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Research Article

Almost (α/η)-ψ_Γ-Contraction in Induced Fuzzy Metric Space and Application to Fredholm Integral Equations

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Abstract. In this paper, we established a new class of almost (α/η)-ψ_Γ-contraction mapping in induced fuzzy metric space (FMS) and then proved the results for existence of fixed point theorem (FPT) for multi-valued mappings (MVMs) on the collection of non-empty closed subsets. In application, we prove the existence theorem for Fredholm integral inclusion (FII). An illustrative example also introduced in support of our main result.

Keywords. Almost (α/η)-ψ_Γ-contraction, Induced fuzzy metric space, Fredholm integral equation

Mathematics Subject Classification (2020). 47S40, 54E50, 54E70

1. Introduction

In nonlinear analysis the *fixed point theory* (FPT), play a key role and divided into two categories such as contraction mappings in *complete metric space* (CMS) and continuous mappings in compact and convex normed space. Zadeh [34] was first to introduced the concept of fuzzy set theory. Kramosil and Michálek [15] set the notion of FMS, which was improved by George and Veeramani [8]. The Banach [4] contractive FPT extended by Gregori and Sapena [10] to fuzzy contractive FPT of CMS. Some FPT's on fuzzy contraction introduced by several researchers, which are the generalization of the results of Gregori and Sapena. For more generalization in this field, see [9, 11, 16–18].

The concept of Hausdorff FMS introduced by Rodríguez-López and Romaguera [25] in 2004. Miheţ [19] in 2008 introduced ψ-CMs in non-Archimedean FMS. After that a number of research papers developed by researchers to prove FPT's for MV contraction mapping in Hausdorff FMS (for more results, see [13, 23]). The concept of α-ψ-CM and *α-admissible mapping* (AM) for *single-valued mapping* (SVM) first time initiated by Samet [29] and proved different FP results. In 2013, Hussain *et al.* [14] found a FPT for α-AM with respect to η on *metric space* (MS). Recently, García *et al.* [7] gave the idea of α^*-η_*-admissible for set-valued mappings in FMS and obtained a random FPT for MVM.

For more developments in FMS's in these regards (see [2, 3, 13, 20, 22, 27]). The motive of this paper is to initiate a new class of almost (α/η)-ψ_Γ-CM in IFMS and to prove the FPT for MVM on the collection of non-empty closed subsets. We also initiated an application of FII and examples in support of our main result. The following are the terminologies, basic definitions and properties of FMS.

Definition 1.1 ([34])**.** Let X be a non-empty. A set $\tilde{A}$ defined as $\tilde{A} = \{(\chi, \mu_{\tilde{A}}(\chi)) : \chi \in X\}$, is said to be FS, where $\mu_{\tilde{A}}(\chi)$ is a function with domain X and values in $[0,1]$.
According to [30], $*$ is function from $[0,1]^2$ to $[0,1]$ called a continuous t-norm such that is $m * n = mn$ and $m * 1 = m$ for every $m \in [0,1]$ and $m * n \le o * p$, whenever $m \le o$ and $n \le p$ for all $m, n, o, p \in [0,1]$. It also satisfy the commutativity $(m * n = n * m)$ and associativity $(m * (n * o)) = ((m * n) * o)$. For examples of continuous t-norm $m * n = mn$ or $m * n = \min(m, n)$, and $m * n = \frac{mn}{\max\{m,n,\lambda\}}$ for $0 < \lambda < 1$.

Definition 1.2 ([8])**.** Let X be a non-empty set and a set $\mathcal{M}$ defined as $\mathcal{M} : X \times X \times (0,\infty) \to [0,1]$ is a FS. A triplet $(X, \mathcal{M}, *)$ with a continuous t-norm '$*$' is said to be a FMS if for all $\chi, \gamma, \zeta \in X$ and $t > 0$, satisfying the following conditions:

$(\mathcal{VM}_1)$ $\mathcal{M}(\chi, \gamma, t) > 0$,

$(\mathcal{VM}_2)$ $\mathcal{M}(\chi, \chi, t) = 1$, and $\mathcal{M}(\chi, \gamma, t) = 1 \Leftrightarrow \chi = \gamma$ for some $t > 0$,

$(\mathcal{VM}_3)$ $\mathcal{M}(\chi, \gamma, t) = \mathcal{M}(\gamma, \chi, t)$,

$(\mathcal{VM}_4)$ $\mathcal{M}(\chi, \zeta, t + s) \ge \mathcal{M}(\chi, \gamma, t) * \mathcal{M}(\gamma, \zeta, s)$,

$(\mathcal{VM}_5)$ $\mathcal{M}(\chi, \gamma, \cdot) : (0,\infty) \to [0,1]$ is continuous.

According to [15], $\mathcal{M}$ is a FS which satisfies $(\mathcal{VM}_3)$ and $(\mathcal{VM}_4)$ while $(\mathcal{VM}_1)$, $(\mathcal{VM}_2)$ and $(\mathcal{VM}_5)$ defined as $\mathcal{M} : X \times X \times (0,+\infty) \to [0,1]$ and replaced by $(\mathcal{VM}_1)$, $(\mathcal{VM}_2)$ and $(\mathcal{VM}_5)$, as $(\mathcal{VM}_1)$ $\mathcal{M}(\chi, \gamma, 0) = 0$, $(\mathcal{VM}_2)$ $\mathcal{M}(\chi, \gamma, t) = 1$ iff $\chi = \gamma$, $(\mathcal{VM}_5)$ $\mathcal{M}(\chi, \gamma, \cdot) : [0,\infty) \to [0,1]$ is left continuous.

Remark 1.1 ([15])**.** It is appropriate to recall that $0 < \mathcal{M}(\chi, \gamma, t) < 1$ for all $t > 0$, yield $\chi \ne \gamma$.

Example 1.1. Let us define continuous t-norm $m, n \in [0,1]$ such that $m * n = mn$ or $m * n = \min(m, n)$ and $\mathcal{M}_d : X \times X \times [0,\infty] \to [0,1]$ as $\mathcal{M}_d(\chi, \gamma, t) = \frac{t}{t + d(\chi,\gamma)}$, for all $\chi, \gamma \in X$ and $t > 0$, then fuzzy metric $\mathcal{M}_d$ induced by the metric d and $(X, \mathcal{M}_d, *)$ is called a FMS where (X, d) is a metric space.

Lemma 1.1 ([15]). *Let $(X,\mathcal{M},*)$ be a FMS. Then for $\chi,\gamma \in X$ and $t>0$, $\mathcal{M}(\chi,\gamma,\cdot)$ is non-decreasing on $(0,\infty)$ and $\lim_{t\to\infty} M(\chi,\gamma,t)=1$, for all $\chi,\gamma \in X$.*

Definition 1.3 ([8]). Let $(X,\mathcal{M},*)$ be a FMS and $\{\chi_n\}$ be a sequence in X, then

(i) $\{\chi_n\}$ is said to be convergent at $\chi \in X$ if for each $\varepsilon>0$, and $t>0$, $\exists\ n_0 \in \mathcal{N}$ such that $\mathcal{M}(\chi_n,\chi,t)>1-\varepsilon$, for all $n_0 \in \mathcal{N}$.

(ii) For all $\varepsilon>0$, and each $t>0$, $\exists\ n_0 \in \mathcal{N}$ for Cauchy sequence $\{\chi_n\}$ such that $\mathcal{M}(\chi_n,\chi_m,t)>1-\varepsilon$, for all $n,m \ge n_0$.

(iii) The $(X,\mathcal{M},*)$ is complete if $\{\chi_n\}$ is convergent or Cauchy sequence.

In 2003 the data dependence problem in FP for MV operator was given by Rus and Sîntămărian [27]. A collection of all non-empty closed sub sets of X denoted by $\mathcal{C}(X)$ or more precisely $\mathcal{CB}(X)$. For $\mathcal{A},\mathcal{B} \in \mathcal{C}(X)$, let $\mathcal{H}:\mathcal{C}(X)\times\mathcal{C}(X)\to[0,\infty)$ be defined by

$$\mathcal{H}(\mathcal{A},\mathcal{B})=\max\left\{\sup_{b\in\mathcal{B}}\left(\inf_{a\in\mathcal{A}} d(a,b)\right),\sup_{a\in\mathcal{A}}\left(\inf_{b\in\mathcal{B}} d(a,b)\right)\right\},$$

where $\mathcal{H}$ is called the Pompieu-Hausdorff functional. Also, $d(\chi,\mathcal{B})=\inf\{d(\chi,\gamma):\gamma\in\mathcal{B})\}$ or $d(\mathcal{A},\gamma)=\inf\{d(\chi,\gamma):\chi\in\mathcal{A})\}$.

The concept of MV contraction introduced by Nadler [20] and introduced the notion of Pompieu-Hausdorff metric to ensure the extant of FP for MV contraction maps. Berinde and Berinde [5] extend it for MV almost contraction.

Definition 1.4 ([5]). A MV mapping $\mathcal{T}:X\to\mathcal{CB}(X)$ is an almost contraction if there are $\delta\in(0,1)$ and $\mathcal{L}\ge 0$, such that for all $\chi,\gamma\in X$, the following inequality hold:

$$\mathcal{H}(\mathcal{T}\chi,\mathcal{T}\gamma)\le\delta d(\chi,\gamma)+\mathcal{L}d(\gamma,\mathcal{T}\chi).$$

A new class of $\mathcal{F}$-contraction established by Wardowski [33].

