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Differential Harnack Estimate for A Nonlinear Parabolic Equation under List's Flow

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Abstract In this paper, we prove a differential Harnack inequality for positive solutions of nonlinear parabolic equations of the type

$$f_t = \Delta f - f\log f + Sf,$$

see (1.9). Here the metric is evolving under the List's flow.

1. Introduction

Let $(M^n, g(t))$, $(t \in [0,T)$ be a solution to the following List's flow which was introduced by B. List [15]:

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\alpha_n d\varphi \otimes d\varphi, \\ \varphi_t = \Delta\varphi, \end{cases} \tag{1.1}$$

where $\alpha_n = \frac{n-1}{n-2}$, φ is a smooth function on M^n and Δ denotes the Laplacian given by $g(t)$. The motivation to study the system (1.1) stems from its connection to general relativity. Let $S_{ij} = R_{ij} - \alpha_n \varphi_i \varphi_j$ be a symmetric two-tensor. Then (1.1) becomes

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2S_{ij}, \\ \varphi_t = \Delta\varphi. \end{cases} \tag{1.2}$$

The following solitons are special solutions of the system (1.2) in the spirit of the definitions for the Hamilton's Ricci flow.

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Definition 1.1. We call a Riemannian manifold (M^n, g) a gradient soliton if there exists a smooth function h such that for some constant ρ, it holds that

$$\begin{cases} S_{ij} + h_{ij} = \rho g_{ij}, \\ \Delta\varphi = \nabla\varphi\nabla h. \end{cases} \tag{1.3}$$

In particular, h is called the potential function. If $\rho > 0$ in (1.3), then (M^n, g) is called a shrinking gradient soliton; if $\rho < 0$, (M^n, g) is called a expanding gradient soliton; and if $\rho = 0$, (M^n, g) is called a steady gradient soliton.

Let $S = g^{ij}S_{ij} = R - \alpha_n|\nabla\varphi|^2$ be the trace of the two-tensor S_{ij}. By the second Bianchi identity, we deduce from (1.3) that

$$\begin{aligned} \frac{1}{2}R_{,i} &= g^{jk}R_{ij,k} \\ &= g^{jk}(\rho g_{ij} - h_{ij} + \alpha_n\varphi_i\varphi_j)_{,k} \\ &= -g^{jk}h_{ij,k} + \frac{\alpha_n}{2}(|\nabla\varphi|^2)_i + \alpha_n(\Delta\varphi)\varphi_i \\ &= -(\Delta h)_{,i} - R_{ij}h^j + \frac{\alpha_n}{2}(|\nabla\varphi|^2)_i + \alpha_n(\Delta\varphi)\varphi_i \\ &= S_{,i} - R_{ij}h^j + \frac{\alpha_n}{2}(|\nabla\varphi|^2)_i + \alpha_n(\Delta\varphi)\varphi_i \end{aligned}$$

which shows that

$$\frac{1}{2}S_{,i} = S_{ij}h^j. \tag{1.4}$$

Here, $R_{,i}$ stands for the covariant derivative for R along the direction i. Hence $(S + |\nabla h|^2 - 2\rho h)_{,i} = 0$ and there is a constant C such that

$$S + |\nabla h|^2 - 2\rho h = C. \tag{1.5}$$

On the other hand, taking the trace of both sides of (1.3) yields

$$S + \Delta h = n\rho. \tag{1.6}$$

Combining (1.5) and (1.6), we arrive at

$$\Delta h - |\nabla h|^2 + 2\rho h = n\rho - C. \tag{1.7}$$

Letting $f = e^{-h}$, then (1.7) can be written as

$$\Delta f + 2\rho f \log f + (n\rho - C)f = 0. \tag{1.8}$$

It is very interesting to find the exact value of the constant C. Hence, it is interesting to study the following nonlinear parabolic equation under the List's flow:

$$f_t = \Delta f + a(t)f \log f + bSf, \tag{1.9}$$

where b is a constant and $a(t)$ is a function depending only on the time t. If we define $f = e^{-u}$, then (1.9) satisfies the evolution equation:

$$u_t = \Delta u - |\nabla u|^2 - bS + a(t)u. \tag{1.10}$$

The differential Harnack inequalities were originated by Li-Yau [17] for positive solutions of the heat equation and have become one of important tools in the study of geometric analysis. Its technique was then brought into the study of geometric evolution equation by Hamilton, see [9]. For the research of differential Harnack inequalities, for example, see [1–4, 6–8, 10–14, 18–20]. In this paper, we prove differential Harnack inequalities for positive solutions of nonlinear parabolic equations of the type (1.9) under the List's flow by using the methods introduced by Cao and Zhang in [5]. Our main results are as follows:

