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Research Article

2D Hexagonal Finite Fuzzy Cellular Automata

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Abstract. In this paper, 2D hexagonal finite fuzzy cellular automata defined by fuzzy transition of local rules based on hexagonal cell structure are studied. The inverse problem of 2D hexagonal finite fuzzy cellular automata is also studied.

Keywords. Two dimensional hexagonal fuzzy cellular automta, Matrix algebra, Null boundary condition, Periodic boundary condition

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1. Introduction

Cellular automata are discrete dynamical that exhibit a variety of dynamical behaviors, although they are formed by simple basic components. *Cellular automata* (CA) were first used for modelling various physical, biological process and on computer science. The concept of cellular automata was initiating the early 1950's by John Von Neumann and Stanislaw Ulam. John Von Neumann [8] showed that a cellular automaton (CA) can be universal.

Cellular automata are also called Cellular Space, Tessellation Automata, Homogeneous structures, Cellular structure, Tessellation structures and Iterative arrays [14].

The study of CA has received remarkable attention in the last few years because CA have been widely investigated in many disciplines (e.g., mathematics, simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, cryptography). Most of the work for CA is done for one-dimensional (1-D) cases. Recently, two dimensional (2-D) CA have attracted much of the interest.

A cellular automata is a model of a system of "cell" objects with the following characteristics.

- The cell live on a grid.
- Each cell has a state. The number of state possibilities is typically finite. The simplest example has the two possibilities of 1 and 0 (otherwise referred to as "ON" and "OFF" (or) "alive" and "dead").
- Each cell has a neighborhood. This can be defined in any number of ways, but it is typically a list of adjacent cells.

In a cellular automaton all cells typically begin in state 0, except for a finite number that are in other states. The nonzero patterns that occur while a cellular automaton is running are called 'configurations'. At each tick of the clock, many of the cells enter a new state and a new configuration develops. It is natural to refer to the sequence of configurations that develop as 'generations'.

Zadeh [15] introduced the notion of fuzzy subset of a set as a tool for representing uncertainty. His ideas have been applied to wide range of scientific areas. Wee [13] applied the ideas of Zadeh in automata theory and language theory. A fuzzy automata is generalization of non deterministic automata. Fuzzy approach has been applied to automata theory in the very early age of fuzzy set theory. The basic idea in the formulation of a fuzzy automata is that, unlike the classical case, the automata switch from one state to another one to certain degree. Santos [9] proposed fuzzy automata as a model of pattern recognition and control systems. Mordeson and Malik [6] gave a detailed account of fuzzy automata and languages in their book (2002). Staimnn and Adalassing [11] dealt with applications of fuzzy automata in the field of clinical monitoring.

The hexagonal finite cellular automata (shortly HFCA) are two dimensional (2D) cellular automata whose cells are of form of a hexagonal. Morita *et al*. [7] introduced this type of cellular automata (CA) and they called it hexagonal partitioned CA (HPCA).

The notation of fuzziness in cellular automata was first applied by Andrew I. Adamatzky [1]. He divide a set of fuzzy cellular automata into 14 classes.

This paper is organized as follows. In Section 2, the concept used in the paper are formally defined. In Section 3, the principle of fuzzy cellular automata is given [1]. We obtain fuzzy transition matrix of the 2D hexagonal fuzzy cellular automata. We fuzzify the neighborhood and local transition rule of a 2D hexagonal finite cellular automata. In Section 4, we obtain the fuzzy matrix of inverse of T_R^0, if the number of columns of T_R^0 are even. We also present some example for the explaining the concepts.

2. Preliminaries

Definition 2.1 ([5]). A periodic boundary CA is the one in which the extreme cells are adjacent to each other.

Definition 2.2 ([5]). A null boundary CA is the one in which the extreme cells are connected to logic zero state.

The state of the cell (i,j) at time t is denoted by $X^t_{(i,j)}$. The state of the cell (i,j) at time $t+1$ is denoted by $X^{(t+1)}_{(i,j)} = Y^{(t)}_{(i,j)}$.