Definition 1.5 ([33]). A mapping $\mathcal{T}:X\to X$ is said to be a SV mapping satisfying

$$\tau+\mathcal{F}(d(\mathcal{T}\chi,\mathcal{T}\gamma))\le\mathcal{F}(d(\chi,\gamma))$$

for all $\chi,\gamma\in X$, with $\mathcal{T}\chi\ne\mathcal{T}\gamma$, where $\tau>0$ and $\mathcal{F}:\mathcal{R}^+\to\mathcal{R}$ is a function that satisfy the following conditions:

($\mathcal{F}_1$) $\mathcal{F}$ is strictly increasing, that is for $\chi,\gamma\in\mathcal{R}^+$, such that $\chi<\gamma\Rightarrow\mathcal{F}(\chi)<\mathcal{F}(\gamma)$.

($\mathcal{F}_2$) for each $\{\chi_n\}\subseteq\mathcal{R}^+$, $\lim_{n\to\infty}\chi_n=0$ iff $\lim_{n\to\infty}\mathcal{F}(\chi_n)=-\infty$,

($\mathcal{F}_3$) $\exists\ 0<\mathcal{P}<1$ so that $\lim_{\chi\to 0^+}\chi^{\mathcal{P}}\mathcal{F}(\chi)=0$.

Recently, Al-Mezel and Ahmed [3] proved a generalized FP results for almost $(\sigma,\mathcal{F}_\sigma)$-contractions with applications to FII. The notion of almost $\mathcal{F}$-contraction in the setting of Hausdorff metric space for fuzzy mappings defined by Al-Mazrooei and Ahmad [2]. In 2019, Chauhan *et al*. [6] proved some FPT's for $\mathcal{S}_{\mathcal{F}}$-contraction in complete FMS and Sezen [31] proved FPT's for new type CMs.

Definition 1.6 ([31]). Let $\mathcal{G}:[0,1]\to\mathcal{R}$ be strictly increasing, continuous mapping and for each sequence $\{a_n\}_{n\in N}$ of positive numbers and $\lim_{n\to\infty} a_n = 1$ if and only if $\lim_{n\to\infty}\mathcal{G}(a_n)=+\infty$. Let Δ be the family of all $\mathcal{G}$ functions. A mapping $\mathcal{T}:X\to X$ is said to be $\mathcal{G}$-contraction if there exists $\delta\in(0,1)$, such that

$$\mathcal{M}(\mathcal{T}\chi,\mathcal{T}\gamma,t)<1\Rightarrow \mathcal{G}\left(\mathcal{M}(\mathcal{T}\chi,\mathcal{T}\gamma,t)\right)\le \mathcal{G}\left(\mathcal{M}(\chi,\gamma,t)\right)+\delta,\quad \text{for all } \chi,\gamma\in X \text{ and } \mathcal{G}\in\Delta.$$

Lemma 1.2 ([11]). *Let* $(X,\mathcal{M},*)$ *be a FMS. Then the* 3*-tuple* $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ *is a IFMS.*

Definition 1.7 ([11]). Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS. Then for all $E,F\in\mathcal{C}(X), t>0$, the IFM $\mathcal{H}_\mathcal{M}(E,F,t):(\mathcal{C}(X))^2\times(0,\infty)\to[0,1]$ is a function defined as

$$\mathcal{H}_\mathcal{M}(E,F,t)=\frac{t}{t+\mathcal{H}(E,F)}=\min\left\{\int_{a\in E}\left(\sup_{b\in F}\mathcal{M}(a,b,t)\right),\inf_{b\in F}\left(\sup_{a\in E}\mathcal{M}(a,b,t)\right)\right\},$$

where $\mathcal{H}$ is the Hausdorff distance metric in the collection $\mathcal{C}(X)$.

Definition 1.8. Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be an IFMS. If there exits $0\le\lambda\le1$, the MV contraction of a mapping $E:X\to\mathcal{C}(X)$ is defined by

$$\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)\ge\lambda\mathcal{M}(\chi,\gamma,t),\quad \text{for all } \chi,\gamma\in X,\ t>0.$$

Definition 1.9 ([11]). Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS and let $\Im$ be non-empty subset of $\mathcal{C}(X)$. A mapping $E:\Im\to\mathcal{C}(X)$ is said to be almost MV contraction for some $\mathcal{L}\ge0$, if there exists $0\le\lambda\le1$ such that

$$\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)\ge\lambda\mathcal{M}(\chi,\gamma,t)+\mathcal{L}\mathcal{M}(\gamma,E\chi,t),\quad \text{for all } \chi,\gamma\in X,\ t>0.$$

Motivated by the concepts of α-AM in [14, 23, 29] the concept of α^*-η_*-admissible for set valued mappings as follows:

Definition 1.10 ([7]). Let $\alpha,\eta:X^2\times(0,\infty)\to[0,1)$ be two functions for non-empty set X. A mapping and $\mathcal{T}:X\to2^X$ is said to be α^*-η_*-AM if for all $\chi,\gamma\in X$, we have

$$\alpha(\chi,\gamma,t)\le\eta(\chi,\gamma,t)\Rightarrow\alpha^*(\mathcal{T}\chi,\mathcal{T}\gamma,t)\le\eta_*(\mathcal{T}\chi,\mathcal{T}\gamma,t),\quad \text{for all } \chi,\gamma\in X,\ t>0,$$

where

$$\alpha^*(\mathcal{T}\chi,\mathcal{T}\gamma,t)=\sup_{\chi\in\mathcal{T}x,\gamma\in\mathcal{T}y}\alpha(\chi,\gamma,t)$$

and

$$\eta^*(\mathcal{T}\chi,\mathcal{T}\gamma,t)=\sup_{\chi\in\mathcal{T}x,\gamma\in\mathcal{T}y}\eta(\chi,\gamma,t).$$

Let $\psi:\mathcal{R}^+\to\mathcal{R}$ where $\psi\in\Psi$ that satisfy the following conditions:

(ω_1) ψ is strictly increasing, that is for $\chi,\gamma\in\mathcal{R}^+$, such that $\chi<\gamma\Rightarrow\psi(\chi)<\psi(\gamma)$,

(ω_2) for all $\{\chi_n\}\subseteq\mathcal{R}^+$ $\lim_{n\to\infty}\chi_n=1\Leftrightarrow\lim_{n\to\infty}\psi(\chi_n)=-\infty$,

(ω_3) $\exists\ 0<\mathcal{P}<1$ so that $\lim_{\chi\to0^+}\chi^\mathcal{P}\psi(\chi)=1$.

2. Implicit Functions

Several metric FP results are proved in simpler way by considering the concept of implicit function contraction type condition. Popa [24] first to take initiative in this direction and solved FPT's.

Let Γ be the set of all continuous functions $\Gamma \in \Gamma$ such that $\Gamma(u_1,u_2,u_3,u_4,u_5):(\mathcal{R}^+)^5 \to \mathcal{R}^+$ be a collection of all continuous function with the following conditions:

(Γ_1) $\Gamma(u,u,v,u+v,1)$, $\Gamma(u,u,v,1,u+v)$, $\Gamma(u,u,u,u,u)$ belong in $(0,1]$,

(Γ_2) for all $(u_1,u_2,u_3,u_4,u_5) \in (\mathcal{R}^+)^5$ and $\partial > 0$, $\Gamma(\partial u_1,\partial u_2,\partial u_3,\partial u_4,\partial u_5) \ge \partial\Gamma(u_1,u_2,u_3,u_4,u_5)$,

(Γ_3) for all $u_i,v_i \in \mathcal{R}^+$, $u_i \ge v_i$, $i=1,2,3,4,5$, $\Gamma(u_1,u_2,u_3,u_4,u_5) \ge \Gamma(v_1,v_2,v_3,v_4,v_5)$ also $\Gamma(u_1,u_2,u_3,u_4,1) \ge \Gamma(v_1,v_2,v_3,v_4,1)$ and $\Gamma(u_1,u_2,u_3,1,u_4) \ge \Gamma(v_1,v_2,v_3,1,v_4)$.

Lemma 2.1 ([11]). *If $\Gamma \in \Gamma$ and $u,v \in \mathcal{R}^+$ are such that*

$$v > \min\{\Gamma(u,u,v,u+v,1),\Gamma(u,u,v,1,u+v),\Gamma(u,v,u,u+v,1),\Gamma(u,v,u,1,u+v)\},$$

then $v > u$.

Lemma 2.2 ([11]). *Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a complete IFMS and $E,F \in \mathcal{C}(X)$ with $\mathcal{H}_\mathcal{M}(E,F,t) > 0$ for any $t > 0$. Then, for all $m > 1$, $t > 0$ and $e \in E$,, such that $\exists\ f = f(e) \in F$ implies $\mathcal{M}(e,f,t) > m\mathcal{H}_\mathcal{M}(E,F,t)$.*

3. Main Results

Theorem 3.1 ([11]). *Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a complete IFMS and the $E: X \to \mathcal{C}(X)$ be an MV-almost contraction, then E has a FP.*

Definition 3.1. Let $\alpha,\eta : X \times X \times (0,\infty) \to [0,1)$ be two functions for non-empty set X. A mapping $E: X \to \mathcal{C}(X)$ is said to be $\frac{\alpha^*}{\eta_*}$-AM if for all $\chi,\gamma \in X$, $t > 0$, we have

$$\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \le 1 \Rightarrow \frac{\alpha^*(E\chi,E\gamma,t)}{\eta_*(E\chi,E\gamma,t)} \le 1,$$

where

$$\alpha^*(E\chi,E\gamma,t) = \sup_{\chi\in E\chi,\gamma\in E\gamma} \alpha(\chi,\gamma,t)$$

and

$$\eta_*(E\chi,E\gamma,t) = \inf_{\chi\in E\chi,\gamma\in E\gamma} \eta(\chi,\gamma,t).$$

Definition 3.2. Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS. A MVM $E: X \to \mathcal{C}(X)$ is said to be an almost (α/η)-ψ_Γ-contraction if there exists $\alpha,\eta : X \times X \times (0,\infty) \to [0,1)$, $\psi_\Gamma \in \Gamma$, $\Gamma \in \Gamma$, $\tau > 0$ and $\mathcal{L} \ge 0$ so that

$$\begin{aligned} &2\tau + \psi_\Gamma\left(\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \cdot \mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)\right) \\ &\quad \ge \psi_\Gamma(\Gamma(\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E\chi,t),\mathcal{M}(\gamma,E\gamma,t),\mathcal{M}(\chi,E\gamma,t),\mathcal{M}(\gamma,E\chi,t))) + \mathcal{L}\mathcal{M}(\gamma,E\chi,t) \end{aligned} \tag{1}$$

for all $\chi,\gamma \in X$, $t > 0$, with $\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) > 0$.