Theorem 1.1. *Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the List's flow* (1.2) *on a compact manifold. Let f be a positive solution to the equation*

$$f_t = \Delta f - f \log f + S f, \tag{1.11}$$

$f = e^{-u}$ and

$$\overline{H} = 2\Delta u - |\nabla u|^2 - 3S - \frac{2n}{t}. \tag{1.12}$$

If $S_t + \frac{1}{t} S + 2\nabla u \nabla S + 2S^{ij} u_i u_j$ and S both are nonnegative, then for all time $t \in (0, T)$,

$$\overline{H} \leq \frac{n}{4}. \tag{1.13}$$

Using the standard method, we can prove the following consequence easily.

Corollary 1.2. *Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the List's flow* (1.2) *on a compact manifold. Let f be a positive solution to the equation*

$$f_t = \Delta f - f \log f + S f. \tag{1.14}$$

Assume that (x_1, t_1) and (x_2, t_2), $0 < t_1 < t_2$ are two points in $M^n \times (0, T)$. Let

$$\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} e^t \left(|\gamma'|^2 + S + \frac{2n}{t} + \frac{n}{4} \right) dt,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2). Then we have

$$e^{t_1} \log f(x_1, t_1) \leq e^{t_2} \log f(x_2, t_2) + \frac{\Gamma}{2}. \tag{1.15}$$

Theorem 1.3. *Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the List's flow* (1.2) *on a compact manifold. Let f be a positive solution to the equation*

$$f_t = \Delta f - \frac{2}{t+2} f \log f + S f, \tag{1.16}$$

$f = e^{-u}$ and

$$\widetilde{H} = 2\Delta u - |\nabla u|^2 - 3S - \frac{2n}{t}. \tag{1.17}$$

If $S_t+\frac{1}{t}S+2\nabla u\nabla S+2S^{ij}u_iu_j$ *and* S *both are nonnegative, then for all time* $t\in(0,T)$,

$$\widetilde{H}\leq 0. \tag{1.18}$$

We remark that our results extend the ones in [5].

2. Proof of Results

First we recall two evolutions under the List's flow (1.2) which can be found in [16]:

Lemma 2.1. *Let* $(M^n,g(t))$ *be a solution to the List's flow* (1.2). *Then the following evolution equations hold:*

(i) $\dfrac{\partial}{\partial t}\Gamma_{ij}^k=g^{kl}(-R_{jl,i}-R_{il,j}+R_{ij,l}+2\alpha_n\varphi_{ij}\varphi_l)$,

(ii) $\dfrac{\partial}{\partial t}S_{ij}=\Delta S_{ij}-R_{il}S_{jl}-R_{jl}S_{il}-2R_{kijl}S_{kl}+2\alpha_n(\Delta\varphi)\varphi_{ij}$.

Next, we prove the following key Lemma 2.2 which will be used late.

Lemma 2.2. *Let* $(M^n,g(t))$ *be a solution to the List's flow* (1.2) *and* u *satisfies* (1.10). *Let*

$$H=\alpha\Delta u-\beta|\nabla u|^2+\gamma S+p\frac{u}{t}+q\frac{n}{t},$$

where α, β, γ, p, q *are all constants. Then* H *satisfies the following evolution equation:*

