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Four Inequalities for the Exponential Integral Functions

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Abstract In this paper, we present four inequalities involving the exponential integral function.

1. Introduction

For any $x > 0$ and $n \in \mathbb{N}_0$, we denote

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt .$$

The function E_n is called the exponential integral function (see [1]).

In 2006, Laforgia and Natalini [2] presented an inequality that

$$E_m(x)E_n(x) \geq E_{\frac{m+n}{2}}(x),$$

where $x > 0$ and $m, n \in \mathbb{N}_0$.

In 2012, Sulaiman [3] showed that E_n is non-increasing for all $n \in \mathbb{N}_0$. Moreover, he also presented following inequalities.

For any $x, y > 0$, $p > 1 = \frac{1}{p} + \frac{1}{q}$, and $m + n, pm, qn \in \mathbb{N}_0$,

$$E_{m+n}\left(\frac{x}{p} + \frac{y}{q}\right) \leq E_{pm}^{1/p}(x) E_{qn}^{1/q}(y). \tag{1.1}$$

For any $x, y > 1$, $n \in \mathbb{N}_0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x + y \leq xy$,

$$E_n(xy) \leq E_n^{1/p}(px) E_n^{1/q}(qy). \tag{1.2}$$

For any $x > 0$, $0 < y < 1$, $n \in \mathbb{N}_0$, $0 < p < 1 = \frac{1}{p} + \frac{1}{q}$ and $x + y \geq xy$,

$$E_n(xy) \geq E_n^{1/p}(px) E_n^{1/q}(qy). \tag{1.3}$$

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For any $x, y > 1$, $n \in \mathbb{N}_0$, $p > 1$, $0 < r < 1$ and $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{r} + \frac{1}{s}$,

$$E_n(xy) \geq E_n^{1/r}\left(\frac{rx^p}{p}\right) E_n^{1/s}\left(\frac{sy^q}{q}\right). \tag{1.4}$$

In this paper, we present the generalizations for inequalities (1.1), (1.2), (1.3) and (1.4).

2. Results

Theorem 2.1. *Assume that* $n-1 \in \mathbb{N}$, $x_1, x_2, \ldots, x_n > 0$, $p_1, p_2, \ldots, p_n > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $m = \sum_{i=1}^n m_i$, *and* $p_1m_1, p_2m_2, \ldots, p_nm_n, m \in \mathbb{N}_0$. *Then*

$$E_m\left(\sum_{i=1}^n \frac{x_i}{p_i}\right) \leq \prod_{i=1}^n E_{p_im_i}^{1/p_i}(x_i).$$

Proof. By the generalized Hölder inequality,

$$\begin{aligned}
E_m\left(\sum_{i=1}^n \frac{x_i}{p_i}\right) &= \int_1^\infty t^{-\sum_{i=1}^n m_i} e^{-t\sum_{i=1}^n \frac{x_i}{p_i}}\, dt \\
&= \int_1^\infty \prod_{i=1}^n t^{-m_i} e^{-t\frac{x_i}{p_i}}\, dt \\
&\leq \prod_{i=1}^n \left(\int_1^\infty t^{-p_im_i} e^{-x_it}\, dt\right)^{1/p_i} \\
&= \prod_{i=1}^n E_{p_im_i}^{1/p_i}(x_i).
\end{aligned}$$

□

We note on Theorem 2.1 that if $n = 2$ then we obtain the inequality (1.1).

Theorem 2.2. *Assume that* $n-1 \in \mathbb{N}, m \in \mathbb{N}_0, x_1, x_2, \ldots, x_n > 1, p_1, p_2, \ldots, p_n > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$ *and* $\sum_{i=1}^n x_i \leq \prod_{i=1}^n x_i$. *Then*

$$E_m\left(\prod_{i=1}^n x_i\right) \leq \prod_{i=1}^n E_m^{1/p_i}(p_ix_i).$$

Proof. Since E_m is non-increasing, it follows that

$$\begin{aligned}
E_m\left(\prod_{i=1}^n x_i\right) &\leq E_m\left(\sum_{i=1}^n x_i\right) \\
&= \int_1^\infty t^{-m} e^{-t\sum_{i=1}^n x_i}\, dt \\
&= \int_1^\infty \prod_{i=1}^n t^{-\frac{m}{p_i}} e^{-x_it}\, dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$E_m\left(\prod_{i=1}^{n} x_i\right) \le \prod_{i=1}^{n}\left(\int_1^{\infty} t^{-m}e^{-p_i x_i t}\,dt\right)^{1/p_i}$$

$$= \prod_{i=1}^{n} E_m^{1/p_i}(p_i x_i)\,.$$

□

We note on Theorem 2.2 that if $n=2$ then we obtain the inequality (1.2).