Theorem 3.2. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E : X \to \mathcal{C}(X)$ be a MV-almost (α/η)-ψ_Γ-CM such that the following claim grasps:*

(i) *E is an $\frac{\alpha^*}{\eta_*}$-AM,*

(ii) *$\exists\ \chi_0 \in X$, and $\chi_1 \in E\chi_0$ with $\frac{\alpha(\chi_0,\chi_1,t)}{\eta(\chi_0,\chi_1,t)} \le 1$,*

(iii) *for any $\{\chi_n\}$ in X, so that $\chi_n \to z$ and $\frac{\alpha(\chi_n,\chi_{n+1},t)}{\eta(\chi_n,\chi_{n+1},t)} \le 1$, for all $n \in \mathcal{N}$, $\Rightarrow \frac{\alpha(\chi_n,z,t)}{\eta(\chi_n,z,t)} \le 1$, for all $n \in \mathcal{N}$, then $\exists\ \chi^* \in X$, such that $\chi^* \in E\chi^*$.*

Proof. By hypothesis (ii), there exist $\chi_0 \in X$, and $\chi_1 \in E\chi_0$ with $\frac{\alpha(\chi_0,\chi_1,t)}{\eta(\chi_0,\chi_1,t)} \le 1$. If $\chi_1 \in E\chi_1$, then χ_1 is a FP of E and so the proof end. Now, suppose that $\chi_1 \notin E\chi_1$. Then $\mathcal{M}(\chi_1,E\chi_1,t) < 1$ and so $\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t) < 1$. From eq. (1), we have

$$
\begin{aligned}
2\tau + \psi_\Gamma(\mathcal{M}(\chi_1,E\chi_1,t)) &\ge 2\tau + \mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t) + \mathcal{L}\mathcal{M}(\chi_1,E\chi_0,t)\\
&\ge 2\tau + \psi_\Gamma\left(\frac{\alpha(\chi_0,\chi_1,t)}{\eta(\chi_0,\chi_1,t)}\cdot\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t)\right) + \mathcal{L}\mathcal{M}(\chi_1,E\chi_0,t),\\
2\tau + \psi_\Gamma(\mathcal{M}(\chi_1,E\chi_1,t)) &\ge 2\tau + \mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t) + \mathcal{L}\mathcal{M}(\chi_1,E\chi_0,t)\\
&\ge \psi_\Gamma(\Gamma(\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\chi_0,E\chi_1,t),\mathcal{M}(\chi_1,\chi_1,t)))\\
&\ge \psi_\Gamma(\Gamma(\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t)+\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\chi_1,\chi_1,t)))
\end{aligned}
$$

which implies

$$\mathcal{M}(\chi_1,E\chi_1,t) > \Gamma(\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t)+\mathcal{M}(\chi_1,E\chi_1,t),1)$$

so that$\mathcal{M}(\chi_1,E\chi_1,t) > \mathcal{M}(\chi_0,\chi_1,t)$. Thus by Lemma 2.1, we obtain

$$
\begin{aligned}
&2\tau + \psi_\Gamma(\mathcal{M}(\chi_1,E\chi_1,t))\\
&\quad\ge \psi_\Gamma(\Gamma(\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t)+\mathcal{M}(\chi_0,\chi_1,t),1))\\
&\quad> \psi_\Gamma(\Gamma(\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_0,\chi_1,t),\mathcal{M}(\chi_1,\chi_0,t),2\mathcal{M}(\chi_0,\chi_1,t),1))\\
&\quad\ge \psi_\Gamma\left(\mathcal{M}(\chi_0,\chi_1,t)\cdot\Gamma\left(1,1,1,2,\frac{1}{\mathcal{M}(\chi_0,\chi_1,t)}\right)\right)\\
&\quad\ge \psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)).
\end{aligned}
$$

Thus

$$2\tau + \psi_\Gamma(\mathcal{M}(\chi_1,E\chi_1,t)) \ge \psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)). \tag{2}$$

Since $\psi_\Gamma \in \Gamma$, so $\exists\ m < 1$, such that

$$\psi_\Gamma(m\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t)) > \psi_\Gamma(\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t)) - \tau. \tag{3}$$

Next as

$$\mathcal{M}(\chi_1,E\chi_1,t) \ge \mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t) > m\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t). \tag{4}$$

Again by Lemma 2.1, $\exists\ \omega_2 \in E\chi_1$ (obviously $\omega_2 \ne \chi_1$), such that

$$\mathcal{M}(\chi_1,\omega_2,t) \ge \mathcal{M}(\chi_1,E\chi_1,t) \tag{5}$$

Thus from eqs. (3), (4) and (5), we have

$$\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t)) \ge \psi_\Gamma(m\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t)) > \psi_\Gamma(\mathcal{H}_{\mathcal{M}}(E\chi_0,E\chi_1,t)) - \tau \tag{6}$$

which implies by eq. (2) that

$$2\tau+\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t))\ge 2\tau+\psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_0,E\chi_1,t))-\tau\ge\psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t))+\tau.$$

Thus, we have

$$\tau+\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t))\ge\psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)). \tag{7}$$

Since

$$\tau+\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t))\ge\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t))\cdot\frac{\alpha(\chi_1,\omega_2,t)}{\eta(\chi_1,\omega_2,t)}\ge 1$$

and $\frac{\alpha}{\eta}$-admissibility of E and eq. (1), we have

$$\begin{aligned}
&2\tau+\psi_\Gamma(\mathcal{M}(\omega_2,E\omega_2,t))\\
&\quad\ge 2\tau+\psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t))\\
&\quad\ge 2\tau+\psi_\Gamma\left(\frac{\alpha(\chi_1,\omega_2,t)}{\eta(\chi_1,\omega_2,t)}\cdot\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t)\right)\\
&\quad\ge\psi_\Gamma(\Gamma(\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)))\\
&\qquad+\mathcal{L}\mathcal{M}(\omega_2,E\chi_1,t)\\
&\quad\ge\psi_\Gamma(\Gamma(\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,\omega_2,t))),\\
&\mathcal{M}(\omega_2,E\omega_2,t)>\mathcal{M}(\chi_1,\omega_2,t).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&2\tau+\psi_\Gamma(\mathcal{M}(\omega_2,E\omega_2,t))\\
&\quad\ge\psi_\Gamma(\Gamma(\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,\omega_2,t))),\\
&\qquad\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\omega_2,\chi_1,t),\mathcal{M}(\chi_1,\omega_2,t)+\mathcal{M}(\omega_2,E\omega_2,t),1)\\
&\quad\ge\psi_\Gamma\left(\mathcal{M}(\chi_1,\omega_2,t)\,\Gamma\left(1,1,1,2,\frac{1}{\mathcal{M}(\chi_1,\omega_2,t)}\right)\right)\\
&\quad\ge\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t)).
\end{aligned}$$

Thus, we get

$$2\tau+\psi_\Gamma(\mathcal{M}(\omega_2,E\omega_2,t))\ge\psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t)). \tag{8}$$

Since $\psi_\Gamma\in\Gamma$, so $\exists\ m<1$, such that

$$\psi_\Gamma(m\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t))>\psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t))-\tau. \tag{9}$$

Next, as

$$\mathcal{M}(\omega_2,E\omega_2,t)\ge\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t)>m\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t). \tag{10}$$

According to Lemma 2.2, $\exists\ \chi_3\in E\omega_2$ as $\chi_3\ne\omega_2$, such that

$$\mathcal{M}(\omega_2,\chi_3,t)\ge\mathcal{M}(\omega_2,E\omega_2,t). \tag{11}$$

Thus, by eqs. (9), (10) and (11), we have

$$\begin{aligned}
\psi_\Gamma(\mathcal{M}(\omega_2,\chi_3,t))&\ge\psi_\Gamma(\mathcal{M}(\omega_2,E\omega_2,t))\\
&\ge\psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t))\\
&\ge\psi_\Gamma(m\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t))
\end{aligned}$$

$$> \psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_1, E\omega_2, t)) - \tau \tag{12}$$

which implies by eq. (12) that

$$\begin{aligned} 2\tau + \psi_\Gamma(\mathcal{M}(\omega_2,\chi_3,t)) &\geq 2\tau + \psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_1, E\omega_2, t)) - \tau \\ &\geq \psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t)) + \tau\,. \end{aligned}$$

Thus, we have

$$\psi_\Gamma(\mathcal{M}(\omega_2,\chi_3,t)) \geq \psi_\Gamma(\mathcal{M}(\chi_1,\omega_2,t)) - \tau\,. \tag{13}$$