$$\begin{aligned}
H_t=&\Delta H-2\nabla H\nabla u+2(\gamma+b\beta)\nabla u\nabla S\\
&-2(\alpha-\beta)\left|u_{ij}-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2\\
&+\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)H-b\alpha\Delta S-2\alpha\alpha_n(\Delta\varphi)\nabla\varphi\nabla u\\
&+\left[-2\beta a(t)+\frac{p}{t}+\beta\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)\right]|\nabla u|^2\\
&+2\beta\alpha_n(\nabla\varphi\nabla u)^2+\left(2\gamma+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2-2\alpha R^{ij}u_iu_j+2\gamma\alpha_n(\Delta\varphi)^2\\
&-\left[\frac{bp}{t}-\frac{\alpha\lambda}{t}+\gamma\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)\right]S+\left(\frac{pa(t)}{t}-\frac{p}{t^2}\right)u\\
&-\frac{qn}{t^2}+\frac{n\lambda^2(\alpha-\beta)}{2t^2}-\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)\left(\frac{pu}{t}+\frac{qn}{t}\right),
\end{aligned}\tag{2.1}$$

where λ *is also a constant.*

Proof. From (1.2) we have

$$\frac{\partial}{\partial t}g^{ij}=2S^{ij}.$$

Hence, we obtain from Lemma 2.1 that

$$\begin{aligned}(\Delta u)_t &= (g^{ij}u_{ij})_t\\ &= 2S^{ij}u_{ij}+\Delta(u_t)-g^{ij}u_k\frac{\partial}{\partial t}\Gamma^k_{ij}\\ &= \Delta(\Delta u)-\Delta(|\nabla u|^2)-b\Delta S+a(t)\Delta u+2S^{ij}u_{ij}-2\alpha_n(\Delta\varphi)\nabla\varphi\nabla u\\ &= \Delta(\Delta u)-2|\nabla^2 u|^2-2\nabla u\nabla\Delta u-b\Delta S+a(t)\Delta u\\ &\quad +2S^{ij}u_{ij}-2R^{ij}u_iu_j-2\alpha_n(\Delta\varphi)\nabla\varphi\nabla u, \end{aligned}\tag{2.2}$$

where in the last equality we used the Bochner formula

$$\Delta(|\nabla u|^2)=2|\nabla^2 u|^2+2\nabla u\nabla\Delta u+2\mathrm{Ric}(\nabla u,\nabla u).$$

Using the Lemma 2.1 again, we arrive at

$$\begin{aligned}(|\nabla u|^2)_t &= \Delta(|\nabla u|^2)-2|\nabla^2 u|^2-2\nabla u\nabla(|\nabla u|^2)-2b\nabla u\nabla S\\ &\quad +2a(t)|\nabla u|^2-2\alpha_n(\nabla\varphi\nabla u)^2,\end{aligned}\tag{2.3}$$

and

$$S_t=\Delta S+2|S_{ij}|^2+2\alpha_n(\Delta\varphi)^2.\tag{2.4}$$

By virtue of (2.2)-(2.4), we have

$$\begin{aligned}H_t &= \alpha[\Delta(\Delta u)-2|\nabla^2 u|^2-2\nabla u\nabla\Delta u-b\Delta S+a(t)\Delta u\\ &\quad +2S^{ij}u_{ij}-2R^{ij}u_iu_j-2\alpha_n(\Delta\varphi)\nabla\varphi\nabla u]\\ &\quad -\beta[\Delta(|\nabla u|^2)-2|\nabla^2 u|^2-2\nabla u\nabla(|\nabla u|^2)-2b\nabla u\nabla S\\ &\quad +2a(t)|\nabla u|^2-2\alpha_n(\nabla\varphi\nabla u)^2]+\gamma[\Delta S+2|S_{ij}|^2+2\alpha_n(\Delta\varphi)^2]\\ &\quad +\frac{p}{t}[\Delta u-|\nabla u|^2-bS+a(t)u]-\frac{pu}{t^2}-\frac{qn}{t^2}\\ &= \Delta H-2\nabla H\nabla u+2(\gamma+b\beta)\nabla u\nabla S-2(\alpha-\beta)|\nabla^2 u|^2+2\alpha S^{ij}u_{ij}\\ &\quad -b\alpha\Delta S+\alpha a(t)\Delta u-2\alpha\alpha_n(\Delta\varphi)\nabla\varphi\nabla u+\left(-2\beta a(t)+\frac{p}{t}\right)|\nabla u|^2\\ &\quad +2\beta\alpha_n(\nabla\varphi\nabla u)^2+2\gamma|S_{ij}|^2-2\alpha R^{ij}u_iu_j+2\gamma\alpha_n(\Delta\varphi)^2-\frac{bp}{t}S\\ &\quad +\left(\frac{pa(t)}{t}-\frac{p}{t^2}\right)u-\frac{qn}{t^2}\\ &= \Delta H-2\nabla H\nabla u+2(\gamma+b\beta)\nabla u\nabla S\\ &\quad -2(\alpha-\beta)\left|u_{ij}-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2\end{aligned}$$

