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Research Article

Fixed Point Results for Cyclic Contractions in Partial Symmetric Spaces

Savita Rathee[1] and Priyanka Gupta*[2]

[1] *Department of Mathematics, M. D. University, Rohtak 124001, Haryana, India*
[2] *Department of Mathematics, Govt. College for Women, Madlauda, Panipat 132103, Haryana, India*

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Abstract. In this paper, we prove fixed point results for various cyclic contractions in partial symmetric spaces. Our results generalize the fixed point results of Asim *et al*. (Fixed point results in partial symmetric spaces with an application, *Axioms* **8**(13) (2019), 1 – 15) proved for the class of partial symmetric spaces for various contractions. Also, we provide an example in the support of proved result.

Keywords. Partial symmetric spaces, Fixed point theorems, Cyclic contractions

Mathematics Subject Classification (2020). 47H10; 54H25

1. Introduction

Banach [4] in 1922, firstly originated the concept of contraction in the field of *Fixed Point Theory* and gave the *Banach Contraction Principle*. This principle has been prolonged and theorized to different class of spaces, such as *metric like spaces* [2], *partial metric spaces* [13], *b-metric spaces* [5] and several others. Sometimes, one may find situations in what respect all the metric circumstances are not essential [1,6,7,14,15]. Inspired by this, several authors rooted *fixed point results* in *symmetric spaces*. A symmetric d on a non-empty set $\mathscr{U}$ is a function $d : \mathscr{U} \times \mathscr{U} \to R^+$ which entertain $d(u,v) = d(v,u)$ and $d(u,v) = 0$ if and only if $u = v$, for all $u,v \in \mathscr{U}$. Different from metric, the symmetric spaces are not continuous. The uniqueness of

*Email: gupta.priyanka989@gmail.com

the limit of a sequence is no longer make sured because of the awol of triangular inequality. Matthews [13] originated the concept of *partial metric spaces* and settled a cognate of the *Banach contraction*, *Kannan-Ciric and Ciric quasi type* fixed point results.

Combining the perceptions of *partial metric spaces and symmetric spaces*, Asim *et al*. [3] originated the class of *Partial Symmetric Spaces* (PSS) and gave *fixed point results* for distinct type of contractions in such spaces. On the other hand, in 2003, the Banach Contraction Principle was generalised by Kirk *et al*. [10] with the use of cyclic contraction and further used by several authors to attain various fixed point results (see e.g., [8, 9]).

This paper deals with a unique impend in the field of cyclic contraction mappings. In this paper, we prove fixed point results for various cyclic contractions in PSS. Our results generalize the several existing results. Also we provide an example in the support of proved result.

2. Preliminaries

Asim *et al*. [3] introduced the PSS for a non-empty set $\mathscr{U}$.

Definition 2.1. Let $\mathscr{U}$ be a non-empty set. $P : \mathscr{U} \times \mathscr{U} \to R_+$ is said to be partial symmetric if, for all $\mu, \nu \in \mathscr{U}$,

(i) $\mu = \nu$ iff $P(\mu,\mu) = P(\nu,\nu) = P(\mu,\nu)$;

(ii) $P(\mu,\mu) \le P(\mu,\nu)$;

(iii) $P(\mu,\nu) = P(\nu,\mu)$.

Then, the pair $(\mathscr{U}, P)$ is said to be PSS.

A PSS $(\mathscr{U}, P)$ diminishes to a symmetric space if $P(\mu,\mu) = 0$, for all $\mu \in \mathscr{U}$. Undoubtedly, every symmetric space is PSS, but not conversely.

Example 2.2 ([3]). Consider $\mathscr{U} = R$ and outline a mapping $P : \mathscr{U} \times \mathscr{U} \to R^+$ for all $\mu, \nu \in \mathscr{U}$ and $p, q > 1$, as follows:

$$P(\mu,\nu) = |\mu - \nu|^p + |\mu - \nu|^q.$$

Then, $(\mathscr{U}, P)$ becomes a PSS.

