangle = \overline{\langle P^i_a \cdot P^m_y + P^j_b \cdot P^n_z, P^l_x\rangle} = \overline{\langle P^i_a \cdot P^m_y, P^l_x\rangle} + \overline{\langle P^j_b . P^n_z, P^l_x\rangle}$

$= \overline{P^i_a \cdot \langle P^m_y, P^l_x\rangle} + \overline{P^j_b \cdot \langle P^n_z, P^l_x\rangle} = \overline{P^i_a} \cdot \langle P^l_x, P^m_y\rangle + \overline{P^j_b} \cdot \langle P^l_x, P^n_z\rangle$. □

Theorem 4.5 (Schwarz inequality)**.** *Let $(X,\langle\cdot\rangle)$ be a multi inner product space. Let $P^l_x, P^m_y, P^n_z \in X$ and P^i_a, P^j_b etc. be multi scalars. Then $|\langle P^l_x, P^m_y\rangle| \le \|P^l_x\| \cdot \|P^m_y\|$.*

Proof. (i) If P^i_a be a multi scalar, then $\langle P^l_x + P^i_a \cdot P^m_y, P^l_x + P^i_a \cdot P^m_y\rangle \ge P^1_0$, i.e.,

$$\langle P^l_x, P^l_x\rangle + P^i_a \cdot \langle P^m_y, P^l_x\rangle + \overline{P^i_a} \cdot \langle P^l_x, P^m_y\rangle + |P^i_a|^2 \langle P^m_x, P^m_y\rangle \ge P^1_0. \tag{4.1}$$

If $\langle P^m_y, P^m_y\rangle = P^1_0$ then $y = \theta$ and $m = 1$ so $\langle P^l_x, P^m_y\rangle = \langle P^l_x, P^1_\theta\rangle = \langle P^l_x, P^1_0 \cdot P^1_\theta\rangle = P^1_0$ and in this case the inequality is proved.

In case $\langle P^m_y, P^m_y\rangle > P^1_0$. Let $P^i_a = -\frac{\langle P^l_x, P^m_y\rangle}{\langle P^m_y, P^m_y\rangle}$. Then we obtain from (4.1),

$$\|P^l_x\|^2 - \frac{|\langle P^l_x, P^m_y\rangle|^2}{\|P^m_y\|^2} - \frac{\langle P^l_x, P^m_y\rangle \cdot \overline{\langle P^l_x, P^m_y\rangle}}{\|P^m_y\|^2} + \frac{|\langle P^l_x, P^m_y\rangle|^2}{\|P^m_y\|^2} \ge P^1_0$$

i.e.,

$$\|P^l_x\|^2 - \frac{\langle P^l_x, P^m_y\rangle \cdot \overline{\langle P^l_x, P^m_y\rangle}}{\|P^m_y\|^2} \ge P^1_0$$

i.e., $|\langle P^l_x, P^m_y\rangle| \le \|P^l_x\| \cdot \|P^m_y\|$. □

Proposition 4.6. *Let $(X,\langle\cdot\rangle)$ be a multi inner product space. Let $P^l_x, P^m_y, P^n_z \in X$ and P^i_a, P^j_b etc. be multi scalars. Then*

(i) $\|P^l_x + P^m_y\|^2 + \|P^l_x - P^m_y\|^2 = P^1_2 \cdot \|P^l_x\|^2 + P^1_2 \cdot \|P^m_y\|^2$ *(Parallelogram law).*

(ii) $\langle P^l_x, P^m_y\rangle = P^1_{\frac{1}{4}}\{\|P^l_x + P^m_y\|^2 - \|P^l_x - P^m_y\|^2 + P^1_i\|P^l_x + P^1_i P^m_y\|^2 - P^1_i\|P^l_x - P^1_i P^m_y\|^2\}$ *(Polarization identity).*