In [10] Irfan Siap, consider the hexagonal information matrix $C^{(t)} = \begin{pmatrix} X^{(t)}_{11} & \cdots & X^{(t)}_{1n} \\ \vdots & \ddots & \vdots \\ X^{(t)}_{m1} & \cdots & X^{(t)}_{mn} \end{pmatrix}$.

The matrix $C^{(t)}$ is called the configuration of the 2D finite CA at time t. He associate planar hexagonal presentations with row vectors by transforming them from $C^{(t)}$ to $([X]_{1\times mn}) = (X^{(t)}_{11}, X^{(t)}_{12}, \cdots, X^{(t)}_{1n}, \cdots, X^{(t)}_{m1}, \cdots, X^{(t)}_{mn})$.
He consider the transition matrix T_R such that $[X]_{1\times mn} \cdot (T_R)_{mn\times mn} = [Y]_{mn\times 1}$, where $([Y]_{mn\times 1}) = (Y^{(t)}_{11}, Y^{(t)}_{12}, \cdots, Y^{(t)}_{1n}, \cdots, Y^{(t)}_{m1}, \cdots, Y^{(t)}_{mn})$.

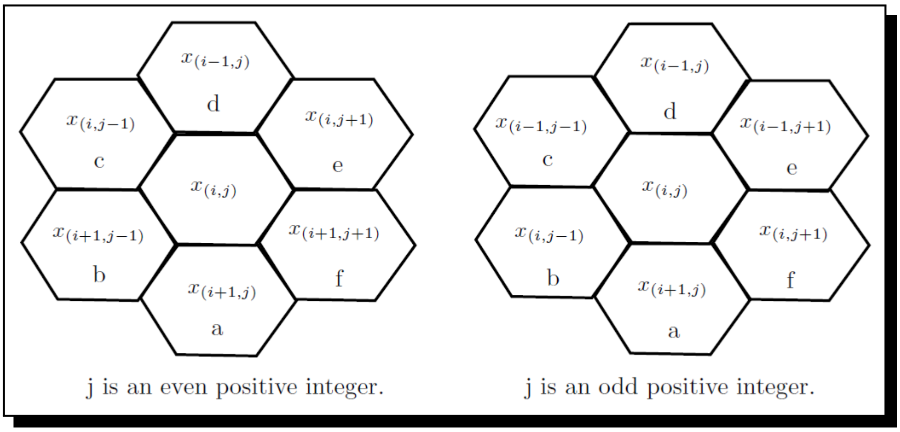

Figure 1. Two configurations of the HCA

If j is an even positive integer, then, we have

$$Y^{(t)}_{(i,j)} = aX^{(t)}_{(i+1,j)} + bX^{(t)}_{(i+1,j-1)} + cX^{(t)}_{(i,j-1)} + dX^{(t)}_{(i-1,j)} + eX^{(t)}_{(i,j+1)} + fX^{(t)}_{(i+1,j+1)} \bmod 3 \tag{2.1}$$

If j is an odd positive integer, then, we have

$$Y^{(t)}_{(i,j)} = aX^{(t)}_{(i+1,j)} + bX^{(t)}_{(i+1,j-1)} + cX^{(t)}_{(i-1,j-1)} + dX^{(t)}_{(i-1,j)} + eX^{(t)}_{(i-1,j+1)} + fX^{(t)}_{(i,j+1)} \bmod 3 \tag{2.2}$$

where $a,b,c,d,e,f \in Z^*_3 = Z_3 \setminus \{0\} = \{1,2\}$ and $X^{(t)}_{(i,j)} \in Z_3$.

The matrix spaces C^t of order $m\times n$ with co-efficient in Z_3 denoted by $M_{n\times m}$ Z_3 are isomorphic to Z^{mn}_3 as vectors spaces. This Z^{mn}_3 is a matrix of row space.