Example 2.3 ([3]). Consider $\mathscr{U} = R^+$ and outline a mapping $P : \mathscr{U} \times \mathscr{U} \to R^+$ for all $\mu, \nu \in \mathscr{U}$ and $p, q > 1$, as follows:

$$P(\mu,\nu) = (\max\{\mu,\nu\})^p + (\max\{\mu,\nu\})^q.$$

Then, $(\mathscr{U}, P)$ becomes a PSS.

Example 2.4 ([3]). Consider $\mathscr{U} = [0,\pi)$ and outline a mapping $P : \mathscr{U} \times \mathscr{U} \to R^+$ for all $\mu, \nu \in \mathscr{U}$ and $\alpha > 0$, as follows:

$$P(\mu,\nu) = \sin|\mu - \nu| + \alpha.$$

Then, $(\mathscr{U}, P)$ becomes a PSS.

Let $(\mathscr{U}, P)$ be a PSS. Then the P-open ball, with centre $\mu \in \mathscr{U}$, radius $\epsilon > 0$ is outlined by

$$B_P(\mu,\epsilon) = \{\nu \in \mathscr{U} : P(\mu,\nu) < P(\mu,\mu) + \epsilon\}.$$

Similarly, for centre $u \in \mathscr{U}$ and radius $\epsilon > 0$, the P-closed ball is outlined by

$$B_P[\mu,\epsilon] = \{v \in \mathscr{U} : P(\mu,\nu) \leq P(\mu,\mu)+\epsilon\}.$$

The collection of all P-open balls, for all $\mu \in \mathscr{U}$ and $\epsilon > 0$,

$$\mu_P = \{B_P(\mu,\epsilon) : \mu \in \mathscr{U}, \epsilon > 0\}$$

forms a basis for topology τ_P on $\mathscr{U}$.

Lemma 2.5. *Let $(\mathscr{U},\tau_P)$ be a topological space and $F : \mathscr{U} \to \mathscr{U}$ is continuous then, for every convergent sequence $\{\mu_n\}$ converges to μ in $\mathscr{U}$, the sequence $\{F\mu_n\}$ converges to $F\mu$.*

Now, we outline some basic definitions which we will use in our subsequent future discussions.

Definition 2.6. A sequence $\{\mu_n\}$ in PSS $(\mathscr{U},P)$ is forenamed to be P-convergent to $\mu \in \mathscr{U}$, with respect to τ_P, if

$$P(\mu,\mu) = \lim_{n\to\infty} P(\mu_n,\mu).$$

Definition 2.7. A sequence $\{\mu_n\}$ in $(\mathscr{U},P)$ is forenamed to be P-Cauchy if $\lim_{m,n\to\infty} P(\mu_n,\mu_m)$ exists and is finite.

Definition 2.8. A PSS $(\mathscr{U},P)$ is forenamed to be P-complete if each one P-Cauchy sequence $\{\mu_n\}$ in $\mathscr{U}$ is P-convergent, with respect to τ_P, to a function $\mu \in \mathscr{U}$ such that

$$P(\mu,\mu) = \lim_{n\to\infty} P(\mu_n,\mu) = \lim_{n,m\to\infty} P(\mu_n,\mu_m).$$

Now, we state some definitions related to PSS on $\mathscr{U}$.

Definition 2.9. Let $(\mathscr{U},P)$ be a PSS. Then

(A1) $\lim_{n\to\infty} P(\mu_n,\mu) = P(\mu,\mu)$ and $\lim_{n\to\infty} P(\mu_n,\nu) = P(\mu,\nu)$ imply $\mu = \nu$, for a sequence $\{\mu_n\}, \mu$ and ν in $\mathscr{U}$.

(A2) A PSS $(P,\mathscr{U})$ is said to be 1-continuous if $\lim_{n\to\infty} P(\mu_n,\mu) = P(\mu,\mu)$ implies that $\lim_{n\to\infty} P(\mu_n,\nu) = P(\mu,\nu)$, for a sequence $\{\mu_n\}$ in $\mathscr{U}$ and $\mu,\nu \in \mathscr{U}$.