Proof. (i)
$$\begin{aligned}\|P_x^l+P_y^m\|^2+\|P_x^l-P_y^m\|^2 &= \langle P_x^l+P_y^m,P_x^l+P_y^m\rangle+\langle P_x^l-P_y^m,P_x^l-P_y^m\rangle\\ &= \langle P_x^l,P_x^l\rangle+\langle P_y^m,P_x^l\rangle+\langle P_x^l,P_y^m\rangle+\langle P_y^m,P_y^m\rangle\\ &\quad+\langle P_x^l,P_x^l\rangle-\langle P_y^m,P_x^l\rangle-\langle P_x^l,P_y^m\rangle+\langle P_y^m,P_y^m\rangle\\ &= P_2^1\langle P_x^l,P_x^l\rangle+P_2^1\langle P_y^m,P_y^m t\rangle\\ &= P_2^1\|P_x^l\|^2+P_2^1\|P_y^m\|^2.\end{aligned}$$

(ii) We have
$$\|P_x^l+P_y^m\|^2=\langle P_x^l+P_y^m,P_x^l+P_y^m\rangle=\|P_x^l\|^2+\|P_y^m\|^2+\langle P_y^m,P_x^l\rangle+\langle P_x^l,P_y^m\rangle\,. \tag{4.2}$$
In (4.2) replace P_y^m by $-P_y^m$, $P_i^1\cdot P_y^m$, $-P_i^1\cdot P_y^m$ then we obtain
$$\|P_x^l-P_y^m\|^2=\|P_x^l\|^2+\|P_y^m\|^2-\langle P_y^m,P_x^l\rangle-\langle P_x^l,P_y^m\rangle$$
$$\|P_x^l+P_i^1\cdot P_y^m\|^2=\|P_x^l\|^2+\|P_y^m\|^2+P_i^1\langle P_y^m,P_x^l\rangle-P_i^1\langle P_x^l,P_y^m\rangle$$
and
$$\|P_x^l-P_i^1P_y^m\|^2=\|P_x^l\|^2+\|P_y^m\|^2-P_i^1\langle P_y^m,P_x^l\rangle+P_i^1\langle P_x^l,P_y^m\rangle$$
or what are same as
$$-\|P_x^l-P_y^m\|^2=-\|P_x^l\|^2-\|P_y^m\|^2+\langle P_y^m,P_x^l\rangle+\langle P_x^l,P_y^m\rangle, \tag{4.3}$$
$$P_i^1\|P_x^l+P_i^1P_y^m\|^2=P_i^1\|P_x^l\|^2+P_i^1\|P_y^m\|^2-\langle P_y^m,P_x^l\rangle+\langle P_x^l,P_y^m\rangle \tag{4.4}$$
and
$$P_i^1\|P_x^l-P_i^1P_y^m\|^2=-P_i^1\|P_x^l\|^2-P_i^1\|P_y^m\|^2-\langle P_y^m,P_x^l\rangle+\langle P_x^l,P_y^m\rangle. \tag{4.5}$$
After adding (4.2), (4.3), (4.4) and (4.5) the right hand side becomes $P_4^1\langle P_x^l,P_y^m\rangle$ and that proves the identity. □

Theorem 4.7. *Let $(X,\langle\cdot\rangle)$ be a multi inner product space. Let us define $\|\cdot\|:X_{pt}\to(m\mathbf{R}^+)_{pt}$ by $\|P_x^l\|=\sqrt{\langle P_x^l,P_x^l\rangle}$, for all $P_x^l\in X_{pt}$. Then $\|\cdot\|$ is a multi norm on X.*

Proof. We have (N1) and (N2). $\|P_x^l\|=\sqrt{\langle P_x^l,P_x^l\rangle}\ge P_0^1$, for all $P_x^l\in X_{pt}$, and $\|P_x^l\|=P_0^1\iff\sqrt{\langle P_x^l,P_x^l\rangle}=P_0^1\iff x=\theta$ and $l=1$ (using (I1)).