3. Fuzzy Transition Matrix of the 2D Hexagonal Fuzzy Cellular Automata

In [10], they obtain the rule matrix of 2D finite CA with hexagonal rule over the field Z_3 under the Null Boundary Condition. This rule matrix [3] T^0_R which takes the tth finite hexagonal configuration C^t of order $m\times n$ to the $(t+1)$th state C^{t+1}.

By using [10], we generalize the CA into FCA.

In this section, we define the fuzzy cellular automata and obtain the fuzzy membership matrix of the rule matrix T_R^0.

Definition 3.1 (Fuzzy Cellular Automata)**.** A fuzzy cellular automata (FCA) is defined by the tuple (A,Q,u,F,M), where

- m dimensional array A of cell.
- Q is a finite set of cell states.
- The neighborhood u is a K-tuple of state of cells.
- $F = \{f \mid f : Q^k \to Q\}$ is a set of local transition rule.
- $M = \{\mu \mid \mu : Q^k \times F \to [0,1]\}$ is a set of grades of the local transition.

Theorem 3.1. *Let $M = (A,Q,u,F,M)$ be a fuzzy cellular automata and $a,b,c,d,e,f \in Z_3^*$, $m \geq 3$ and n be an even positive integer. Then, the rule matrix $F = T_R^0$ from Z_3^{mn} to Z_3^{mn} which takes the tth finite hexagonal configuration $C^{(t)}$ of order $m \times n$ to the $(t+1)$th state $C^{(t+1)}$ is given. We will prove that corresponding fuzzy matrix of grades of the local transition is given by*

$$F = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1,n-1} & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2,n-1} & t_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{m,n-1} & t_{mn} \end{pmatrix}_{m \times n}.$$