(A3) A PSS $(\mathscr{U},P)$ is said to be continuous if $\lim_{n\to\infty} P(\mu_n,\mu) = P(\mu,\mu)$ and $\lim_{n\to\infty} P(\mu_n,v) = P(\mu,\nu)$ imply that $\lim_{n\to\infty} P(\mu_n,\nu_n) = P(\mu,\nu)$ where $\{\mu_n\}$ and $\{\nu_n\}$ are sequences in $\mathscr{U}$ and $\mu,\nu \in \mathscr{U}$.

(A4) $\lim_{n\to\infty} P(\mu_n,\mu) = P(\mu,\mu)$ and $\lim_{n\to\infty} P(\mu_n,\nu_n) = P(\mu,\mu)$ imply $\lim_{n\to\infty} P(\nu_n,\mu) = P(\mu,\mu)$, for a sequence $\{\mu_n\},\{\nu_n\}$ and μ in $\mathscr{U}$.

(A5) $\lim_{n\to\infty} P(\mu_n,\nu_n) = P(\mu,\mu)$ and $\lim_{n\to\infty} P(\nu_n,\eta_n) = P(\mu,\mu)$ imply $\lim_{n\to\infty} P(\mu_n,\eta_n) = P(\mu,\mu)$, for a sequence $\{\mu_n\},\{\nu_n\}$ and $\{\eta_n\}$ in $\mathscr{U}$.

Definition 2.10. Let f be any self mapping defined on a non-empty set $\mathscr{U}$ then $\mu \in \mathscr{U}$ is said to be a fixed point of f if $f\mu = \mu$.

Definition 2.11 ([11])**.** Let $\mathscr{A}$ and $\mathscr{B}$ be non-empty subsets of a set $\mathscr{U}$. A map $T : \mathscr{A} \cup \mathscr{B} \to \mathscr{A} \cup \mathscr{B}$ is said to be a cyclic map if $T(\mathscr{A}) \subseteq \mathscr{B}$ and $T(\mathscr{B}) \subseteq \mathscr{A}$.

Definition 2.12 ([12])**.** Let $(\mathscr{U},d)$ be a complete metric space and T, S be two mappings. Then T and S are said to be weakly compatible if they commute at their coincidence point $\mu \in \mathscr{U}$, that is, $T\mu = S\mu$ implies $TS\mu = ST\mu$.

3. Main Results

In this section, we prove some fixed point theorems in the relation of a *symmetric space*.
Let $(\mathscr{U},P)$ be a PSS and $\mathscr{A}:\mathscr{U}\to\mathscr{U}$. Then, for every $\mu\in\mathscr{U}$ and for all $i,j\in N$, we define

$$\Xi(P,\mathscr{A},\mu)=\sup\{P(\mathscr{A}^i\mu,\mathscr{A}^j\mu):i,j\in N\}. \tag{3.1}$$

Definition 3.1. Let C_1 and C_2 be non-empty subsets of PSS $(\mathscr{U},P)$. A cyclic map $\mathscr{A}:C_1\cup C_2\to C_1\cup C_2$ is said to be cyclic k-contraction if for all $k\in[0,1)$ such that

$$P(\mathscr{A}\mu,\mathscr{A}\nu)\le kP(\mu,\nu) \tag{3.2}$$

for all $\mu\in C_1$ and $\nu\in C_2$.

Theorem 3.2. *Let C_1 and C_2 be non-empty closed subsets of a complete PSS $(\mathscr{U},P)$. Let $\mathscr{A}$ be a cyclic mapping that fulfills a cyclic k-contraction condition for some $k\in[0,1)$. If there exists $\mu_0\in\mathscr{U}$ such that $\Xi(P,\mathscr{A},\mu_0)<\infty$ and either $(\mathscr{U},P)$ fulfills the* (A1) *property or $\mathscr{A}$ is continuous. Then $\mathscr{A}$ has exactly one fixed point $\mu\in\mathscr{U}$ such that $P(\mu,\mu)=0$.*

