Communications in Mathematics and Applications
Vol. 13, No. 1, pp. 19–26, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i1.1689

http://www.rgnpublications.com

Research Article

A New Definition of Fractional Derivatives With Mittag-Leffler Kernel of Two Parameters

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Received: October 2, 2021 **Accepted:** December 9, 2021

Abstract. In this paper, a new fractional derivative with Mittag-Leffler kernel of two parameters is proposed. Several functional properties of this derivative are explained and applied to solve the fractional time Fourier's law equation.

Keywords. Fractional derivatives, Non-local kernel, Non-singular kernel, Mittag-Leffler function of two parameters, Fractional time Fourier's law equation

Mathematics Subject Classification (2020). 26A33, 34K37

1. Introduction

A derivative was presented by Atangana and Baleanu and followed by some valuable results [1–4]. The fractional derivatives introduced by Caputo and Riemann-Liouville have been applied in several real world problems by enormous success [6, 7] however there are lot of things to do in this topic. So, we proposed the new definition of fractional derivatives with Mittag-Leffler kernel of two parameters.

2. Preliminaries

2.1 Definition

Let $f \in H^1(a,b)$, $a < b$, $a \in [-\infty, t)$ and $\alpha \in [0,1]$ then, the Caputo fractional time derivative with non-singular kernel [6, 7] is defined by

$$ {}_aD_t^{(\alpha)} f(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau, \tag{2.1}$$

where $N(\alpha)$ is a normalization function such that $N(0) = N(1) = 1$.

2.2 Mittag Leffler Function

The Mittag-Leffler function [9, 17] is the solution of fractional ordinary differential equation given by

$$\frac{d^{\alpha}y}{dx^{\alpha}} = ay, \quad 0 < \alpha < 1. \tag{2.2}$$

The Mittag-Leffler function and its general form are then considered as non-local functions.

Let us consider the following generalized Mittag-Leffler function of two parameters:

$$t^{\beta-1}E_{\alpha,\beta}(-t^{\alpha}) = t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(\alpha k+\beta)}. \tag{2.3}$$

Using a kernel which is non-local and non-singular, namely the Mittag-Leffler function, Atangana and Baleanu proposed a new definition of fractional derivatives as below:

2.3 Definition

Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$ then, the definitions of the AB fractional derivative with non-local and non-singular kernel [2] are given by:

$$^{ABC}{}_aD_t^{\alpha} f(t) = \frac{N(\alpha)}{1-\alpha}\int_a^t f'(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau, \tag{2.4}$$

$$^{ABR}{}_aD_t^{\alpha} f(t) = \frac{N(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau \tag{2.5}$$

with $N(\alpha)$ being a normalisation function mentioned in Caputo fractional time derivative.

3. A New Definition of Fractional Derivative

By changing the kernel $E_{\alpha}\left[-\frac{\alpha t^{\alpha}}{1-\alpha}\right]$ with the Mittag-Leffler function $t^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]$ of two parameter and $\frac{N(\alpha)}{1-\alpha}$ by $\frac{B(\alpha,\beta)}{2-\alpha-\beta}$, we obtain the following new definition of fractional derivative:

$${}_aD_t^{\alpha,\beta} f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_a^t f'(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau. \tag{3.1}$$

Thus, the following derivative is proposed.

3.1 Definition

Let $f \in H^1(a,b)$, $a < b$, $0 \le \alpha$, $\beta \le 1$ then, the definition of the new fractional derivative is given:

$$^{SABC}{}_aD_t^{\alpha,\beta} f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_a^t f'(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau, \tag{3.2}$$

where $B(\alpha,\beta)$ is a normalization function such that $B(\alpha,\beta) = N(\alpha+\beta-1)$.

If $\alpha+\beta = 1$, the original function will not be recovered except when at the origin the function vanishes. To avoid this problem, we intend Definition 3.2.

3.2 Definition

Let $f \in H^1(a,b), a < b, 0 \le \alpha, \beta \le 1$ then, the definition of the new fractional derivative is given:

$$^{SABR}{}_aD_t^{\alpha,\beta} f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t f(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau. \tag{3.3}$$

Obviously, equations (3.2) and (3.3) have a non-local and non-singular kernel.

4. Properties of the New Derivatives

In this section, initially we present the relation between both derivatives defined above using Laplace transform.