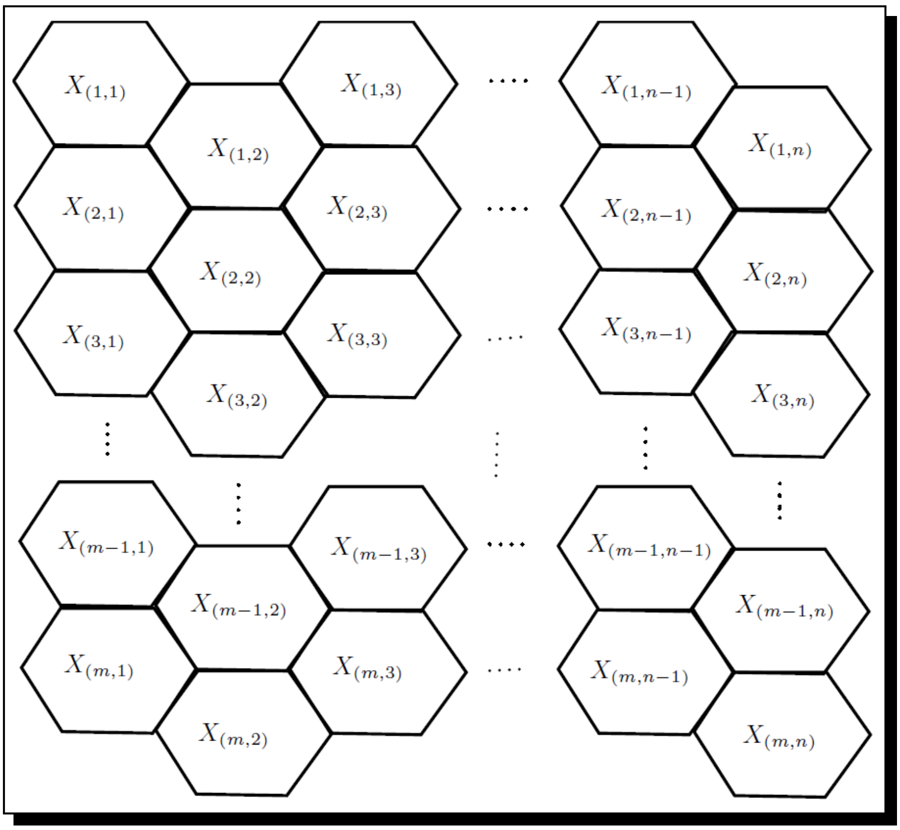

Figure 2. Hexagonal fuzzy cellular automata of order $m \times n$

Proof. Let $M=(A,Q,u,F,M)$ be a fuzzy cellular automata, where
A = 2 dimensional array of cells,
$Q=\{X_{i,j}\}$, $i=1,2,3,\dots,m$ and $j=1,2,3,\dots,n$ is a finite set of cell state,
u is a neighborhood of state of cells, if j is an even then the neighborhood of X_{ij} is given in eq. (2.1), if j is an odd then the neighborhood of X_{ij} is given in eq. (2.2),
$f:Q^6\times Q$ be a local transition rule,
μ is the set of grades of the local transition $\mu:Q^6\times F\to[0,1]$ is given by

$$\mu((aX_{(2,1)},fX_{(1,2)}),f((aX_{(2,1)},fX_{(1,2)}),(y_{(1,1)})))=\frac{y_{(1,1)}}{100}=t_{(1,1)}$$

$$\mu((aX_{(2,2)},bX_{(2,1)},cX_{(1,1)},eX_{(1,3)},fX_{(2,3)}),f((aX_{(2,2)},bX_{(2,1)},cX_{(1,1)},eX_{(1,3)},fX_{(2,3)}),(y_{(1,2)})))=\frac{y_{(1,2)}}{100}=t_{(1,2)}$$

$$\mu((aX_{(2,3)},bX_{(1,2)},fX_{(1,4)}),f((aX_{(2,3)},bX_{(1,2)},fX_{(1,4)}),(y_{(1,3)})))=\frac{y_{(1,3)}}{100}=t_{(1,3)}$$

$$\vdots$$

$$\mu((aX_{(2,n-1)},bX_{(1,n-2)},fX_{(1,n)}),f((aX_{(2,n-1)},bX_{(1,n-2)},fX_{(1,n)}),(y_{(1,n-1)})))=\frac{y_{(1,n-1)}}{100}=t_{(1,n-1)}$$

$$\mu((aX_{(2,n)},bX_{(2,n-1)},cX_{(1,n-1)}),f((aX_{(2,n)},bX_{(2,n-1)},cX_{(1,n-1)}),(y_{(1,n)})))=\frac{y_{(1,n)}}{100}=t_{(1,n)}$$

Consider $2\le k\le m-1$

$$\mu((aX_{k+1,1},dX_{k-1,1},eX_{k-1,2},fX_{k,2}),f((aX_{k+1,1},dX_{k-1,1},eX_{k-1,2},fX_{k,2}),(y_{(k,1)})))=\frac{y_{k,1}}{100}=t_{(k,1)}$$

$$\mu((aX_{k+1,2},bX_{k+1,1},cX_{k,1},dX_{k-1,2},eX_{k,3},fX_{k+1,3}),\\ f((aX_{k+1,2},dX_{k+1,1},cX_{k,1},dX_{k-1,2},eX_{k,3},fX_{k+1,3}),(y_{k,2})))=\frac{y_{(k,2)}}{100}=t_{(k,2)}$$

$$\vdots$$

$$\mu((aX_{k+1,n-1},bX_{k,n-2},cX_{k-1,n-2},dX_{k-1,n-1},eX_{k-1,n},fX_{k,n}),\\ f((aX_{k+1,n-1},bX_{k,n-2},cX_{k-1,n-2},dX_{k-1,n-1},eX_{k-1,n},fX_{k,n}),(y_{k,n-1})))=\frac{y_{(k,n-1)}}{100}=t_{(k,n-1)}$$