Proof. Let $\mu_0\in C_1\cup C_2$. Then either $\mu_0\in C_1$ or $\mu_0\in C_2$. Let $\mu_0\in A$. Since $\mathscr{A}C_1\subseteq C_2$, we have $\mathscr{A}\mu_0\in C_2$. Thus $\exists\ \mu_1\in C_2$ with $\mathscr{A}\mu_0=\mu_1$. Also, $\mathscr{A}C_2\subseteq C_1$, we have $\mathscr{A}\mu_1\in A$. Thus there exists $\mu_2\in A$ such that $\mu_2=\mathscr{A}\mu_1$. Continuing in this way, we can build up a sequence in $C_1\cup C_2$ by $\mu_n=\mathscr{A}\mu_{n-1}=\mathscr{A}^n\mu_0$. Now, if $\mu_{n+1}=\mu_n$ for all $n\in N$ then the result follows immediately. Suppose that $\mu_{n+1}\neq\mu_n$ for all $n\in N$.
Now, from (3.2), for all $i,j\in N$, we have

$$P(\mathscr{A}^{n+i}\mu_0,\mathscr{A}^{n+j}\mu_0)\le kP(\mathscr{A}^{n-1+i}\mu_0,\mathscr{A}^{n-1+j}\mu_0).$$

The above inequality comes true for all $i,j\in N$; so, by eq. (3.1), we have

$$\Xi(P,\mathscr{A},\mathscr{A}^n\mu_0)\le k\Xi(P,\mathscr{A},\mathscr{A}^{n-1}\mu_0).$$

Repeating the procedure again and again, we have (for every $n\in N$)

$$\Xi(P,\mathscr{A},\mathscr{A}^n\mu_0)\le k^n\Xi(P,\mathscr{A},\mu_0). \tag{3.3}$$

Let $n,m\in N$, such that $m=n+p$ for some $p\in N$. Using eq. (3.3),we have

$$P(\mathscr{A}^n\mu_0,\mathscr{A}^{n+p}\mu_0)\le\Xi(P,\mathscr{A},\mathscr{A}^n\mu_0)\le k^n\Xi(P,,\mathscr{A},\mu_0).$$

As $\Xi(P,\mathscr{A},\mu_0)<\infty$ and $k\in[0,1)$, we have

$$\lim_{n,m\to\infty}P(\mu_n,\mu_m)=0,$$

so that $\{\mu_n\}=\{\mathscr{A}^n\mu_0\}$ is a P-Cauchy sequence in $\mathscr{U}$. Due to the P-completeness of $\mathscr{U}$, there exists $\mu\in\mathscr{U}$ such that $\{\mu_n\}$ P-converges to μ. We note that $\{\mathscr{A}^{2n}\mu_0\}$ is a sequence in C_1 and $\{\mathscr{A}^{2n-1}\mu_0\}$ is a sequence in C_2 in such a way that both sequences tend to the same limit μ. Since C_1 and C_2 are closed, we have $\mu\in C_1\cap C_2$, and then $C_1\cap C_2\neq\phi$.

Presume that $\mathscr{A}$ is continuous. Then

$$\mu = \lim_{n\to\infty} \mu_{n+1} = \mathscr{A}(\lim_{n\to\infty} \mu_n) = \mathscr{A}\mu.$$

Alternatively, presume that $(\mathscr{U}, P)$ fulfills (A1) property. Now, we have $P(\mathscr{A}\mu_n, \mathscr{A}\mu) \le P(\mu_n, \mu)$, which on acquiring $n \to \infty$, implies that $\lim_{n\to\infty} P(\mu_{n+1}, \mathscr{A}\mu) = 0$. Thus, using the property (A1), $\mathscr{A}\mu = \mu$. Thus, μ is a fixed point of $\mathscr{A}$.
To prove the uniqueness of the fixed point, let on conflicting that there exists $\mu, \nu \in \mathscr{U}$ such that $\mathscr{A}\mu = \mu$, $\mathscr{A}\nu = \nu$. Then by definition of cyclic k-contraction,

$$P(\mu, \nu) = P(\mathscr{A}\mu, \mathscr{A}\nu) \le kP(\mu, \nu),$$

a contradiction. Hence $\mu = \nu$, that is, μ is a exactly one fixed point of $\mathscr{A}$. Finally, we prove that $P(\mu, \mu) = 0$. Since $\mathscr{A}$ is cyclic k-contraction mapping, we have

$$P(\mu, \mu) = P(\mathscr{A}\mu, \mathscr{A}\mu) \le kP(\mu, \mu).$$

This gives $P(\mu, \mu) < 0$, a contradiction. Hence $P(\mu, \mu) = 0$. This finalizes the proof. □