$$
\begin{aligned}
&+\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)H-b\alpha\Delta S-2\alpha\alpha_n(\Delta\varphi)\nabla\varphi\nabla u\\
&+\left[-2\beta a(t)+\frac{p}{t}+\beta\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)\right]|\nabla u|^2\\
&+2\beta\alpha_n(\nabla\varphi\nabla u)^2+\left(2\gamma+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2-2\alpha R^{ij}u_iu_j+2\gamma\alpha_n(\Delta\varphi)^2\\
&-\left[\frac{bp}{t}-\frac{\alpha\lambda}{t}+\gamma\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)\right]S+\left(\frac{pa(t)}{t}-\frac{p}{t^2}\right)u\\
&-\frac{qn}{t^2}+\frac{n\lambda^2(\alpha-\beta)}{2t^2}-\left(a(t)-\frac{2(\alpha-\beta)\lambda}{\alpha t}\right)\left(\frac{pu}{t}+\frac{qn}{t}\right).
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 1.1. Choosing $\alpha=2$, $\beta=1$, $\gamma=-3$, $p=0$, $q=-2$ and $\lambda=2$ in the above Lemma 2.2, we have

$$
\begin{aligned}
\overline{H}_t=&\Delta\overline{H}-2\nabla\overline{H}\nabla u-4\nabla u\nabla S-2\left|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\right|^2-\left(1+\frac{2}{t}\right)\overline{H}-2\Delta S\\
&-4\alpha_n(\Delta\varphi)\nabla\varphi\nabla u+\left(1-\frac{2}{t}\right)|\nabla u|^2+2\alpha_n(\nabla\varphi\nabla u)^2-4|S_{ij}|^2\\
&-4R^{ij}u_iu_j-6\alpha_n(\Delta\varphi)^2-\left(3+\frac{2}{t}\right)S-\frac{2n}{t}\\
=&\Delta\overline{H}-2\nabla\overline{H}\nabla u-2\left|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\right|^2-\left(1+\frac{2}{t}\right)\overline{H}+\left(1-\frac{2}{t}\right)|\nabla u|^2\\
&-3S-\frac{2n}{t}-2\alpha_n(\Delta\varphi+\nabla\varphi\nabla u)^2-2\left(S_t+\frac{S}{t}+2\nabla u\nabla S+2S^{ij}u_iu_j\right)\\
\le&\Delta\overline{H}-2\nabla\overline{H}\nabla u-\frac{2}{n}\left(\Delta u-S-\frac{n}{t}\right)^2-\left(1+\frac{2}{t}\right)\overline{H}+\left(1-\frac{2}{t}\right)|\nabla u|^2\\
&-3S-\frac{2n}{t}-2\alpha_n(\Delta\varphi+\nabla\varphi\nabla u)^2-2\left(S_t+\frac{S}{t}+2\nabla u\nabla S+2S^{ij}u_iu_j\right),
\end{aligned}
\tag{2.5}
$$

where in the last inequality we used the Cauchy-Schwarz inequality. By the definition in (1.12), we have