$$\mu((aX_{k+1,n},bX_{k+1,n-1},cX_{k,n-1},dX_{k-1,n}),\\ f((aX_{k+1,n},bX_{k+1,n-1},cX_{k,n-1},dX_{k-1,n})))=\frac{y_{(k,n)}}{100}=t_{(k,n)}$$

$$\mu((dX_{m-1,1},eX_{m-1,2},fX_{m,2}),f((dX_{m-1,1},eX_{m-1,2},fX_{m,2}),(y_{m,1})))=\frac{y_{(m,1)}}{100}=t_{(m,1)}$$

$$\mu((cX_{m,1},dX_{m-1,2},eX_{m,3}),f((cX_{m,1},dX_{m-1,2},eX_{m,3}),(y_{m,2})))=\frac{y_{(m,2)}}{100}=t_{(m,2)}$$

$$\vdots$$

$$\mu((bX_{m,n-2},cX_{m-1,n-2},dX_{m-1,n-1},eX_{m-1,n},fX_{m,n}),\\ f((bX_{m,n-2},cX_{m-1,n-2},dX_{m-1,n-1},eX_{m-1,n},fX_{m,n}),(y_{m,n-1})))=\frac{y_{(m,n-1)}}{100}=t_{(m,n-1)}$$

$$\mu((cX_{m,n-1},dX_{m-1,n}),f((cX_{m,n-1},dX_{m-1,n}),(y_{m,n})))=\frac{y_{(m,n)}}{100}=t_{(m,n)}$$

□

Theorem 3.2. *Let $M = (A,Q,u,F,M)$ be a fuzzy cellular automata and $a,b,c,d,e,f \in Z_3^*$, $m \geq 3$ and n be an odd positive integer. Then, the rule matrix T_R^1 from Z_3^{mn} to Z_3^{mn} which takes the tth finite hexagonal configuration $C^{(t)}$ of order $m \times n$ to the $(t+1)$th state $C^{(t+1)}$ is given. We will prove that corresponding fuzzy matrix of grades of the local transition is given by*

$$F = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1,n-1} & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2,n-1} & t_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{m,n-1} & t_{mn} \end{pmatrix}_{m\times n}.$$

Proof. The proof of Theorem 3.2 can be obtained by following similar steps as in the proof of Theorem 3.1. □

Example 3.1. Let $m = 3$ and $n = 4$ the fuzzy matrix of 2D finite fuzzy CA with hexagonal rule over Z_3 be as follows.

$$F = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{pmatrix},$$

where $a,b,c,f = 1$ and $d,e = 2$ then $X_{11} = X_{14} = X_{23} = X_{31} = X_{33} = 1$, $X_{12} = X_{21} = X_{22} = X_{24} = X_{34} = 2$, $X_{13} = X_{32} = 0$.