Definition 3.3. Let $(\mathscr{U}, P)$ be a PSS. A mapping $\mathscr{A} : \mathscr{U} \to \mathscr{U}$ is forenamed to be a Kannan-Ciric type k-contraction if: for all $\mu, \nu \in \mathscr{U}$,

$$P(\mathscr{A}\mu, \mathscr{A}\nu) \le k \max\{P(\mu, \mathscr{A}\mu), P(\nu, \mathscr{A}\nu)\}, \quad \text{where } k \in [0,1).$$

Inspired by the above definition, we establish cyclic Kannan-Ciric k-contraction.

Definition 3.4. Let C_1 and C_2 be non-empty subsets of PSS $(\mathscr{U}, P)$. A cyclic map $\mathscr{A} : C_1 \cup C_2 \to C_1 \cup C_2$ is said to be cyclic Kannan-Ciric k-contraction if for $k \in [0,1)$

$$P(\mathscr{A}\mu, \mathscr{A}\nu) \le k \max\{P(\mu, \mathscr{A}\mu), P(\nu, \mathscr{A}\nu)\}, \tag{3.4}$$

for all $\mu \in C_1$ and $\nu \in C_2$.

Theorem 3.5. *Let C_1 and C_2 be non-empty closed subsets of a complete PSS $(\mathscr{U}, P)$. Let $\mathscr{A}$ be a cyclic mapping that fulfills a cyclic Kannan-Ciric k-contraction condition for some $k \in [0,1)$. If there exists $\mu_0 \in \mathscr{U}$ such that $\Xi(P, \mathscr{A}, \mu_0) < \infty$ and either $(\mathscr{U}, P)$ fulfills the (A1) property or $\mathscr{A}$ is continuous. Then $\mathscr{A}$ has exactly one fixed point $\mu \in \mathscr{U}$ such that $P(\mu, \mu) = 0$.*

Proof. Let $\mu_0 \in C_1 \cup C_2$. Then either $\mu_0 \in C_1$ or $\mu_0 \in C_2$. Let $\mu_0 \in C_1$. Since $\mathscr{A}C_1 \subseteq C_2$, we have $\mathscr{A}\mu_0 \in C_2$. Thus $\exists\ \mu_1 \in C_2$ with $\mathscr{A}\mu_0 = \mu_1$. Also, $\mathscr{A}C_2 \subseteq C_1$, we have $\mathscr{A}\mu_1 \in C_1$. Thus $\exists\ \mu_2 \in C_1$ such that $\mu_2 = \mathscr{A}\mu_1$. Pursuing in this way, we can build up a sequence in $C_1 \cup C_2$ by $\mu_n = \mathscr{A}\mu_{n-1} = \mathscr{A}^n\mu_0$. Now, if $\mu_{n+1} = \mu_n$ for all $n \in N$ then the result follows immediately. Suppose that $\mu_{n+1} \neq \mu_n$ for all $n \in N$.
Now, from (3.4), for all $i, j \in N$, we have

$$\begin{aligned} P(\mu_n, \mu_{n+1}) &= P(\mathscr{A}\mu_{n-1}, \mathscr{A}\mu_n) \le k \max\{P(\mu_{n-1}, \mathscr{A}\mu_{n-1}), P(\mu_n, \mathscr{A}\mu_n)\} \\ &\le k \max\{P(\mu_{n-1}, \mu_n), P(\mu_n, \mu_{n+1})\}. \end{aligned}$$

Presume that $\max\{P(\mu_{n-1}, \mu_n), P(\mu_n, \mu_{n+1})\} = P(\mu_n, \mu_{n+1})$, then

$$P(\mu_n, \mu_{n+1}) \le kP(\mu_n, \mu_{n+1}),$$

a contradiction (since $k \in [0,1)$).