Thus, as in eq. (13), we have $\{x_n\}$ in X so that $\chi_{n+1} \in E\chi_n$ and $\frac{\alpha(\chi_n,\chi_{n+1})}{\eta(\chi_n,\chi_{n+1})} \leq 1$, for all $n \in \mathcal{N}$. Furthermore, for all $n \in \mathcal{N}$

$$\psi_\Gamma(\mathcal{M}(\chi_n,\chi_{n+1},t)) \geq \psi_\Gamma(\mathcal{M}(\chi_{n-1},\chi_n,t)) - \tau\,. \tag{14}$$

Therefore, by eq. (14), we have

$$\begin{aligned} \psi_\Gamma(\mathcal{M}(\chi_n,\chi_{n+1},t)) &\geq \psi_\Gamma(\mathcal{M}(\chi_{n-1},\chi_n,t)) - \tau \\ &\geq \psi_\Gamma(\mathcal{M}(\chi_{n-2},\chi_{n-1},t)) - 2\tau \\ &\;\;\vdots \\ &\geq \psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)) - n\tau\,. \end{aligned} \tag{15}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \psi_\Gamma(\mathcal{M}(\chi_n,\chi_{n+1},t)) = -\infty,$$

which jointly with (ω_2) gives

$$\lim_{n\to\infty} \mathcal{M}(\chi_n,\chi_{n+1},t) = 1.$$

Thus, from (ω_3), $\exists\ \mathcal{P} \in (0,1)$ so that

$$\lim_{n\to\infty} [\mathcal{M}(\chi_n,\chi_{n+1},t)]^{\mathcal{P}} \cdot \psi_\Gamma(\mathcal{M}(\chi_n,\chi_{n+1},t)) = 1. \tag{16}$$

From eqs. (15) and (16), we obtain

$$\begin{aligned} &[\mathcal{M}(\chi_n,\chi_{n+1},t)]^{\mathcal{P}} \cdot \psi_\Gamma(\mathcal{M}(\chi_n,\chi_{n+1},t)) - [\mathcal{M}(\chi_n,\chi_{n+1},t)]^{\mathcal{P}} \cdot \psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)) \\ &\quad \geq [\mathcal{M}(\chi_n,\chi_{n+1},t)]^{\mathcal{P}} \cdot [\psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)) + n\tau] - [\mathcal{M}(\chi_n,\chi_{n+1},t)]^{\mathcal{P}} \cdot \psi_\Gamma(\mathcal{M}(\chi_0,\chi_1,t)) \\ &\quad \geq n\tau[\mathcal{M}(\chi_n,\chi_{n+1},t)]^{\mathcal{P}} \\ &\quad \geq 1\,. \end{aligned}$$

For $n \to \infty$, implies

$$\lim_{n\to\infty} n[\mathcal{M}(\chi_n,\chi_{n+1},t)]^{p} = 1\,. \tag{17}$$

So that

$$\begin{aligned} &\lim_{n\to\infty} n^{\frac{1}{p}}[\mathcal{M}(\chi_n,\chi_{n+1},t)] = 1 \\ \Rightarrow\ &\sum_{n=1}^{\infty} \mathcal{M}(\chi_n,\chi_{n+1},t) \end{aligned}$$

converges i.e. a Cauchy sequence $\{\chi_n\}$. As $(\mathcal{C}(X), \mathcal{H}_\mathcal{M}, *)$ is complete, so that $\exists\ \chi^* \in X$, such that $\lim_{n\to\infty} \chi_n = \chi^*$.

For $\chi^* \in E\chi^*$ as fixed point, from (iii), we have $\frac{\alpha(\chi_n,\chi^*,t)}{\eta(\chi_n,\chi^*,t)} \le 1$, for all $n \in \mathcal{N}$. Assume on the contrary that $\chi_n \notin E\chi^*$, then $\exists\ n_0 \in \mathcal{N}$ and $\{\chi_{n_k}\}$ of $\{\chi_n\}$ so that $\mathcal{M}(\chi_{n_{k+1}},E\chi^*,t) > 0$, for all $n_k \ge n_0$. Now, using (1) with $\chi = \chi_{n_{k+1}}$ and $\chi = \chi^*$, we have

$$\begin{aligned}
&2\tau + \psi_\Gamma(\mathcal{M}(\chi_{n_{k+1}},E\chi^*,t))\\
&\quad \ge 2\tau + \psi_\Gamma(\mathcal{M}_{\mathcal{H}}(E\chi_{n_k},E\chi^*,t))\\
&\quad \ge 2\tau + \psi_\Gamma\left(\frac{\alpha(\chi_{n_k},\chi^*,t)}{\eta(\chi_{n_k},\chi^*,t)}\mathcal{M}_{\mathcal{H}}\left(E\chi_{n_k},E\chi^*,t\right)\right)\\
&\quad \ge \psi_\Gamma(\Gamma(\mathcal{M}(\chi_{n_k},\chi^*,t),\mathcal{M}(\chi_{n_k},E\chi_{n_k},t),\mathcal{M}(\chi^*,E\chi^*,t),\mathcal{M}(\chi_{n_k},E\chi^*,t)+\mathcal{M}(E\chi_{n_k},\chi^*,t))).
\end{aligned}$$

By (ω_1), we get

$$\mathcal{M}(\chi_{n_{k+1}},E\chi^*,t) > \psi_\Gamma(\Gamma(\mathcal{M}(\chi_{n_k},\chi^*,t),\mathcal{M}(\chi_{n_k},\chi_{n_{k+1}},t),\mathcal{M}(\chi^*,E\chi^*,t),\mathcal{M}(\chi_{n_k},E\chi^*,t),\mathcal{M}(\chi^*,\chi_{n_{k+1}},t)))$$

$$\mathcal{M}(\chi^*,E\chi^*,t) \ge \Gamma(1,1,\mathcal{M}(\chi^*,E\chi^*,t),\mathcal{M}(\chi^*,E\chi^*,t),1)$$

by Lemma 2.1, that is $1 > \mathcal{M}(\chi^*,E\chi^*,t) > 1$, a contradiction.
Hence

$$\mathcal{M}(\chi^*,E\chi^*,t) = 1 \Rightarrow \chi^* \in E\chi^*. \qquad \square$$

Theorem 3.3. *Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS and $E : X \to \mathcal{C}(X)$ be a MV-almost (α/η)-ψ_Γ-contractive mapping if their exists $\alpha,\eta : X \times X \times (0,\infty) \to [0,1)$, $\psi_\Gamma \in \Gamma$, $\Gamma \in S$, $\tau > 0$ and $\mathcal{L} \ge 0$ so that*

$$\begin{aligned}
&2\tau + \psi_\Gamma\left(\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\cdot\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)\right)\\
&\quad \ge (\psi_\Gamma(\Gamma(\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E\chi,t),\mathcal{M}(\gamma,Ey,t),\mathcal{M}(\chi,E\gamma,t),\mathcal{M}(\gamma,E\chi,t)))+\mathcal{L}\mathcal{M}(\gamma,E\chi,t))^k
\end{aligned}$$

for all $\chi,\gamma \in X$, $t > 0$, $k \in (0,1)$ with $\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) > 0$ and the following assertions holds:

(i) *E is an $\frac{\alpha^*}{\eta_*}$-AM,*

(ii) *$\exists\ \chi_0 \in X$, and $\chi_1 \in E\chi_0$ with $\dfrac{\alpha(\chi_0,\chi_1,t)}{\eta(\chi_0,\chi_1,t)} \le 1$,*

(iii) *for any $\{\chi_n\}$ in X, so that $\chi_n \to \mathcal{Z}$ and $\dfrac{\alpha(\chi_n,\chi_{n+1},t)}{\eta(\chi_n,\chi_{n+1},t)} \le 1$, for all $n \in \mathcal{N}$*
$\Rightarrow \dfrac{\alpha(\chi_n,\mathcal{Z},t)}{\eta(\chi_n,\mathcal{Z},t)} \le 1$, *for all $n \in \mathcal{N}$, then $\exists\ \chi^* \in X$, such that $\chi^* \in E\chi^*$.*

Proof. From eq. (1) and for all $\chi,\gamma \in X$, $t > 0$, $k \in (0,1)$ with $\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) > 0$

$$\begin{aligned}
&2\tau + \psi_\Gamma\left(\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\cdot\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)\right)\\
&\quad \ge (\psi_\Gamma(\Gamma(\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E\chi,t),\mathcal{M}(\gamma,E\gamma,t),\mathcal{M}(\chi,E\gamma,t),\mathcal{M}(\gamma,E\chi,t)))+\mathcal{L}\mathcal{M}(\gamma,E\chi,t))^k
\end{aligned}$$

since $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \le 1$, $\psi_\Gamma \in \Gamma$, $\mathcal{L} = 0$ and if $k = 0$, so that

$$2\tau + \mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) > \psi_\Gamma(\Gamma(\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E\chi,t),\mathcal{M}(\gamma,E\gamma,t),\mathcal{M}(\chi,E\gamma,t),\mathcal{M}(\gamma,E\chi,t)))$$

$\tau + \mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) \ge \mathcal{M}(\chi,\gamma,t) \Rightarrow \chi \in E\chi$ by $\Gamma(t_1,t_2,t_3,t_4,t_5) = t_1$ in Theorem 3.2. $\square$

4. Consequences

Corollary 4.1. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E:X\to\mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma\in\Gamma$, $\tau>0$, such that*

$$2\tau+\psi_\Gamma\left(\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\cdot\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)\right)\ge(\psi_\Gamma(\mathcal{M}(\chi,\gamma,t)))^k$$

for all $\chi,\gamma\in X$, $t>0$, $k\in(0,1)$ and $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\le 1$ with $\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)<1$. Then $\chi^*\in X$, such that $\chi^*=E\chi^*$.