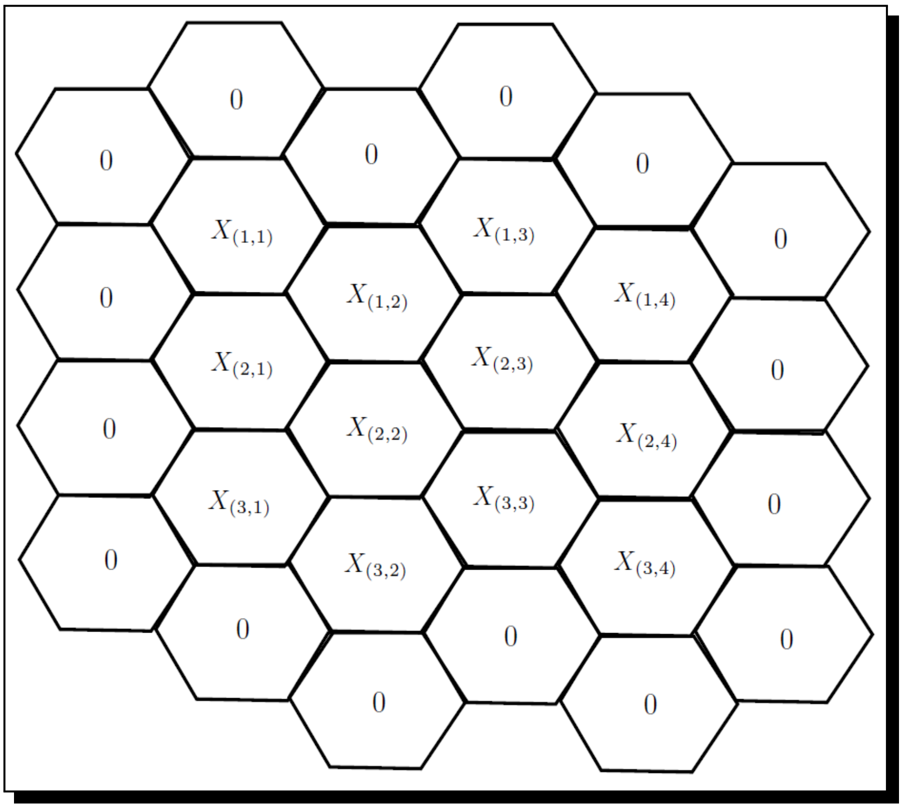

Figure 3. Information matrix of the HFFCA under NBC for $m = 3$, $n = 4$

Solution. By using Theorem 3.1, we have the following fuzzy matrix of grades of the local transition is given by

$\mu((aX_{21},0,0,0,0,fX_{12}),f((aX_{21},0,0,0,0,fX_{12}),(y_{11}))=\frac{y_{11}}{100}=t_{11}$

$\mu((2,0,0,0,0,2),f((2,0,0,0,0,2),(4)))=\frac{4}{100}=0.04$

$\mu((aX_{22},bX_{21},cX_{11},0,eX_{13},fX_{23}),f((aX_{22},bX_{21},cX_{11},0,eX_{13},fX_{23}),(y_{11}))=\frac{y_{12}}{100}=t_{12}$

$\mu((2,2,1,0,0,1),f((2,2,1,1),(6)))=\frac{6}{100}=0.06$

$\mu((aX_{23},bX_{12},0,0,0,fX_{14}),f((aX_{23},bX_{12},0,0,0,fX_{14}),(y_{13}))=\frac{y_{13}}{100}=t_{13}$

$\mu((1,2,0,0,0,1),f((1,2,1),(5)))=\frac{5}{100}=0.05$

$\mu((aX_{24},bX_{23},cX_{13},0,0,0),f((aX_{24},bX_{23},cX_{13},0,0,0),(y_{14}))=\frac{y_{14}}{100}=t_{14}$

$\mu((2,1,1,0,0,0),f((2,1,1),(4)))=\frac{4}{100}=0.04$

$\mu((aX_{31},0,0,dX_{11},eX_{12},fX_{22}),f((aX_{31},0,0,dX_{11},eX_{12},fX_{22}),(y_{21}))=\frac{y_{21}}{100}=t_{21}$

$\mu((1,0,0,2,4,2),f((1,2,4,2),(9)))=\frac{9}{100}=0.09$

$\mu((aX_{32},bX_{31},cX_{21},dX_{12},eX_{23},fX_{33}),f((aX_{32},bX_{31},cX_{21},dX_{12},eX_{23},fX_{33}),(y_{22}))=\frac{y_{22}}{100}=t_{22}$

$\mu((0,1,2,4,2,2),f((1,2,4,2,2),(11)))=\frac{11}{100}=0.11$

$\mu((aX_{33},bX_{22},cX_{12},dX_{13},eX_{14},fX_{24}),f((aX_{33},bX_{22},cX_{12},dX_{13},eX_{14},fX_{24}),(y_{23}))=\frac{y_{23}}{100}=t_{23}$

$\mu((1,2,2,0,2,2),f((1,2,2,2,2),(9)))=\frac{9}{100}=0.09$

$\mu((aX_{34},bX_{33},cX_{23},dX_{14},0,0),f(aX_{34},bX_{33},cX_{23},dX_{14},0,0),(y_{24}))=\frac{y_{24}}{100}=t_{24}$