Thus, $\max\{P(\mu_{n-1},\mu_n),P(\mu_n,\mu_{n+1})\}=P(\mu_{n-1},\mu_n)$. Therefore,

$$P(\mu_n,\mu_{n+1})=kP(\mu_{n-1},\mu_n),\quad \text{for all } n\in N.$$

Thus inductively, we have

$$P(\mu_n,\mu_{n+1})=k^nP(\mu_0,\mu_1),\quad \text{for all } n\in N.$$

On taking the limit as $n\to\infty$, we get

$$\lim_{n\to\infty}P(\mu_n,\mu_{n+1})=0. \tag{3.5}$$

Now, we affirm that $\{\mu_n\}$ is a P-Cauchy sequence. From eq. (3.4), we have, for $n,m\in N$,

$$\begin{aligned}P(\mu_n,\mu_m)&=P(\mathscr{A}\mu_{n-1},\mathscr{A}\mu_{m-1})\\&\le k\max\{P(\mu_{n-1},\mathscr{A}\mu_{n-1}),P(\mu_{m-1},\mathscr{A}\mu_{m-1})\}\\&\le k\max\{P(\mu_{n-1},\mu_n),P(\mu_{m-1},\mu_m)\}.\end{aligned}$$

By taking the limit as $n,m\to\infty$ and using (3.5), we have

$$\lim_{n,m\to\infty}P(\mu_n,\mu_m)=0.$$

Hence $\{\mu_n\}$ is a P-Cauchy sequence. Due to the P-completeness of $\mathscr{U}$, $\exists\ \mu\in\mathscr{U}$ such that $\{\mu_n\}$ P-converges to μ. We note that $\{\mathscr{A}^{2n}\mu_0\}$ is a sequence in C_1 and $\{\mathscr{A}^{2n-1}\mu_0\}$ is a sequence in C_2 in such a way that both sequences tend to the same limit μ. Since C_1 and C_2 are closed, we have $\mu\in C_1\cap C_2$, and then $C_1\cap C_2\neq\phi$.
Presume that $\mathscr{A}$ is continuous. Then

$$\mu=\lim_{n\to\infty}\mu_{n+1}=\mathscr{A}(\lim_{n\to\infty}\mu_n)=\mathscr{A}\mu.$$

So, μ is a fixed point of $\mathscr{A}$.
To prove the uniqueness of the fixed point, let on contrary that there exists $\mu,\nu\in\mathscr{U}$ such that $\mathscr{A}\mu=\mu,\mathscr{A}\nu=\nu$. Then by definition of cyclic Kannan-Ciric k-contraction,

$$\begin{aligned}P(\mu,\nu)&=P(\mathscr{A}\mu,\mathscr{A}\nu)\\&\le k\max\{P(\mu,\mathscr{A}\mu),P(\nu,\mathscr{A}\nu)\}\\&=k\max\{P(\mu,\mu),P(\nu,\nu)\}.\end{aligned}$$

So, either $P(\mu,\nu)\le kP(\mu,\mu)$ or $P(\mu,\nu)\le kP(\nu,\nu)$, which is a conflict. Therefore, μ is exactly one fixed point of $\mathscr{A}$. Finally, we show that $P(\mu,\mu)=0$. Since $\mathscr{A}$ is cyclic Kannan-Ciric k-contraction mapping, we have

$$\begin{aligned}P(\mu,\mu)&=P(\mathscr{A}\mu,\mathscr{A}\mu)\\&\le k\max\{P(\mu,\mathscr{A}\mu),P(\mu,\mathscr{A}\mu)\}\\&=k\max\{P(\mu,\mu),P(\mu,\mu)\}\\&=kP(\mu,\mu).\end{aligned}$$

This gives $P(\mu,\mu)<0$, a contradiction. Hence $P(\mu,\mu)=0$. This finalizes the proof. □

Now, we raise some fixed point results for cyclic k-weak contraction within the context of PSS.

Definition 3.6. Let $(\mathscr{U},P)$ be a PSS and $\mathscr{A}:\mathscr{U}\to\mathscr{U}$. Then $\mathscr{A}$ is forenamed to be k-weak contraction if, for all $\mu,\nu\in\mathscr{U}$, and $k\in[0,1)$,

$$P(\mathscr{A}\mu,\mathscr{A}\nu)\le k\max\{P(\mu,\nu),P(\mu,\mathscr{A}\mu),P(\nu,\mathscr{A}\nu),P(\mu,\mathscr{A}\nu),P(\nu,\mathscr{A}\mu)\}.$$

Inspired by the above definition, we give the definition of cyclic k-weak contraction.