(N3): $\|P_a^i\cdot P_x^l\|=\sqrt{\langle P_a^i\cdot P_x^l,P_a^i\cdot P_x^l\rangle}=\sqrt{P_a^i\cdot\langle P_x^l,P_a^i\cdot P_x^l\rangle}=\sqrt{P_a^i\cdot\overline{P_a^i}\cdot\langle P_x^l,P_x^l\rangle}=|P_a^i|\cdot\sqrt{\langle P_x^l,P_x^l\rangle}=|P_a^i|\cdot\|P_x^l\|$ for all $P_x^l\in X_{pt}$ and for every multi scalar P_a^i (using (I2)).

(N4): We have, $\|P_x^l+P_y^m\|^2=\langle P_x^l+P_y^m,P_x^l+P_y^m\rangle=\langle P_x^l,P_x^l\rangle+\langle P_x^l,P_y^m\rangle+\langle P_y^m,P_x^l\rangle+\langle P_y^m,P_y^m\rangle$, i.e.,
$$\|P_x^l+P_y^m\|^2=\|P_x^l\|^2+\|P_y^m\|^2+\langle P_y^m,P_x^l\rangle+\langle P_x^l,P_y^m\rangle\,. \tag{4.6}$$
We have
$$|\langle P_x^l,P_y^m\rangle|\le\|P_x^l\|\cdot\|P_y^m\|$$
and so
$$|\langle P_y^m,P_x^l\rangle|=\overline{|\langle P_x^l,P_y^m\rangle|}\le\|P_x^l\|\cdot\|P_y^m\|\,.$$

So,

$$|\langle P^l_x,P^m_y\rangle + \langle P^m_y,P^l_x\rangle| \le |\langle P^l_x,P^m_y\rangle| + |\langle P^m_y,P^l_x\rangle| \le 2\|P^l_x\|\cdot\|P^m_y\|.$$

But,

$$\langle P^l_x,P^m_y\rangle + \langle P^m_y,P^l_x\rangle = \overline{\langle P^l_x,P^m_y\rangle} + \langle P^l_x,P^m_y\rangle,$$

is a multi real number.

Hence,

$$\langle P^l_x,P^m_y\rangle + \langle P^m_y,P^l_x\rangle \le 2\|P^l_x\|\cdot\|P^m_y\|.$$

From (4.6),

$$\|P^l_x+P^m_y\|^2 \le \|P^l_x\|^2 + \|P^m_y\|^2 + 2\|P^l_x\|\cdot\|P^m_y\| = (\|P^l_x\| + \|P^m_y\|)^2.$$

Hence, $\|P^l_x+P^m_y\| \le \|P^l_x\| + \|P^m_y\|$.

Hence, $\|\cdot\|$ is a multi norm on X. □

Remark 4.8. From the above theorem it follows that every 'multi inner product space' is also a 'multi normed linear space' with the multi norm defined as above. With the help of this multi norm, we can introduce a 'multi metric' on X by the formula,

$$d(P^l_x,P^m_y) = \|P^l_x - P^m_y\| = \sqrt{\langle P^l_x - P^m_y, P^l_x - P^m_y\rangle}, \quad \text{for all } P^l_x,P^m_y \in X_{pt}.$$

Remark 4.9. The converse of the above theorem is however not true. For example consider $X = l_p (p \ge 1, p \ne 2)$. Then X is a normed linear space with respect to the norm $\|x\| = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}$ for $x = \{\xi_i\}$ of l_p. Let P^l_x be a multi point of the multi vector space X over V with $w_X = w$. Let us define $\|\ \| : X_{pt} \to (m\mathbf{R}^+)_{pt}$ such that $\|P^l_x\| = P^l_{\|x\|}$, for all $P^l_x \in X_{pt}$. Then $\|\ \|$ is an Mnorm over X and $(X, \|\ \|)$ is an Mnormed linear space. Then $d : X_{pt} \times X_{pt} \to (m\mathbf{R}^+)_{pt}$ defined by $d(P^l_x,P^m_y) = \|P^l_x - P^m_y\|$, for all $P^l_x,P^m_y \in X_{pt}$ is a multi metric on X.