Theorem 4.1. *If $\beta = 1$, the derivatives ${}^{SABC}{}_aD_t^{\alpha,\beta} f(t)$ are equal to ${}^{ABC}{}_aD_t^{\alpha} f(t)$ and ${}^{\mathrm{SABR}}{}_aD_t^{\alpha,\beta} f(t)$ are equal to ${}^{ABR}{}_aD_t^{\alpha} f(t)$.*

Proof. The proofs are just by observation of definitions of derivatives. □

Theorem 4.2. *Let $f \in H^1(a,b)$, $a < b$ and $\alpha,\beta \in [0,1]$ then the following relation is obtained:*

$$ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) = {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) + H_0(t). \tag{4.1}$$

Proof. By using the equation (3.2) and the Laplace transform [9] applied on both sides, we get

$$\mathscr{L}\left\{{}^{SABC}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta}[p\mathscr{L}\{f(t),p\} - f(0)]}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}}.$$

Therefore, we have

$$\mathscr{L}\left\{{}^{SABC}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta+1}\mathscr{L}\{f(t),p\}}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}} - \frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta} f(0)}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{4.2}$$

Similarly, by using the equation (3.3) and the Laplace transform [9] applied on both sides, we get

$$\mathscr{L}\left\{{}^{SABR}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta} p \left[\frac{p^{\alpha-\beta}}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}} \mathscr{L}\{f(t),p\}\right].$$

Therefore, we have

$$\mathscr{L}\left\{{}^{SABR}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta+1}\mathscr{L}\{f(t),p\}}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{4.3}$$

Using equation (4.2), we have

$$\mathscr{L}\left\{{}^{SABC}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \mathscr{L}\left\{{}^{SABR}{}_0D_t^{\alpha,\beta} f(t), p\right\} - \frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta} f(0)}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{4.4}$$

Applying the inverse Laplace on both sides of eq. (4.4) we obtain

$$ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) = {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) - \frac{B(\alpha,\beta)}{2-\alpha-\beta} f(0) t^{\beta-1} E_{\alpha,\beta}\left(-\frac{\alpha+\beta-1}{2-\alpha-\beta} t^\alpha\right). \tag{4.5}$$

This implies that

$$ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) = {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) + H_0(t),$$

where

$$H_0(t) = -\frac{B(\alpha,\beta)}{2-\alpha-\beta} f(0) t^{\beta-1} E_{\alpha,\beta}\left(-\frac{\alpha+\beta-1}{2-\alpha-\beta} t^\alpha\right).$$

This completes the proof. □

Theorem 4.3. *Let f be a continuous function defined on a closed interval $[a,b]$. Then the following inequality is obtained on $[a,b]$:*

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) \right\| < \frac{B(\alpha,\beta)}{2-\alpha-\beta} K. \tag{4.6}$$

Proof. Using $\|h(t)\| = \max_{a\le t\le b} |h(t)|$, we have

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t)\right\| = \left\| \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right\|$$
$$< \frac{B(\alpha,\beta)}{2-\alpha-\beta}\left\|\frac{d}{dt}\int_0^t f(\tau)d\tau\right\|$$
$$= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\|f(t)\|.$$

This implies that

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t)\right\| < \frac{B(\alpha,\beta)}{2-\alpha-\beta}K,$$

where

$$K = \|f(t)\|.$$

This completes the proof. □

Theorem 4.4. *The S.A.B. derivative in Riemann and Caputo sense possess the Lipschitz condition. In other words, for a given functions f and h, the following inequalities can be established*

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABR}{}_0D_t^{\alpha,\beta} h(t)\right\| \le H(t)\|f(t)-h(t)\| \tag{4.7}$$

and

$$\left\| {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABC}{}_0D_t^{\alpha,\beta} h(t)\right\| \le H(t)\|f(t)-h(t)\|. \tag{4.8}$$

Proof.