$$
|\nabla u|^2=2\left(\Delta u-S-\frac{n}{t}\right)-\overline{H}-S.
$$

Hence, (2.5) yields

$$
\begin{aligned}
\overline{H}_t\le&\Delta\overline{H}-2\nabla\overline{H}\nabla u-\frac{2}{n}\left(\Delta u-S-\frac{n}{t}\right)^2+2\left(\Delta u-S-\frac{n}{t}\right)\\
&-\left(2+\frac{2}{t}\right)\overline{H}-\frac{2}{t}|\nabla u|^2-4S-\frac{2n}{t}-2\alpha_n(\Delta\varphi+\nabla\varphi\nabla u)^2
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(S_t+\frac{S}{t}+2\nabla u\nabla S+2S^{ij}u_iu_j\right)\\
=&\Delta\overline{H}-2\nabla\overline{H}\nabla u-\frac{2}{n}\left(\Delta u-S-\frac{n}{t}-\frac{n}{2}\right)^2-\left(2+\frac{2}{t}\right)\overline{H}\\
&-\frac{2}{t}|\nabla u|^2-4S+\frac{n}{2}-\frac{2n}{t}-2\alpha_n(\Delta\varphi+\nabla\varphi\nabla u)^2\\
&-2\left(S_t+\frac{S}{t}+2\nabla u\nabla S+2S^{ij}u_iu_j\right).
\end{aligned}
\tag{2.6}
$$

Under the assumption in Theorem 1.1, we derive from (2.6)

$$
\overline{H}_t\leq\Delta\overline{H}-2\nabla\overline{H}\nabla u-\left(2+\frac{2}{t}\right)\overline{H}+\frac{n}{2}-\frac{2n}{t}.
\tag{2.7}
$$

It is easy to see that for t small enough that

$$
\overline{H}<0.
$$

Moreover, for any $T_0<T$, we assume that the maximum in $(0,T_0]$ is taken at t_0. Then at the maximum value point, we have

$$
\left(2+\frac{2}{t_0}\right)\overline{H}\leq\frac{n}{2}-\frac{2n}{t_0}
$$

which means that

$$
\overline{H}\leq\frac{n}{4}\left(1-\frac{5}{t_0+1}\right)\leq\frac{n}{4}.
\tag{2.8}
$$

We complete the proof of Theorem 1.1. □

Proof of Theorem 1.3. Choosing $\alpha=2$, $\beta=1$, $\gamma=-3$, $p=0$, $q=-2$ and $\lambda=2$ in the above Lemma 2.2 and noticing that $a(t)=-\frac{2}{t+2}$, we have

$$
\begin{aligned}
\widetilde{H}_t=&\Delta\widetilde{H}-2\nabla\widetilde{H}\nabla u-4\nabla u\nabla S-2\left|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\right|^2-\left(\frac{2}{t+2}+\frac{2}{t}\right)\widetilde{H}\\
&-2\Delta S-4\alpha_n(\Delta\varphi)\nabla\varphi\nabla u+\left(\frac{2}{t+2}-\frac{2}{t}\right)|\nabla u|^2+2\alpha_n(\nabla\varphi\nabla u)^2\\
&-4|S_{ij}|^2-4R^{ij}u_iu_j-6\alpha_n(\Delta\varphi)^2-\left(\frac{6}{t+2}+\frac{2}{t}\right)S-\frac{4n}{t^2+2t}\\
=&\Delta\widetilde{H}-2\nabla\widetilde{H}\nabla u-2\left|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\right|^2-\left(\frac{2}{t+2}+\frac{2}{t}\right)\widetilde{H}\\
&+\left(\frac{2}{t+2}-\frac{2}{t}\right)|\nabla u|^2-\frac{6}{t+2}S-\frac{4n}{t^2+2t}-2\alpha_n(\Delta\varphi+\nabla\varphi\nabla u)^2\\
&-2\left(S_t+\frac{S}{t}+2\nabla u\nabla S+2S^{ij}u_iu_j\right).
\end{aligned}
\tag{2.9}
$$

Under the assumption in Theorem 1.3, we derive from (2.9)

$$
\widetilde{H}_t\leq\Delta\widetilde{H}-2\nabla\widetilde{H}\nabla u-\left(\frac{2}{t+2}+\frac{2}{t}\right)\widetilde{H}-\frac{4n}{t^2+2t}.
\tag{2.10}
$$

It is easy to see that for t small enough that

$$\widetilde{H} < 0.$$

Therefore, we obtain $\widetilde{H} \leq 0$ by applying maximum principle. It completes the proof of Theorem 1.3. □

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