Proof. Considering $\Gamma\in\Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\le 1$ given $\mathcal{L}=0$ and $\Gamma(t_1,t_2,t_3,t_4,t_5)=t_1$ in Theorem 3.2. □

Corollary 4.2. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E_1,E_2:X\to\mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose there exists $\psi_\Gamma\in\Gamma$, $\tau>0$ such that*

$$\begin{aligned}&2\tau+\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(E_1\chi,E_2\gamma,t))\\&\quad\ge\psi_\Gamma\left(\min\left\{\mathcal{M}(\chi,\gamma,t),\frac{\mathcal{M}(\chi,E_1\chi,t)+\mathcal{M}(\gamma,E_2\gamma,t)}{2},\frac{\mathcal{M}(\chi,E_2\gamma,t)+\mathcal{M}(\gamma,E_1\chi,t)}{2}\right\}\right),\end{aligned}$$

for all $\chi,\gamma\in X$, $t>0$ with $\mathcal{H}_{\mathcal{M}}(E_1\chi,E_2\gamma,t))>0$. Then there exists $\chi^*\in X$, such that $\chi^*\in(E_1\cap E_2)\chi^*$.

Proof. Considering $\Gamma\in\Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\le 1$ given $\mathcal{L}=0$ and $\Gamma(t_1,t_2,t_3,t_4,t_5)=\min\{t_1,\frac{t_2+t_3}{2},\frac{t_4+t_5}{2}\}$ in Theorem 3.2. □

Corollary 4.3. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E:X\to\mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma\in\Gamma$, $\tau>0$ such that*

$$2\tau+\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t))\ge\psi_\Gamma(\mathcal{M}(\chi,\gamma,t)),\quad\text{for all } \chi,\gamma\in X,\, t>0$$

with $\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)<1$. Then according the result of Banach-type (α/η)-ψ_Γ-contraction in IFMS, $\exists\ \chi^*\in X$, such that $\chi^*=E\chi^*$.

Proof. Considering $\Gamma\in\Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\le 1$ given $\mathcal{L}=0$ and $\Gamma(u_1,u_2,u_3,u_4,u_5)=u_1$ in Theorem 3.2. □

Corollary 4.4. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E:X\to\mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma\in\Gamma$, $\tau>0$, such that*

$$2\tau+\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(\chi,E\gamma,t))\ge\psi_\Gamma(\mathcal{M}(\chi,E\chi,t)+\mathcal{M}(\gamma,\gamma t)),\quad\text{for all } \chi,\gamma\in X,\, t>0$$

with $\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)<1$. Then according the result of Kannan-type (α/η)-ψ_Γ-contraction in IFMS, $\exists\ \chi^*\in X$, such that $\chi^*=E\chi^*$.

Proof. Considering $\Gamma\in\Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\le 1$ given $\mathcal{L}=0$ and $\Gamma(u_1,u_2,u_3,u_4,u_5)=u_2+u_3$ in Theorem 3.2. □

Corollary 4.5. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E:X\to\mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma\in\Gamma$, $\tau>0$, such that*

$$2\tau+\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t))\ge\psi_\Gamma(\mathcal{M}(\chi,E\gamma,t)+\mathcal{M}(\gamma,E\chi t)),$$

for all $\chi,\gamma\in X$, $t>0$ with $\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)<1$. Then according the result of Chatterjea-type

(α/η)-ψ_Γ-contraction in IFMS, $\exists\ \chi^* \in X$, such that $\chi^* = E\chi^*$.

Proof. Considering $\Gamma \in \Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \le 1$, $\mathcal{L} = 0$ and $\Gamma(u_1,u_2,u_3,u_4,u_5) = u_4 + u_5$ in Theorem 3.2. □

Corollary 4.6. *Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS and $E : X \to \mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma \in \Gamma$, $\tau > 0$ and non negative real numbers $\Omega_1,\Omega_2,\Omega_3,\Omega_4$ and Ω_5 with $\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5 \le 1$, such that*

$$2\tau + \psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)) \ge \psi_\Gamma(\Omega_1\mathcal{M}(\chi,\gamma,t)+\Omega_2\mathcal{M}(\chi,E\chi,t)+\Omega_3\mathcal{M}(\gamma,E\gamma,t)+\Omega_4\mathcal{M}(\chi,E\gamma,t)+\Omega_5\mathcal{M}(\gamma,E\chi,t)),$$

for all $\chi,\gamma \in X$, $t > 0$ with $\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) < 1$. Then according the result of Hardy-Roger-type (α/η)-ψ_Γ-contraction in IFMS, $\exists\ \chi^* \in X$, such that $\chi^* = E\chi^*$.

Proof. Considering $\Gamma \in \Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \le 1$, $\mathcal{L} = 0$ and $\Gamma(u_1,u_2,u_3,u_4,u_5) = \omega_1u_1+\omega_2u_2+\omega_3u_3+\omega_4u_4+\omega_5u_5$ in Theorem 3.2. □

Corollary 4.7. *Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS and $E : X \to \mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma \in \Gamma$, $\tau > 0$, such that*

$$2\tau + \psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)) \ge \psi_\Gamma\left(\min\left\{\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E\chi,t),\mathcal{M}(\gamma,E\gamma,t),\frac{\mathcal{M}(\chi,E\gamma,t)+\mathcal{M}(\gamma,E\chi,t)}{2}\right\}\right),$$

for all $\chi,\gamma \in X$, $t > 0$ with $\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) < 1$. Then according the result of Cirić-type (α/η)-ψ_Γ-contraction in IFMS, $\exists\chi^* \in X$, such that $\chi^* = E\chi^*$.

Proof. Considering $\Gamma \in \Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \le 1$ given $\mathcal{L} = 0$ and $\Gamma(u_1,u_2,u_3,u_4,u_5) = \min\left\{u_1,u_2,u_3,\frac{u_4+u_5}{2}\right\}$ in Theorem 3.2. □

Corollary 4.8. *Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS and $E : X \to \mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs. Suppose $\exists\ \psi_\Gamma \in \Gamma$, $\tau > 0$, such that*

$$2\tau + \psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t)) \ge \psi_\Gamma(\min\{\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E\chi,t),\mathcal{M}(\gamma,E\gamma,t),\mathcal{M}(\chi,E\gamma,t),\mathcal{M}(\gamma,E\chi,t)\}),$$

for all $\chi,\gamma \in X$, $t > 0$ with $\mathcal{H}_\mathcal{M}(E\chi,E\gamma,t) < 1$. Then according the result of Cirić-type (α/η)-ψ_Γ-contraction in IFMS, $\exists\ \chi^* \in X$, such that $\chi^* = E\chi^*$.

Proof. Considering $\Gamma \in \Gamma$, $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} \le 1$ given $\mathcal{L} = 0$ and $\Gamma(u_1,u_2,u_3,u_4,u_5) = \min\{u_1,u_2,u_3,u_4,u_5\}$ in Theorem 3.2. □

Example 4.1. Let $X = \mathcal{N} \cup \{0\}$ be endowed with the usual fuzzy metric $\mathcal{M}(\chi,\gamma,t) = \frac{t}{t+|\chi-\gamma|}$, for all $\chi,\gamma \in X$, $t > 0$. Define $\alpha,\eta : X \times X \to [0,\infty)$ by

$$\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} = \begin{cases} 1, & \text{if } \chi,\gamma \in \{0,1\}, \\ \frac{1}{2}, & \text{if } \chi,\gamma > 1, \\ 0, & \text{if otherwise.} \end{cases}$$

Let $(\mathcal{C}(X),\mathcal{H}_\mathcal{M},*)$ be a IFMS. A mapping and $E : X \to \mathcal{C}(X)$ be a AM-almost (α/η)-ψ_Γ-CMs defined by

$$E\chi=\begin{cases}\{0,1\}, & \text{if } \chi=0,1,\\ \{\chi-1,\chi\}, & \text{if } \chi>1.\end{cases}$$

If $\exists\ 0\le\lambda\le1$ and some $\mathcal{L}\ge0$, such that $\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)\ge\lambda\cdot\mathcal{H}_{\mathcal{M}}(\chi,\gamma,t)+\mathcal{L}\cdot\mathcal{H}_{\mathcal{M}}(\gamma,E\chi,t)$, for all $\chi,\gamma\in X$, $t>0$. Then E has a FP. Again let $\psi_\Gamma:\mathcal{R}^+\to\mathcal{R}$ defined by $\psi_\Gamma(t):t+\log t$, for all $t\in R^+\tau=1/2$, $\psi_\Gamma:(\mathcal{R}^+)^5\to\mathcal{R}^+$ and $\Gamma(u_1,u_2,u_3,u_4,u_5)=u_1$ then

$$\frac{\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)}{\Gamma(u_1,u_2,u_3,u_4,u_5)}e^{\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)-\Gamma(u_1,u_2,u_3,u_4,u_5)}\ge e^{-\tau},\quad \text{for all } \chi,\gamma\in X,\ t>0$$

with $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)<1$. Here if $\chi,\gamma\in\{0,1\}$, $t>0$, then (1) is applicable for $\mathcal{L}=0$. Also, if $\chi,\gamma>1$, with $\chi\ne\gamma$, $t>0$, then $e^{-1/2}>\frac{1}{2}e^{-\frac{1}{2}(\frac{t}{t+|\chi-\gamma|})}$.
Again if χ or $\gamma\in\{0,1\}$ and χ or γ with $\chi\ne\gamma$ then $\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)}\mathcal{H}_{\mathcal{M}}(E\chi,E\gamma,t)=1$. Thus the contractive condition is satisfied trivially so E is an almost (α/η)-ψ_Γ-contraction.