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABR}{}_0D_t^{\alpha,\beta} h(t)\right\|$$
$$= \left\| \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau \right.$$
$$\left. -\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t h(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right\|.$$

Using the Lipschitz condition of the first order derivative, we can find a small positive constant ϕ_1 such that:

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABR}{}_0D_t^{\alpha,\beta} h(t)\right\|$$
$$< \frac{B(\alpha,\beta)\phi_1}{2-\alpha-\beta}t^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]\left\|\int_0^t f(\tau)d\tau - \int_0^t h(\tau)d\tau\right\| \tag{4.9}$$

and then the following result is obtained

$$\left\| {}^{SABR}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABR}{}_0D_t^{\alpha,\beta} h(t)\right\| < \frac{B(\alpha,\beta)\phi_1}{2-\alpha-\beta}t^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]\|f(t)-h(t)\|\,t$$
$$= H(t)\|f(t)-h(t)\|,$$

where

$$H(t) = \frac{B(\alpha,\beta)\phi_1}{2-\alpha-\beta}t^{\beta}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]$$

which produces the requested result. The proof of (4.8) can be obtained similarly. □

5. Derivation of New Fractional Integral

Let for a natural number n, f be an n-times differentiable and $f^{(k)}(0)=0$, for $k=1,2,3,\dots,n$, then by observation we get

$$^{SABC}{}_0D_t^{\alpha,\beta}\left[\frac{d^n}{dt^n}f(t)\right]=\frac{d^n}{dt^n}\left[^{SABC}{}_0D_t^{\alpha,\beta}f(t)\right]. \tag{5.1}$$

Let

$$^{SABC}{}_0D_t^{\alpha,\beta}f(t)=u(t) \tag{5.2}$$

$$\Rightarrow\quad \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau=u(t)$$

$$\Rightarrow\quad \mathscr{L}\left\{\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau,p\right\}=\mathscr{L}\{u(t),p\}$$

Using the convolution theorem, we get

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha-\beta}[p\mathscr{L}\{f(t),p\}]}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}=\mathscr{L}\{u(t),p\}$$

$$\Rightarrow\quad \mathscr{L}\{f(t),p\}=\frac{2-\alpha-\beta}{B(\alpha,\beta)}\frac{1}{p^{-\beta+1}}\mathscr{L}\{u(t),p\}+\frac{\alpha+\beta-1}{B(\alpha,\beta)}\frac{1}{p^{\alpha-\beta+1}}\mathscr{L}\{u(t),p\}.$$

By taking the inverse Laplace transform, we get the unique solution

$$f(t)=\begin{cases}\dfrac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^t u(y)(t-y)^{-\beta}dy+\dfrac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^t u(y)(t-y)^{\alpha-\beta}dy & \beta\neq1\\ \dfrac{1-\alpha}{N(\alpha)}u(t)+\dfrac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_0^t u(y)(t-y)^{\alpha-1}dy & \beta=1\end{cases} \tag{5.3}$$

5.1 Definition

The fractional integral in two parameters corresponds to the new fractional derivative with non-local and non-singular kernel is defined as

$$^{S}_{a}I_t^{\alpha,\beta}f(t)=\begin{cases}\dfrac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_a^t f(y)(t-y)^{-\beta}dy+\dfrac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_a^t f(y)(t-y)^{\alpha-\beta}dy & \beta\neq1\\ \dfrac{1-\alpha}{N(\alpha)}f(t)+\dfrac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy & \beta=1\end{cases} \tag{5.4}$$

When $\alpha=0$, $\beta=1$ we recover the initial function, and if $\alpha=1$, $\beta=1$, we obtain the ordinary integral.

6. Application to Thermal Science

Fractional time Fourier's Law Equation. The Fourier's law [8, 10, 13] is defined by the classical parabolic equation as

$$\chi\frac{\partial^2T(x,t)}{\partial x^2}-\frac{\partial T(x,t)}{\partial x}=0, \tag{6.1}$$

where

$$\chi=\frac{k}{\rho C_p}.$$

χ is the thermal diffusivity,k is the thermal conductivity, ρ is density, C_p is the specific heat capacity, and T is the temperature conduction in a planar medium with constant properties. Taking into consideration eq. (6.1) and assuming that time derivative is fractional and space derivative is ordinary, the temporal fractional equation will be as follows

$$^{SABC}{}_0D_t^{\alpha,\beta}T(x,t)-\chi\frac{\partial^2T(x,t)}{\partial x^2}=0. \tag{6.2}$$

A particular solution to eq. (6.2) will be in the following form

$$T(x,t)=T_0e^{-i\bar{k}x}u(t). \tag{6.3}$$

Substituting eq. (6.3) into eq. (6.2), we obtain

$$T_0e^{-i\bar{k}x SABC}{}_0D_t^{\alpha,\beta}u(t)-\chi(-i\bar{k})^2T_0e^{-i\bar{k}x}u(t)=0$$

$$\Rightarrow\quad T_0e^{-i\bar{k}x}\left[^{SABC}{}_0D_t^{\alpha,\beta}u(t)+\chi\bar{k}^2u(t)\right]=0$$

$$^{SABC}{}_0D_t^{\alpha,\beta}u(t)+\omega u(t)=0 \tag{6.4}$$

where $\omega=\chi\bar{k}^2\sigma_t^{2-\alpha-\beta}$ is the angular frequency.