We now show that the multi norm $\|\cdot\|$ of X cannot be obtained from a multi inner product. We verify this by showing that this multi norm does not satisfy parallelogram law. Let us consider multi points P^1_x,P^1_y of X such that $x = \{1,1,0,0,\dots\}$, $y = \{1,-1,0,0,\dots\}$. Then $\|P^1_x\| = \|P^1_y\| = P^1_{\sqrt[p]{2}}$ and $\|P^1_x + P^1_y\| = \|P^1_x - P^1_y\| = P^1_2$. We observe that if $p \ne 2$, then the parallelogram law does not hold. Hence, for $p \ne 2$, X is not a multi inner product space.

5. Multi Hilbert Space and Its Properties

Definition 5.1. A multi inner product space $(X, \langle\cdot\rangle)$ is said to be complete if it is complete with respect to the multi metric defined by multi inner product. A complete multi inner product space is said to be a multi Hilbert space.

Example 5.2. The multi inner product space $(X, \langle\cdot\rangle)$ on the multi vector space X, defined as in Example 4.2, is a multi Hilbert space. We recall that l_2 is a complete metric space with respect to the metric $\rho(x,y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2\right)^{\frac{1}{2}}$ for $x = \{\xi_i\}$, $y = \{\eta_i\}$ of l_2. Let P^l_x, P^m_y be any two multi points of X, then x, y are the elements of l_2. We can easily verify that the mapping $d : X_{pt} \times X_{pt} \to m\mathbf{R}_{pt}$ defined by $d(P^l_x, P^m_y) = P^{\max\{l,m\}}_{\rho(x,y)}$ is a multi metric on X.

Let $\{P^{i_n}_{x_n}\}$ be a Cauchy sequence of multi points in (X,d). Then corresponding to every $\epsilon > 0$, $\exists$ $k \in N$ such that $d(\{P^{i_n}_{x_n}\}, \{P^{i_m}_{x_m}\}) \le P^1_{\frac{\epsilon}{2}}$, for all $m,n \ge k$, i.e., $\rho(x_n, x_m)^{\max\{i_n,i_m\}} \le P^1_{\frac{\epsilon}{2}}$, for all $m,n \ge k$. Thus $P_{\rho(x_n,x_m)} \le \frac{\epsilon}{2}$ and hence $\{x_n\}$ is a Cauchy sequence of elements in l_2. Since l_2 is complete, $\{x_n\}$ is convergent and let $\{x_n\} \to x$. Then $x \in l_2$ and hence $P^k_x \in X_{pt}$, for any $1 \le k \le w_x$. Clearly, $\{P^{i_n}_{x_n}\}$ converges to P^k_x and consequently, (X,d) is complete.

Now we have,

$$\begin{aligned}
\sqrt{\langle P^l_x - P^m_y, P^l_x - P^m_y\rangle} &= P^{\max\{l,m\}}_{\langle x-y, x-y\rangle} \\
&= P^{\max\{l,m\}}_{\left(\sum\limits_{i=1}^{\infty} (\xi_i - \eta_i)\overline{(\xi_i - \eta_i)}\right)^{1/2}} \\
&= P^{\max\{l,m\}}_{\left(\sum\limits_{i=1}^{\infty} |\xi_i - \eta_i|^2\right)^{\frac{1}{2}}} \\
&= d(P^l_x, P^m_y).
\end{aligned}$$

Therefore, $d(P^l_x, P^m_y) = \sqrt{\langle P^l_x - P^m_y, P^l_x - P^m_y\rangle}$, for all $P^l_x, P^m_y \in X_{pt}$. Thus the multi metric d is induced by multi inner product $\langle\cdot\rangle$ and the multi metric space (X,d) is complete. Hence $(X,\langle\cdot\rangle)$ is a multi Hilbert space.

Theorem 5.3. *A multi Banach space is a multi Hilbert space if and only if the parallelogram law holds.*

Proof. First, we consider the underlying multi vector space over a real vector space. We have seen that in every multi inner product space parallelogram law holds and also every multi inner product space is also a multi normed linear space and in both spaces completeness is based on the completeness of the multi metric obtain from them. Thus, it is clear that a multi Hilbert space is a multi Banach space in which parallelogram law holds.