For $\chi_0=1$, we have $\chi_1=1\in E\chi_0$ such that $\frac{\alpha(\chi_0,\chi_1,t)}{\eta(\chi_0,\chi_1,t)}<1$. Also, E is strict $\frac{\alpha}{\eta}$-admissible and for $\{\chi_n\}\subseteq X=1$ so that $\chi_n\to\chi$ as $n\to\infty$ and $\frac{\alpha(\chi_n,\chi_{n+1},t)}{\eta(\chi_n,\chi_{n+1},t)}<1$, for all $n\in\mathcal{N}$, therefore by Theorem 3.3, E has a FP in X.

Let us consider the sequence $\{\chi_n\}_{n=1}^{\infty}$ where $\chi_n=1-\frac{1}{n}$, for all $n\in\mathcal{N}$, and $X=\{\chi_n:n\in\mathcal{N}\}$. We also define $\psi_\Gamma:\mathcal{R}^+\to\mathcal{R}$ by $\psi_\Gamma=(1-\exp(1-\chi_n)^{-a})$, $a>0$. Let $E\chi_n=1-\frac{1}{n-1}$, then $\lim\limits_{n\to\infty}\chi_n=\lim\limits_{n\to\infty}(1-\frac{1}{n})=1$, and $\lim\limits_{n\to\infty}\psi_\Gamma(\chi_n)=\lim\limits_{n\to\infty}(1-\exp(1-\chi_n)^{-a})=\lim\limits_{n\to\infty}(1-\exp(1-1+\frac{1}{n})^{-a})=1-\exp(0)^{-a}=-\infty$.
Now, we have $m>n$ and $\chi=\chi_m$, $\gamma=\chi_n$

$$\begin{aligned}\mathcal{H}_{\mathcal{M}}(E\chi_m,E\chi_n,t)&=\frac{t}{t+|E\chi_m,E\chi_n|}\\&=\frac{t}{t+\left|1-\frac{1}{m-1}-1+\frac{1}{n-1}\right|}\\&=\frac{t}{t+\left|\frac{1}{n-1}-\frac{1}{m-1}\right|}\\&>\frac{t}{t+\left|\frac{1}{m}-\frac{1}{n}\right|}+\frac{t}{t+\left|\frac{1}{n}-\frac{1}{m-1}\right|}\\&>\lambda\mathcal{H}_{\mathcal{M}}(\chi_m,\chi_n,t)+\mathcal{L}\mathcal{H}_{\mathcal{M}}(\chi_n,E\chi_m,t),\end{aligned}$$

$\mathcal{L}\ge0$ and for existence of $0\le\lambda\le1$. Since $\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\le1$. So that

$$\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\mathcal{H}_{\mathcal{M}}(E\chi_m,E\chi_n,t)>\lambda\cdot\mathcal{H}_{\mathcal{M}}(\chi_m,\chi_n,t)+\mathcal{L}\cdot\mathcal{H}_{\mathcal{M}}(\chi_n,E\chi_m,t)$$

for some $\mathcal{L}\ge0$ and for existence of $0\le\lambda\le1$. Since $\psi_\Gamma:\mathcal{R}^+\to\mathcal{R}$ is strictly increasing and for all continuous function $\Gamma(u_1,u_2,u_3,u_4,u_5):(\mathcal{R}^+)^5\to\mathcal{R}^+$, we have

$$\psi_\Gamma\left(\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\mathcal{H}_{\mathcal{M}}(E\chi_m,E\chi_n,t)\right)>\psi_\Gamma(\lambda\mathcal{H}_{\mathcal{M}}(\chi_m,\chi_n,t)+\mathcal{L}\mathcal{H}_{\mathcal{M}}(\chi_n,E\chi_m,t))$$

which implies for some $\tau>0$,

$$2\tau+\psi_\Gamma\left(\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\mathcal{H}_{\mathcal{M}}(E\chi_m,E\chi_n,t)\right)\ge\lambda\cdot\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(\chi_m,\chi_n,t))+\mathcal{L}\cdot\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(\chi_n,E\chi_m,t)).$$

5. Applications

Different kind of applications in differential and integral equations is wide applications of FP for multivalued mappings. As an application in support our theorem, we take Fedholm integral equation.

Let $\mathcal{F}:[0,1]\times[0,1]\times\mathcal{R}\to\mathcal{F}(\mathcal{R})$ be a MV operator, where $\mathcal{F}(\mathcal{R})$ denotes a family of non-empty compact and convex subset of $\mathcal{L}$. Consider the integral equation

$$\chi(\ell)=f(\ell)+\int_0^1\zeta(\ell,s)F(\ell,s,\chi(s))ds, \tag{18}$$

where $\ell\in[0,1]$ and $f\in[0,1]\to\mathcal{R}$ a continuous function. Let $X=\mathcal{C}[0,1]$ be the class of all real valued continuous functions. Let $(\mathcal{C}[0,1],\mathcal{H}_{\mathcal{M}^\circ},*)$ be a IFMS where $\mathcal{H}_{\mathcal{M}^\circ}:X\times X\times\mathcal{R}^+\to[0,1]$ defined by

$$\mathcal{H}_{\mathcal{M}^\circ}(\chi,\gamma,t)=\left(\min_{s\in[0,1]}\left\{\frac{t}{t+\left|\chi(s)-\gamma(s)\right|}\right\}\right)=\frac{t}{t+|\chi(s)-\gamma(s)|}, \tag{19}$$

for all $\chi,\gamma\in\mathcal{C}[0,1]$, $t>0$. Let the following conditions hold:

(p_1) $F(\ell,s,\chi(s))$ is upper semi continuous $[0,1]^2$ for the operator $\mathcal{F}:[0,1]\times[0,1]\times\mathcal{R}\to\mathcal{F}(\mathcal{R})$, for all $\chi\in\mathcal{C}[0,1]$,

(p_2) there exists some continuous function $\zeta:[0,1]\times[0,1]\to[0,+\infty)$, such that

$$\begin{aligned}|\mathcal{F}(\ell,s,\forall\ \chi(s))-\mathcal{F}(\ell,s,\gamma(s))| \quad &\geq\zeta(\ell,s)\Big\{\min_{s\in[0,1]}\Big\{\frac{t}{t+|\chi(s)-\gamma(s)|},\frac{t}{t+|\chi(s)-\mathcal{F}(t,s,\chi(s))|},\\ &\frac{t}{t+|\gamma(s)-\mathcal{F}(t,s,\gamma(s))|},\frac{t}{t+|\chi(s)-\mathcal{F}(t,s,\gamma(s))|},\\ &\frac{t}{t+|\gamma(s)-\mathcal{F}(t,s,\chi(s))|}\Big\}\Big\},\end{aligned}$$

for all $\ell,s\in[0,1]$, $\chi,\gamma\in\mathcal{C}[0,1]$, $t>0$,

(p_3) there exists a $\tau>0$, such that $\inf_{s\in[0,1]}\int_0^1\zeta(\ell,s)ds\geq e^{-2\tau}$.

Theorem 5.1. *With the grasps (p_1),(p_2) and (p_3), the equation*

$$\chi(\ell)=f(\ell)+\int_0^1\zeta(\ell,s)F(\ell,s,\chi(s))ds$$

has a solution in $\mathcal{L}[0,1]$.

Proof. Let $E:X\to\mathcal{C}[0,1]$ be the MVM in a IFMS $(\mathcal{C}[0,1],\mathcal{H}_{\mathcal{M}^\circ},*)$, such that

$$E\chi=\left\{\gamma\in\mathcal{C}[0,1]:\gamma(\ell)=f(\ell)+\int_0^1\mathcal{F}(\ell,s,\chi(s))ds,\ \ell\in[0,1]\right\}.$$

Let $\chi\in\mathcal{C}[0,1]$. For the MV operator $f(\ell,s,\chi(s))\in\mathcal{F}(\ell,s,\chi(s))$ for all $\ell,s\in[0,1]$. This follows that

$$f(\ell)+\int_0^1 f(\ell,s,\chi(s))ds\in E\chi$$

thus, $E\chi\neq\phi$ For $\tau>0,\psi_\Gamma\in\Gamma$ and $\frac{\alpha(\chi_1,\omega_2,t)}{\eta(\chi_1,\omega_2,t)}=1$, i.e.

$$\begin{aligned}&2\tau+\psi_\Gamma(\mathcal{H}_\mathcal{M}(E\chi_1,E\omega_2,t))\\ &\quad\geq\psi_\Gamma(\min\{\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)\}),\end{aligned}$$

for all $\chi_1,\omega_2 \in X$, $t>0$. Then from eq. (18) E on $\mathcal{C}[0,1]$ implies $E\chi \in \mathcal{C}[0,1]$.