Definition 3.7. Let C_1 and C_2 be non-empty subsets of PSS $(\mathscr{U},P)$. A cyclic map $\mathscr{U}:C_1\cup C_2\to C_1\cup C_2$ is forenamed to be cyclic k-weak contraction if for all $k\in[0,1)$,

$$P(\mathscr{A}\mu,\mathscr{A}\nu)\le k\max\{P(\mu,\nu),P(\mu,\mathscr{A}\mu),P(\nu,\mathscr{A}\nu),P(\mu,\mathscr{A}\nu),P(\nu,\mathscr{A}\mu)\},$$

for all $\mu\in C_1$ and $\nu\in C_2$.

Theorem 3.8. *Let C_1 and C_2 be non-empty closed subsets of a complete PSS $(\mathscr{U},P)$. Let $\mathscr{A}$ be a cyclic mapping that fulfills the condition of a cyclic k-weak contraction for some $k\in[0,1)$. If there exists $\mu_0\in\mathscr{U}$ such that $\Xi(P,\mathscr{A},\mu_0)<\infty$ and $\mathscr{A}$ is continuous. Then $\mathscr{A}$ has exactly one fixed point $\mu\in\mathscr{U}$ such that $P(\mu,\mu)=0$.*

Proof. Let $\mu_0\in C_1\cup C_2$. Then either $\mu_0\in C_1$ or $\mu_0\in C_2$. Let $\mu_0\in C_1$. Since $\mathscr{A}C_1\subseteq C_2$, we have $\mathscr{A}\mu_0\in C_2$. Thus, there exists $\mu_1\in C_2$ with $\mathscr{A}\mu_0=\mu_1$. Also, $\mathscr{A}C_2\subseteq C_1$, we have $\mathscr{A}\mu_1\in C_1$. Thus there exists $\mu_2\in C_1$ such that $\mu_2=\mathscr{A}\mu_1$. Continuing in this way, we can build up a sequence in $C_1\cup C_2$ by $\mu_n=\mathscr{A}\mu_{n-1}=\mathscr{A}^n\mu_0$. Now, if $\mu_{n+1}=\mu_n$ for all $n\in N$ then the result follows immediately. Presume that $\mu_{n+1}\ne\mu_n$ for all $n\in N$.

Let n be an arbitrary positive integer. Since $\mathscr{A}$ is a cyclic k-weak contraction, for all $i,j\in N$, we have

$$\begin{aligned}P(\mathscr{A}^{n+i}\mu_0,\mathscr{A}^{n+j}\mu_0)\le k\max\{&P(\mathscr{A}^{n-1+i}\mu_0,\mathscr{A}^{n-1+j}\mu_0),P(\mathscr{A}^{n-1+i}\mu_0,\mathscr{A}^{n+i}\mu_0),\\ &P(\mathscr{A}^{n-1+j}\mu_0,\mathscr{A}^{n+j}\mu_0),P(\mathscr{A}^{n-1+i}\mu_0,\mathscr{A}^{n+j}\mu_0),\\ &P(\mathscr{A}^{n-1+j}\mu_0,\mathscr{A}^{n+i}\mu_0)\}.\end{aligned}$$

The above inequality holds for all $i,j\in N$, therefore, by eq. (3.1), we have

$$\Xi(P,\mathscr{A},\mathscr{A}^n\mu_0)\le k\,\Xi(P,\mathscr{A},\mathscr{A}^{n-1}\mu_0).$$

Repeating the procedure again and again, we have (for every $n\in N$)

$$\Xi(P,\mathscr{A},\mathscr{A}^n\mu_0)\le k^n\,\Xi(P,\mathscr{A},\mu_0). \tag{3.6}$$

Let $n,m\in N$, such that $m=n+p$ for some $p\in N$. Using (3.6), we have

$$\begin{aligned}P(\mathscr{A}^n\mu_0,\mathscr{A}^{n+p}\mu_0)&\le\Xi(P,\mathscr{A},\mathscr{A}^n\mu_0)\\ &\le k^n\,\Xi(P,\mathscr{A},\mu_0).\end{aligned}$$