$\mu((2,1,1,2,0,0),f((2,1,1,2),(6)))=\frac{6}{100}=0.06$

$\mu((0,0,0,dX_{21},eX_{22},fX_{32}),f((0,0,0,dX_{21},eX_{22},fX_{32}),(y_{31}))=\frac{y_{31}}{100}=t_{31}$

$\mu((0,0,0,4,4,0),f((4,4),(8)))=\frac{8}{100}=0.08$

$\mu((0,0,cX_{31},dX_{22},eX_{33},0),f((0,0,cX_{31},dX_{22},eX_{33},0),(y_{32}))=\frac{y_{32}}{100}=t_{32}$

$\mu((0,0,1,4,2,0),f((1,4,2),(7)))=\frac{7}{100}=0.07$

$\mu((0,bX_{32},cX_{22},dX_{23},eX_{24},fX_{34}),f((0,bX_{32},cX_{22},dX_{23},eX_{24},fX_{34}),(y_{33}))=\frac{y_{33}}{100}=t_{33}$

$\mu((0,0,2,2,4,2),f((2,2,4,2),(10)))=\frac{10}{100}=0.1$

$\mu((0,0,cX_{33},dX_{24},0,0),f((0,0,cX_{33},dX_{24},0,0),(y_{34}))=\frac{y_{34}}{100}=t_{34}$

$\mu((0,0,1,4,0,0), f((1,4),(5))) = \dfrac{5}{100} = 0.05$

The fuzzy matrix of the rule matrix

$$F = \begin{pmatrix} 0.04 & 0.06 & 0.04 & 0.04 \\ 0.09 & 0.11 & 0.09 & 0.06 \\ 0.08 & 0.007 & 0.1 & 0.05 \end{pmatrix}.$$

4. Reversibility of HFFCA

Definition 4.1 (Reversibility of CA)**.** A cellular automata rule is called reversible if there a another rule that makes the automaton retrace is computation steps backwards in time [4].

There is no specific algorithm that would decide whether a given local rule is reversible or not in [3]. In this paper, by using the matrix algebra it is shown that HFCA is reversible if the number of columns of T_R^0 are even and not reversible the number of columns are odd.

So in this section, we obtain fuzzy matrix of the inverse matrix of T_R^0.

Example 4.1. Let $m = 3$ and $n = 2$ the fuzzy matrix of 2D finite fuzzy CA with hexagonal rule over Z_3 be as follows.

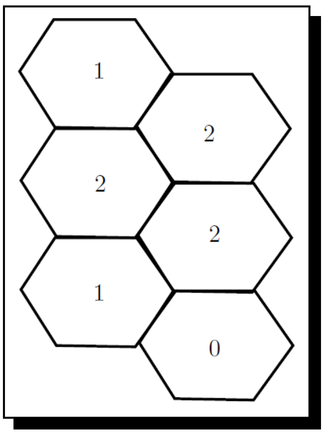

Figure 4. Information matrix of the HFFCA under NBC for $m = 3$, $n = 2$

where $a,b,c,f = 1$ and $d,e = 2$.

The rule matrix of the cellular automata $T_R^0 = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$

Since the columns are even in [10], it is reversible.

Solution. By using Theorem 3.1, for given X values at time t, $X = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$. We fine the corresponding Y values at time $t+1$, $Y = \begin{pmatrix} 4 & 5 \\ 9 & 7 \\ 8 & 5 \end{pmatrix}$ using T_R^0 matrix.

Then, Y matrix converted to the row matrix $\begin{pmatrix} 4 & 5 & 9 & 7 & 8 & 5 \end{pmatrix}$.