Let $\gamma_1 \in E\chi_1$ be arbitrary such that

$$\gamma_1(\ell)=f(\ell)+\int_0^1 \mathcal{F}(\ell,s,\chi_1(s))ds,$$

for all $\ell \in [0,1]$ holds. It implies that, for all $\ell,s \in [0,1]$, $\exists\ f(\ell,s,\chi_1(s)) \in \mathcal{F}(\ell,s,\chi_1(s))$, such that

$$\gamma_1(\ell)=f(\ell)+\int_0^1 f(\ell,s,\chi_1(s))ds,$$

for all $\ell \in [0,1]$. For all $\chi_1,\omega_2 \in X$, $t>0$, it follows from (p_2) that

$$\begin{aligned}&\mathcal{H}_{\mathcal{M}}(|(f(\ell,s,\chi_1(s))-f(\ell,s,\omega_2(s)),t)|)\\ &\quad\geq \zeta(\ell,s)\Big\{\min_{s\in[0,1]}\Big\{\frac{t}{t+|\chi_1(s)-\omega_2(s)|},\frac{t}{t+|\chi_1(s)-\mathcal{F}(t,s,\chi_1(s))|},\frac{t}{t+|\omega_2(s)-\mathcal{F}(t,s,\omega_2(s))|},\\ &\qquad\qquad \frac{t}{t+|\chi_1(s)-\mathcal{F}(t,s,\omega_2(s))|},\frac{t}{t+|\omega_2(s)-\mathcal{F}(t,s,\chi_1(s))|}\Big\}\Big\}.\end{aligned}$$

This implies that $\exists\ \mathcal{Z}(\ell,s) \in \mathcal{F}(\ell,s,\omega_2(s))$, such that for all $\ell,s \in [0,1]$.

$$\begin{aligned}|f(\ell,s,\chi_1(s))-\mathcal{Z}(\ell,s)| \geq \zeta(\ell,s)\Big\{\min_{s\in[0,1]}\Big\{&\frac{t}{t+|\chi_1(s)-\omega_2(s)|},\frac{t}{t+|\chi_1(s)-\mathcal{F}(t,s,\chi_1(s))|},\\ &\frac{t}{t+|\omega_2(s)-\mathcal{F}(t,s,\omega_2(s))|},\frac{t}{t+|\chi_1(s)-\mathcal{F}(t,s,\omega_2(s))|},\\ &\frac{t}{t+|\omega_2(s)-\mathcal{F}(t,s,\chi_1(s))|}\Big\}\Big\}\end{aligned}$$

Now, with MVM, E defined by

$$E(\ell,s)=\mathcal{F}(\ell,s,\omega_2(s))\cap\left\{m\in\mathcal{R}:\left|f(\ell,s,\chi_1(s))-m\right|\geq\zeta(\ell,s)\cdot\left(\frac{t}{t+\left|\chi_1(s)-\omega_2(s)\right|}\right)\right\}.$$

Hence, by (p_1), F is upper semi continuous, it implies that $\exists\ f(\ell,s,\omega_2(s)):[0,1]\times[0,1]\to\mathcal{R}$ such that $f(\ell,s,\omega_2(s)) \in E(\ell,s)$ for $\ell,s \in [0,1]$. Then

$$\gamma_2(\ell)=f(\ell)+\int_0^1 f(\ell,s,\omega_2(s))ds$$

satisfies that

$$\gamma_2(\ell)\in f(\ell)+\int_0^1 \mathcal{F}(\ell,s,\omega_2(s))ds, \ell\in[0,1].$$

That is $\gamma_2 \in E\omega_2$ and

$$\begin{aligned}|\gamma_1(\ell)-\gamma_2(\ell)| &\geq \int_0^1 |f(\ell,s,\chi_1(s))-f(\ell,s,\omega_2(s))|ds\\ &\geq \int_0^1 \zeta(\ell,s)|\chi_1(s)-\omega_2(s)|ds\\ &\geq \min_{\ell\in[0,1]}\int_0^1 \zeta(\ell,s)\Big\{\min_{s\in[0,1]}\Big\{\frac{t}{t+|\chi_1(s)-\omega_2(s)|},\frac{t}{t+|\chi_1(s)-\mathcal{F}(t,s,\chi_1(s))|},\\ &\qquad \frac{t}{t+|\omega_2(s)-\mathcal{F}(t,s,\omega_2(s))|},\frac{t}{t+|\chi_1(s)-\mathcal{F}(t,s,\omega_2(s))|},\\ &\qquad \frac{t}{t+|\omega_2(s)-\mathcal{F}(t,s,\chi_1(s))|}\Big\}\Big\}ds\end{aligned}$$

$$\geq e^{-2\tau}\min\{\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)\},$$

for all $\ell, s \in [0,1]$. Hence, we have

$$\mathcal{H}_{\mathcal{M}}(\gamma_1,\gamma_2,t) \geq e^{-2\tau}\min\{\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)\}.$$

Changing the task of χ_1 and ω_2, we get

$$\mathcal{H}_{\mathcal{M}}(E\chi_1,E\omega_2,t) \geq e^{-2\tau}\min\{\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)\}.$$

Taking natural log on both sides, we have

$$2\tau+\log\{\mathcal{H}_{\mathcal{M}}(E\chi_1,E\omega_2,t)\} \geq \log(\min\{\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)\}).$$

Taking $\psi_\Gamma \in \Gamma$ defined by $\psi_\Gamma(t)=\log(t)$ for $t>0$, we have

$$2\tau+\psi_\Gamma\{\mathcal{H}_{\mathcal{M}}(E\chi_1,E\omega_2,t)\} \geq \psi_\Gamma(\min\{\mathcal{M}(\chi_1,\omega_2,t),\mathcal{M}(\chi_1,E\chi_1,t),\mathcal{M}(\omega_2,E\omega_2,t),\mathcal{M}(\chi_1,E\omega_2,t),\mathcal{M}(\omega_2,E\chi_1,t)\}).$$

According Theorem 5.1 and the function $\Gamma \in S$ given by $\Gamma(u_1,u_2,u_3,u_4,u_5)=\min\{u_1,u_2,u_3,u_4,u_5\}$, the given integral inclusion in eq. (18) has a solution. □

Theorem 5.2. *Let $(\mathcal{C}(X),\mathcal{H}_{\mathcal{M}},*)$ be a IFMS and $E_1,E_2 : X \to \mathcal{C}(X)$ be a MVM if there exists $\psi_\Gamma \in \Gamma$, $\Gamma \in S$ so that*

$$\psi_\Gamma(\mathcal{H}_{\mathcal{M}}(E_1\chi,E_2\gamma,t)) \geq (\psi_\Gamma(\Gamma(\mathcal{M}(\chi,\gamma,t),\mathcal{M}(\chi,E_1\chi,t),\mathcal{M}(\gamma,E_2\gamma,t),\mathcal{M}(\chi,E_2\gamma,t),\mathcal{M}(\gamma,E_1\chi,t))))^k$$

for all $\chi,\gamma \in X$, $t>0$ and $k \in (0,1)$. Then E_1 and E_2 have CFP.

Proof. Here we define $\alpha : X \to (0,1]$ and $F_1,F_2 : X \to (X)$ by

$$F_1(\chi)(\ell)=\begin{cases}\alpha(\chi), & \text{if } \ell \in E_1\chi,\\ 0, & \text{if } \ell \notin E_1\chi\end{cases}$$

and

$$F_2(\chi)(\ell)=\begin{cases}\alpha(\chi), & \text{if } \ell \in E_2\chi,\\ 0, & \text{if } \ell \notin E_2\chi.\end{cases}$$

Then

$$(F_1(\chi))_{\alpha(\chi)}=\{\ell \in F_1(\chi)(\ell) \geq \alpha(\chi)\}=E_1\chi$$

and

$$(F_2(\chi))_{\alpha(\chi)}=\{\ell \in F_2(\chi)(\ell) \geq \alpha(\chi)\}=E_2\chi.$$

Also, from Theorem 5.2, x^* is a FP i.e. $\chi^* \in E_1\chi$ and $\chi^* \in E_2\chi$, which implies that $\chi^* \in E_1\chi \cap E_2\chi$, so that χ^* is a CFP in E_1 and E_2. □

Example 5.1. Let $X = \mathcal{N}$ be endowed with the usual fuzzy metric $\mathcal{M}(\chi,\gamma,t) = \frac{t}{t+|\chi-\gamma|}$, for all $\chi,\gamma \in X$, $t > 0$. Define $\alpha,\eta : X \times X \to [0,\infty)$ by

$$\frac{\alpha(\chi,\gamma,t)}{\eta(\chi,\gamma,t)} = \begin{cases} 1, & \text{if } \chi,\gamma \in \{0,1\}, \\ \frac{1}{2}, & \text{if } \chi,\gamma > 1, \\ 0, & \text{if otherwise.} \end{cases}$$

Let us consider the sequence $\{\chi_n\}_{n=1}^{\infty}$ where $\chi_n = 1-\frac{1}{n}$, for all $n \in \mathcal{N}$, and $X = \{\chi_n : n \in \mathcal{N}\}$. We also define $\psi_\Gamma : \mathcal{R}^+ \to \mathcal{R}$ by $\psi_\Gamma = (1-\exp(1-\chi_n)^{-a})$, $a > 0$. Let $E\chi_n = 1-\frac{1}{n-1}$, then $\lim\limits_{n\to\infty} \chi_n = \lim\limits_{n\to\infty}(1-\frac{1}{n}) = 1$, and $\lim\limits_{n\to\infty} \psi_\Gamma(\chi_n) = \lim\limits_{n\to\infty}(1-\exp(1-\chi_n)^{-a}) = \lim\limits_{n\to\infty}(1-\exp(1-1+\frac{1}{n})^{-a}) = 1-\exp(0)^{-a} = -\infty$.