The numerical approximation to eq. (6.4) is given by

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^t u'(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^\alpha}{2-\alpha-\beta}\right]d\tau+\omega u(t)=0.$$

By taking Laplace transform on both sides we get

$$\mathscr{L}\left\{^{SABC}{}_0D_t^{\alpha,\beta}u(t),p\right\}+\omega\mathscr{L}\{u(t),p\}=0$$

$$\Rightarrow\quad \frac{B(\alpha,\beta)}{2-\alpha-\beta}\,\frac{p^{\alpha-\beta}[p\mathscr{L}\{u(t),p\}-u(0)]}{p^\alpha+\frac{\alpha+\beta-1}{2-\alpha-\beta}}+\omega\mathscr{L}\{u(t),p\}=0$$

$$\Rightarrow\quad \frac{B(\alpha,\beta)}{2-\alpha-\beta}\,\frac{[p^{\alpha-\beta+1}\mathscr{L}\{u(t),p\}-u(0)p^{\alpha-\beta}]}{p^\alpha+\frac{\alpha+\beta-1}{2-\alpha-\beta}}+\omega\mathscr{L}\{u(t),p\}=0$$

$$\Rightarrow\quad \frac{B(\alpha,\beta)p^{\alpha-\beta+1}}{(2-\alpha-\beta)p^\alpha+\alpha+\beta-1}\mathscr{L}\{u(t),p\}-\frac{B(\alpha,\beta)p^{\alpha-\beta}u(0)}{(2-\alpha-\beta)p^\alpha+\alpha+\beta-1}+\omega\mathscr{L}\{u(t),p\}=0$$

$$\Rightarrow\quad \left[\frac{B(\alpha,\beta)p^{\alpha-\beta+1}}{(2-\alpha-\beta)p^\alpha+\alpha+\beta-1}+\omega\right]\mathscr{L}\{u(t),p\}-\frac{B(\alpha,\beta)p^{\alpha-\beta}u(0)}{(2-\alpha-\beta)p^\alpha+\alpha+\beta-1}=0$$

$$\Rightarrow\quad \left[\frac{B(\alpha,\beta)p^{\alpha-\beta+1}+\left((2-\alpha-\beta)p^\alpha+\alpha+\beta-1\right)\omega}{(2-\alpha-\beta)p^\alpha+\alpha+\beta-1}\right]\mathscr{L}\{u(t),p\}-\frac{B(\alpha,\beta)p^{\alpha-\beta}u(0)}{(2-\alpha-\beta)p^\alpha+\alpha+\beta-1}=0$$

$$\Rightarrow\quad \left[B(\alpha,\beta)p^{\alpha-\beta+1}+\left((2-\alpha-\beta)p^\alpha+\alpha+\beta-1\right)\omega\right]\mathscr{L}\{u(t),p\}=B(\alpha,\beta)p^{\alpha-\beta}u(0)$$

$$\Rightarrow\quad \mathscr{L}\{u(t),p\}=\frac{B(\alpha,\beta)p^{\alpha-\beta}u(0)}{B(\alpha,\beta)p^{\alpha-\beta+1}+\left((2-\alpha-\beta)p^\alpha+\alpha+\beta-1\right)\omega}$$

$$\Rightarrow\quad \mathscr{L}\{u(t),p\}=\frac{u(0)}{p}\,\frac{1}{1+\frac{((2-\alpha-\beta)p^\beta+(\alpha+\beta-1)p^{\beta-\alpha})\omega}{B(\alpha,\beta)}}$$

$$\Rightarrow\quad \mathscr{L}\{u(t),p\}=\frac{u(0)}{p}\sum_{k=0}^{\infty}\left[-\frac{\left((2-\alpha-\beta)p^\beta+(\alpha+\beta-1)p^{\beta-\alpha}\right)\omega}{B(\alpha,\beta)}\right]^k$$