We now suppose that, X be a multi Banach space where the parallelogram law holds. We introduce a multi inner product on X by

$$\langle P^l_x, P^m_y\rangle = P^1_{\frac{1}{4}}\{\|P^l_x + P^m_y\|^2 - \|P^l_x - P^m_y\|^2\}.$$

Clearly, $\langle P^l_x, P^l_x\rangle \ge P^1_0$, and $\langle P^l_x, P^l_x\rangle = P^1_0$ if and only if $P^l_x = P^1_\theta$. Also, $\langle P^l_x, P^l_x\rangle = \|P^l_x\|^2$ and $\langle P^l_x, P^m_y\rangle = \langle P^m_y, P^l_x\rangle$. We should now verify the other two axioms

(I3): $\langle P^i_a.P^l_x, P^m_y\rangle = P^i_a.\langle P^l_x, P^m_y\rangle$, for all $P^l_x, P^m_y \in X_{pt}$

and for every multi scalar P^i_a,

(I4): for all $P^l_x, P^m_y, P^n_z \in X_{pt}$, $\langle P^l_x + P^m_y, P^n_z\rangle = \langle P^l_x, P^n_z\rangle + \langle P^m_y, P^n_z\rangle$.

By parallelogram law we obtain,

$$\|P^r_u + P^s_v + P^t_w\|^2 + \|P^r_u + P^s_v - P^t_w\|^2 = P^1_2\|P^r_u + P^s_v\|^2 + P^1_2\|P^t_w\|^2$$

and

$$\|P_u^r - P_v^s + P_w^t\|^2 + \|P_u^r - P_v^s - P_w^t\|^2 = P_2^1\|P_u^r - P_v^s\|^2 + P_2^1\|P_w^t\|^2.$$

On subtraction, we obtain

$$\begin{aligned}&\|P_u^r + P_v^s + P_w^t\|^2 + \|P_u^r + P_v^s - P_w^t\|^2 - \|P_u^r - P_v^s + P_w^t\|^2 - \|P_u^r - P_v^s - P_w^t\|^2 \\ &\quad = P_2^1\|P_u^r + P_v^s\|^2 - P_2^1\|P_u^r - P_v^s\|^2.\end{aligned}$$

By the definition of multi inner product, we have

$$\langle P_u^r + P_v^s, P_w^t\rangle + \langle P_u^r - P_v^s, P_w^t\rangle = P_2^1\langle P_u^r, P_v^s\rangle.$$

If we set $P_x^l = P_u^r + P_v^s$, $P_y^m = P_u^r - P_v^s$ and $P_z^n = P_w^t$, then we obtain,

$$\langle P_x^l, P_z^n\rangle + \langle P_y^m, P_z^n\rangle = \langle P_x^l + P_y^m, P_z^n\rangle.$$

The relation $\langle P_a^i.P_x^l, P_y^m\rangle = P_a^i.\langle P_x^l, P_y^m\rangle$ for all $P_x^l, P_y^m \in X_{pt}$ and for every multi scalar P_a^i, can be proved similarly.

In the complex case, we take the multi inner product of P_x^l, P_y^m as

$$\langle P_x^l, P_y^m\rangle = P_{\frac{1}{4}}^1\{\|P_x^l + P_y^m\|^2 - \|P_x^l - P_y^m\|^2 + P_i^1\|P_x^l + P_i^1.P_y^m\|^2 - P_i^1\|P_x^l - P_i^1.P_y^m\|^2\}.$$

The theorem can be proved similarly. □

6. Conclusions

Operator theory plays an important role in development of functional analysis and it is used in various branches of mathematics and sciences. In this paper, an attempt has been made to introduce multi inner product on multi linear (vector) space. One can further extend the study of multi inner product spaces. Research on multi Hilbert space and operator theory on multi inner product space can be of special interest.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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