Now, we have $m > n$ and $\chi = \chi_m$, $\gamma = \chi_n$

$$\begin{aligned} \mathcal{H}_\mathcal{M}(E\chi_m,E\chi_n,t) &= \frac{t}{t+|E\chi_m,E\chi_n|} = \frac{t}{t+\left|1-\frac{1}{m-1}-1+\frac{1}{n-1}\right|} \\ &= \frac{t}{t+\left|\frac{1}{n-1}-\frac{1}{m-1}\right|} > \frac{t}{t+\left|\frac{1}{m}-\frac{1}{n}\right|} + \frac{t}{t+\left|\frac{1}{n}-\frac{1}{m-1}\right|} \\ &> \lambda\mathcal{H}_\mathcal{M}(\chi_m,\chi_n,t) + \mathcal{L}\mathcal{H}_\mathcal{M}(\chi_n,E\chi_m,t) \end{aligned}$$

$\mathcal{L} \ge 0$ and for existence of $0 \le \lambda \le 1$. Since $\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)} \le 1$. So that

$$\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\mathcal{H}_\mathcal{M}(E\chi_m,E\chi_n,t) > \lambda\cdot\mathcal{H}_\mathcal{M}(\chi_m,\chi_n,t) + \mathcal{L}\cdot\mathcal{H}_\mathcal{M}(\chi_n,E\chi_m,t)$$

for some $\mathcal{L} \ge 0$ and for existence of $0 \le \lambda \le 1$. Since $\psi_\Gamma : \mathcal{R}^+ \to \mathcal{R}$ is strictly increasing and for all continuous function $\Gamma(u_1,u_2,u_3,u_4,u_5) : (\mathcal{R}^+)^5 \to \mathcal{R}^+$, we have

$$\psi_\Gamma\left(\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\mathcal{H}_\mathcal{M}(E\chi_m,E\chi_n,t)\right) > \psi_\Gamma(\lambda\mathcal{H}_\mathcal{M}(\chi_m,\chi_n,t) + \mathcal{L}\mathcal{H}_\mathcal{M}(\chi_n,E\chi_m,t))$$

which implies for some $\tau > 0$,

$$2\tau + \psi_\Gamma\left(\frac{\alpha(\chi_m,\chi_n,t)}{\eta(\chi_m,\chi_n,t)}\mathcal{H}_\mathcal{M}(E\chi_m,E\chi_n,t)\right) \ge \lambda\cdot\psi_\Gamma(\mathcal{H}_\mathcal{M}(\chi_m,\chi_n,t)) + \mathcal{L}\cdot\psi_\Gamma(\mathcal{H}_\mathcal{M}(\chi_n,E\chi_m,t)).$$

6. Conclusions

We define a new class of almost (α/η)-ψ_Γ-CMs for IFMS and prove the FPT's in that context. Our results in IFMS are extension of some well known results of contraction. We also support our result with the help of an example and an application of Fredholm integral equation, which is significantly contributed in FPT and will set new association among the young researchers those who are working on the contraction of fuzzy mappings.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M.S. Abdullahi and A. Azam, L-fuzzy fixed point theorems for L-fuzzy mappings via β_{F_L}-admissible with applications, *Journal of Uncertainty Analysis and Applications* **5** (2017), Article number: 2 (2017), DOI: 10.1186/s40467-017-0056-5.

[2] A.E. Al-Mazrooei and J. Ahmad, Fuzzy fixed point results of generalized almost F-contraction, *Journal of Mathematics and Computer Science* **18**(2) (2018), 206 – 215, DOI: 10.22436/jmcs.018.02.08.

[3] S.A. Al-Mezel and J. Ahmad, Generalized fixed point results for almost $(\alpha,F\sigma)$-contraction with applications to Fredholm Integral Inclusions, *Symmetry* **11**(9) (2019), 1068, DOI: 10.3390/sym11091068.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3**(1) (1922), 133 – 181, URL: https://eudml.org/doc/213289.

[5] M. Berinde and V. Berinde, On a general class of multi-valued weakly Picard mappings, *Journal of Mathematical Analysis and Applications* **326**(2) (2007), 772 – 782, DOI: 10.1016/j.jmaa.2006.03.016.

[6] S.S. Chauhan, M. Imdad, G. Kaur and A. Sharma, Some fixed point theorems for S_F-contraction in complete fuzzy metric spaces, *Afrika Matematika* **30** (2019), 651 – 662, DOI: 10.1007/s13370-019-00673-4.

[7] J.G. García, S. Romaguera and M. Sanchis, An identification theorem for the completion of the Hausdorff fuzzy metric, *Fuzzy Sets and Systems* **227** (2013), 96 – 106, DOI: 10.1016/j.fss.2013.04.012.

[8] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Set Systems* **64**(3) (1994), 395 – 439, DOI: 10.1016/0165-0114(94)90162-7.

[9] M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems* **27**(3) (1988), 385 – 389, DOI: 10.1016/0165-0114(88)90064-4.

[10] V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems* **125**(2) (2002), 245 – 253, DOI: 10.1016/S0165-0114(00)00088-9.

[11] M. Gregori and S. Romaguera, Some properties of fuzzy metric spaces, *Fuzzy Sets and Systems* **115**(3) (2000), 485 – 489, DOI: 10.1016/S0165-0114(98)00281-4.

[12] V. Gregori, S. Morillas and A. Sapena, Examples of fuzzy metrics and applications, *Fuzzy Sets and Systems* **170**(1) (2011), 95 – 111, DOI: 10.1016/j.fss.2010.10.019.

[13] N. Hussain, A. Latif and I. Iqbal, Fixed point results for generalized F-contractions in moduler metric and fuzzy metric spaces, *Fixed Point Theory and Applications* **2015** (2015), Article number: 158, DOI: 10.1186/s13663-015-0407-1.

[14] N. Hussain, P. Salimi and A. Latif, Fixed point results for single and set-valued α-η-ψ-contractive mappings, *Fixed Point Theory and Applications* **2013** (2013), Article number: 212, DOI: 10.1186/1687-1812-2013-212.

[15] I. Kramosil and J. Michálek, Fuzzy metrics and statistical metric spaces, *Kybernetika* **11** (1975), 326 – 334, URL: https://www.kybernetika.cz/content/1975/5/336/paper.pdf.

[16] S. Kutukch, S. Sharma and H. Tokgoz, A fixed point theorem in fuzzy metric space, *International Journal of Mathematical Analysis* **1**(18) (2007), 861 – 872.

[17] Y. Liu and Z. Li, Coincidence point theorems in probabilistic and fuzzy metric spaces, *Fuzzy Sets and Systems* **158**(1) (2007), 58 – 70, DOI: 10.1016/j.fss.2006.07.010.

[18] D. Miheţ, A class of contractions in fuzzy metric spaces, *Fuzzy Sets and Systems* **161**(8) (2010), 1131 – 1137, DOI: 10.1016/j.fss.2009.09.018.

[19] D. Miheţ, Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces, *Fuzzy Sets and Systems* **159**(6) (2008), 739 – 744, DOI: 10.1016/j.fss.2007.07.006.

[20] S.B. Nadler (Jr.), Multi-valued contraction mappings, *Pacific Journal of Mathematics* **30** (1969), 475 – 488, URL: https://msp.org/pjm/1969/30-2/pjm-v30-n2-p12-s.pdf.

[21] M. Nazam, M. Arsad and M. Postolach, Coincidence and common fixed point theorems for four mappings satisfying (α_s,F)-contraction, *Nonlinear Analysis and Model Control* **23**(5) (2018), 664 – 690, DOI: 10.15388/NA.2018.5.3.

[22] S. Phiangsungnoen, Y.J. Cho and P. Kumam, Fixed point results for modified various contractions in fuzzy metric spaces via α-admissiable, *Filomat* **30**(7) (2016), 1869 – 1881, DOI: 10.2298/FIL1607869P.

[23] S. Phiangsungnoen, P. Thounthong and P. Kumam, Fixed point results in fuzzy metric spaces via α and β_k admissible mappings with application to integral types, *Journal of Intelligent & Fuzzy Systems* **34**(1) (2018), 467 – 475, DOI: 10.3233/JIFS-17350.

[24] V. Popa, A general fixed point theorem for weakly compatible mappings in compact metric spaces, *Turkish Journal of Mathematics* **25** (2001), 465 – 474, URL: https://dergipark.org.tr/en/download/article-file/127661.

[25] J. Rodríguez-López and S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems* **147**(2) (2004), 273 – 283, DOI: 10.1016/j.fss.2003.09.007.

[26] A. Roldán, J. Martínez-Moreno, C. Roldán and Y.J. Cho, Multidimensional coincidence point results for compatible mappings in partial order fuzzy metric space, *Fuzzy Sets and Systems* **251** (2014), 71 – 82, DOI: 10.1016/j.fss.2013.10.009.

[27] I.A. Rus and A. Sîntămărian, Data dependence of the fixed point set of some multivalued weakly Picard operators, *Nonlinear Analysis: Theory, Methods & Applications* **52**(8) (2003), 1947 – 1959, DOI: 10.1016/S0362-546X(02)00288-2.

[28] R.K. Saini, B. Singh and A. Jain, A fixed point theorem with implicit relations in fuzzy metric space, *International Mathematical Forum* **5** (2010), 2485 – 2496.

[29] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Analysis: Theory, Methods & Applications* **75**(4) (2012), 2154 – 2165, DOI: 10.1016/j.na.2011.10.014.

[30] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific Journal of Mathematics **10**(1) (1960), 313 – 334, URL: https://projecteuclid.org/journals/pacific-journal-of-mathematics/volume-10/issue-1/Statistical-metric-spaces/pjm/1103038644.full.

[31] M.S. Sezen, Fixed point theorems for new type contractive mappings, *Journal of Functional Spaces* **2019** (2019), Article ID 2153563, DOI: 10.1155/2019/2153563.

[32] M. Sgroi and C. Vetro, Multi-valued F-contractions and solution of certain functional and integral equations, *Filomat* **27**(7) (2013), 1259 – 1268, DOI: 10.2298/FIL1307259S.

[33] D. Wardowaski, Fixed points for new type of contractive mappings in complete metric spaces, *Fixed Point Theory and Applications* **2012** (2012), Article number: 94, DOI: 10.1186/1687-1812-2012-94.

[34] L.A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 –353, DOI: 10.1016/S0019-9958(65)90241-X.