The inverse of rule matrix T_R^0 is given by

$$\begin{pmatrix} -0.5714 & 0.7143 & 0.2857 & 0.2857 & -0.5714 & -1.1429 \\ 0.7143 & -1.1429 & 1.1429 & -0.8571 & 1.7143 & -0.5714 \\ 0.1429 & 0.5714 & -0.5714 & 0.4286 & -0.8571 & 0.2857 \\ 0.1429 & -0.4286 & 0.4286 & -0.5714 & 1.1429 & 0.2857 \\ -0.1429 & 0.4286 & -0.4286 & 0.5714 & -1.1429 & 0.7143 \\ -0.2857 & -0.1429 & 0.1429 & 0.1429 & 0.7143 & -0.5714 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 & 9 & 7 & 8 & 5 \end{pmatrix}$$
$$\begin{pmatrix} -0.5714 & 0.7143 & 0.2857 & 0.2857 & -0.5714 & -1.1429 \\ 0.7143 & -1.1429 & 1.1429 & -0.8571 & 1.7143 & -0.5714 \\ 0.1429 & 0.5714 & -0.5714 & 0.4286 & -0.8571 & 0.2857 \\ 0.1429 & -0.4286 & 0.4286 & -0.5714 & 1.1429 & 0.2857 \\ -0.1429 & 0.4286 & -0.4286 & 0.5714 & -1.1429 & 0.7143 \\ -0.2857 & -0.1429 & 0.1429 & 0.1429 & 0.7143 & -0.5714 \end{pmatrix}$$

$\mu((2.8572, -5.7145, 5.1426, -3.0002, 3.4288, -0.7145),$
$$f((2.8572, -5.7145, 5.1426, -3.0002, 3.4288, -0.7145, (1.0006))) = \frac{x_{11}}{100} = s_{11} = \frac{1.0006}{100} = 0.01$$

$\mu((1.1428, 5.7145, 5.1426, 3.0002, 3.4288, 0.7145),$
$$f((1.1428, 5.7145, 5.1426, 3.0002, 3.4288, 0.7145), (2.0006))) = \frac{x_{12}}{100} = s_{12} = \frac{2.0006}{100} = 0.02$$

$\mu((1.1428, 5.7145, -5.1426, 3.0002, -3.4288, 0.7145),$
$$f((1.1428, 5.7145, -5.1426, 3.0002, -3.4288, 0.7145), (2.0006))) = \frac{x_{21}}{100} = s_{21} = \frac{2.0006}{100} = 0.02$$

$\mu((1.1428, -4.2855, 3.8574, 3.9998, 4.5712, 0.7145),$
$$f((1.1428, -4.2855, 3.8574, 3.9998, 4.5712, 0.7145), (2.0006))) = \frac{x_{22}}{100} = s_{22} = \frac{2.0006}{100} = 0.02$$

$\mu((-2.2856, 8.5715, -7.7139, 8.0003, -9.1432, 3.5715),$
$$f((-2.2856, 8.5715, -7.7139, 8.0003, -9.1432, 3.5715), (1.0006))) = \frac{x_{31}}{100} = s_{31} = \frac{1.0006}{100} = 0.01$$

$\mu((-4.5716, -2.857, 2.5713, 1.9999, 5.7144, -2.857),$
$$f((-4.5716, -2.857, 2.5713, 1.9999, 5.7144, -2.857), (0))) = \frac{x_{32}}{100} = s_{32} = \frac{0}{100} = 0$$

The fuzzy matrix of the inverse matrix $\begin{pmatrix} 0.01 & 0.02 \\ 0.02 & 0.02 \\ 0.01 & 0 \end{pmatrix}$.

5. Conclusion

Fuzzification of 2D hexagonal finite cellular automata are studied over the field Z_3. First, the definition of principle of fuzzy cellular automata is given. The fuzzy matrix of the transition T_R^0 matrix of 2D hexagonal finite cellular automata is computed. If the number of column of transition matrix T_R^0 is even, then the fuzzy matrix of the inverse matrix T_R^0 is also computed. In future by using above concept, a fuzzy mathematical model for predicting spread of the forest fire may be obtained.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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