Theorem 2.3. *Assume that* $n \in \mathbb{N}$, $x_1,\ldots,x_n > 0$, $0 < y < 1$, $m \in \mathbb{N}_0$, $0 < p_1,\ldots,p_n < 1 = \left(\sum_{i=1}^{n}\frac{1}{p_i}\right)+\frac{1}{q}$ *and* $y+\sum_{i=1}^{n} x_i \ge y\prod_{i=1}^{n} x_i$. *Then*

$$E_m\left(y\prod_{i=1}^{n} x_i\right) \ge E_m^{1/q}(qy)\prod_{i=1}^{n} E_m^{1/p_i}(p_i x_i)\,.$$

Proof. Since E_m is non-increasing, it follows that

$$E_m\left(y\prod_{i=1}^{n} x_i\right) \ge E_m\left(y+\sum_{i=1}^{n} x_i\right)$$

$$= \int_1^{\infty} t^{-m}e^{-t(y+\sum_{i=1}^{n} x_i)}\,dt$$

$$= \int_1^{\infty} t^{-\frac{m}{q}}e^{-yt}\prod_{i=1}^{n} t^{-\frac{m}{p_i}}e^{-x_i t}\,dt\,.$$

By the generalized reverse Hölder inequality,

$$E_m\left(y\prod_{i=1}^{n} x_i\right) \ge \left(\int_1^{\infty} t^{-m}e^{-qyt}\,dt\right)^{1/q}\prod_{i=1}^{n}\left(\int_1^{\infty} t^{-m}e^{-p_i x_i t}\,dt\right)^{1/p_i}$$

$$= E_m^{1/q}(qy)\prod_{i=1}^{n} E_m^{1/p_i}(p_i x_i)\,.$$

□

We note on Theorem 2.3 that if $n=1$ then we obtain the inequality (1.3).

Theorem 2.4. *Assume that* $n \in \mathbb{N}$, $x_1,\ldots,x_n, y > 1$, $m \in \mathbb{N}_0$, $p_1,\ldots,p_n > 1$, $0 < r_1,\ldots,r_n < 1$ *and* $\left(\sum_{i=1}^{n}\frac{1}{p_i}\right)+\frac{1}{q} = 1 = \left(\sum_{i=1}^{n}\frac{1}{r_i}\right)+\frac{1}{s}$. *Then*

$$E_m\left(y\prod_{i=1}^{n} x_i\right) \ge E_m^{1/s}\left(\frac{sy^q}{q}\right)\prod_{i=1}^{n} E_m^{1/r_i}\left(\frac{r_i x_i^{p_i}}{p_i}\right)$$

Proof. We note that

$$y\prod_{i=1}^{n} x_i \le \left(\sum_{i=1}^{n}\frac{x_i^{p_i}}{p_i}\right)+\frac{y^q}{q}\,.$$

Since E_m is non-increasing, it follows that

$$E_m\left(y\prod_{i=1}^{n}x_i\right)\geq E_m\left(\frac{y^q}{q}+\sum_{i=1}^{n}\frac{x_i^{p_i}}{p_i}\right)$$

$$=\int_1^\infty t^{-m}e^{-t\left(\frac{y^q}{q}+\sum_{i=1}^{n}\frac{x_i^{p_i}}{p_i}\right)}dt$$

$$=\int_1^\infty t^{-\frac{m}{s}}e^{-\frac{y^q}{q}t}\prod_{i=1}^{n}t^{-\frac{m}{r_i}}e^{-\frac{x_i^{p_i}}{p_i}t}dt\,.$$

By the generalized reverse Hölder inequality,

$$E_m\left(y\prod_{i=1}^{n}x_i\right)\geq\left(\int_1^\infty t^{-m}e^{-s\frac{y^q}{q}t}dt\right)^{1/s}\prod_{i=1}^{n}\left(\int_1^\infty t^{-m}e^{-r_i\frac{x_i^{p_i}}{p_i}t}dt\right)^{1/r_i}$$

$$=E_m^{1/s}\left(\frac{sy^q}{q}\right)\prod_{i=1}^{n}E_m^{1/r_i}\left(\frac{r_ix_i^{p_i}}{p_i}\right)\,. \qquad \square$$

We note on Theorem 2.4 that if $n=1$ then we obtain the inequality (1.4).

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