$$\Rightarrow\quad \mathscr{L}\{u(t),p\}=u(0)\sum_{k=0}^{\infty}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k\left(1+\frac{\alpha+\beta-1}{2-\alpha-\beta}p^{-\alpha}\right)^k p^{\beta k-1}$$

$$\Rightarrow \quad u(t)=u(0)\sum_{k=0}^{\infty}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k\int_0^t\left[\delta(\tau)+\frac{k(\alpha+\beta-1)}{\Gamma(\alpha)(2-\alpha-\beta)}\tau^{\alpha-1}\right]\frac{(t-\tau)^{-\beta k}}{\Gamma(1-\beta k)}d\tau$$

$$\Rightarrow \quad u(t)=u(0)\sum_{k=0}^{\infty}\frac{1}{\Gamma(1-\beta k)}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k\left\{\left(t^{-\beta k+1}-t^{-\beta k}\right)\right.$$
$$\left.+\frac{k(\alpha+\beta-1)}{\Gamma(\alpha)(2-\alpha-\beta)}\frac{\Gamma(\alpha)\Gamma(-\beta k+1)}{\Gamma(\alpha-\beta k+1)}t^{\alpha-\beta k+1}\right\}$$

$$\Rightarrow \quad u(t)=u(0)\sum_{k=0}^{\infty}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k\left\{\frac{(t-1)t^{-\beta k}}{\Gamma(1-\beta k)}+\frac{k(\alpha+\beta-1)}{(2-\alpha-\beta)}\frac{t^{\alpha-\beta k+1}}{\Gamma(\alpha-\beta k+1)}\right\}$$

$$\Rightarrow \quad u(t)=u(0)\left\{(t-1)E_{-\beta,1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{-\beta}\right]\right.$$
$$\left.+\frac{\alpha+\beta-1}{2-\alpha-\beta}t^{\alpha+1}E_{-\beta,\alpha+1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{-\beta}\right]\right\}.$$

Therefore, the relation (6.3) gives us

$$T(x,t)=T_0e^{-i\bar{k}x}u(0)\left\{(t-1)E_{-\beta,1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{-\beta}\right]\right.$$
$$\left.+\frac{\alpha+\beta-1}{2-\alpha-\beta}t^{\alpha+1}E_{-\beta,\alpha+1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{-\beta}\right]\right\}. \tag{6.5}$$

This is an alternative representation of the fractional-time Fourier's law equation using the concept of derivative with two fractional orders α and β.
For $\alpha=1$, we have

$$T(x,t)=T_0e^{-i\bar{k}x}u(0)\left\{(t-1)E_{-\beta,1}\left[-\omega\frac{(1-\beta)}{B(1,\beta)}t^{-\beta}\right]+\frac{\beta}{1-\beta}t^2E_{-\beta,2}\left[-\omega\frac{(1-\beta)}{B(1,\beta)}t^{-\beta}\right]\right\}. \tag{6.6}$$

Further, by substituting $\omega=\chi\bar{k}^2\sigma_t^{1-\beta}=\frac{\bar{k}^2}{T_\beta}$, where χ is a time constant or thermal diffusion coefficient, we have

$$T(x,t)=T_0e^{-i\bar{k}x}u(0)\left\{(t-1)E_{-\beta,1}\left[-\bar{k}^2\frac{(1-\beta)}{T_\beta B(1,\beta)}t^{-\beta}\right]+\frac{\beta}{1-\beta}t^2E_{-\beta,2}\left[-\bar{k}^2\frac{(1-\beta)}{T_\beta B(1,\beta)}t^{-\beta}\right]\right\}. \tag{6.7}$$

This equation represents the fractional-time Fourier's law equation using the concept of derivative with only one fractional order β.

7. Conclusions

The aim of this paper was to suggest new derivative with Mittag-Leffler kernel of two parameters. To achieve this goal, we make use the generalized Mittag-Leffler function of two parameters. One derivative is based upon the Atangana-Baleanu Caputo viewpoint and the second on the Atangana-Baleanu Riemann-Liouville approach. We derive the fractional integral associate using the Laplace transform operator. The new derivative was used to find the better solution of Fractional time Fourier's law equation.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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