As $\Xi(P,\mathscr{A},\mu_0)<\infty$ and $k\in[0,1)$, we have

$$\lim_{n,m\to\infty}P(\mu_n,\mu_m)=0,$$

so that $\{\mu_n\}=\{\mathscr{A}^n\mu_0\}$ is a P-Cauchy sequence in $\mathscr{U}$. Due to the P-completeness of $\mathscr{U}$, there exists $\mu\in\mathscr{U}$ such that $\{\mu_n\}$ P-converges to μ. We note that $\{\mathscr{A}^{2n}\mu_0\}$ is a sequence in C_1 and $\{\mathscr{A}^{2n-1}\mu_0\}$ is a sequence in C_2 in such a way that both sequences tend to the same limit μ. Since C_1 and C_2 are closed, we have $\mu\in C_1\cap C_2$, and then $C_1\cap C_2\ne\phi$.

Presume that $\mathscr{A}$ is continuous. Then

$$\mu = \lim_{n\to\infty} \mu_{n+1} = \mathscr{A}(\lim_{n\to\infty} \mu_n) = \mathscr{A}\mu.$$

So, μ is a fixed point of $\mathscr{A}$.

To prove the uniqueness of the fixed point, let on contrary that there exists $\mu, \nu \in \mathscr{U}$ such that $\mathscr{A}\mu = \mu$, $\mathscr{A}\nu = \nu$. Then by definition of cyclic k-weak contraction,

$$\begin{aligned} P(\mu,\nu) &= P(\mathscr{A}\mu,\mathscr{A}\nu) \\ &\le k\max\{P(\mu,\nu),P(\mu,\mathscr{A}\mu),P(\nu,\mathscr{A}\nu),P(\mu,\mathscr{A}\nu),P(\nu,\mathscr{A}\mu)\} \\ &= k\max\{P(\mu,\nu),P(\mu,\mu),P(\nu,\nu),P(\mu,\nu),P(\nu,\mu)\}. \end{aligned}$$

By using the definition of PSS, we have

$$P(\mu,\nu) \le kP(\mu,\nu) < P(\mu,\nu),$$

a contradiction, and so $P(\mu,\nu) = 0$ which implies that $\mu = \nu$. Thus, μ is a unique fixed point of $\mathscr{A}$. Finally, we show that $P(\mu,\mu) = 0$. Since $\mathscr{A}$ is cyclic k-weak contraction mapping, we have

$$\begin{aligned} P(\mu,\mu) &= P(\mathscr{A}\mu,\mathscr{A}\mu) \\ &\le k\max\{P(\mu,\mu),P(\mu,\mathscr{A}\mu),P(\mu,\mathscr{A}\mu),P(\mu,\mathscr{A}\mu),P(\mu,\mathscr{A}\mu)\} \\ &= k\max\{P(\mu,\mu),P(\mu,\mu),P(\mu,\mu),P(\mu,\mu),P(\mu,\mu)\} \\ &= kP(\mu,\mu). \end{aligned}$$

This gives $P(\mu,\mu) < 0$, a contradiction. Hence $P(\mu,\mu) = 0$. This finalizes the proof. □

Example 3.9. Let $\mathscr{U} = [-1,1]$ and a partial symmetric $P : \mathscr{U} \times \mathscr{U} \to R^+$ outlined as

$$P(\mu,\nu) = \max\{\mu,\nu\} \quad \text{for all } \mu,\nu \in \mathscr{U}.$$

Suppose that $C_1 = [-1,0]$ and $C_2 = [0,1]$. Outline $\mathscr{A} : C_1 \cup C_2 \to C_1 \cup C_2$ by

$$\mathscr{A}\mu = -\frac{\mu}{2} \quad \text{for all } \mu \in \mathscr{U}.$$

Note that $(\mathscr{U}, P)$ is a complete PSS and C_1, C_2 are closed in $\mathscr{U}$.

If $\mu \in C_1$, $-1 \le \mu \le 0$, then $0 \le -\frac{\mu}{2} \le \frac{1}{2}$, i.e., $\mathscr{A}\mu \in C_2$, i.e., $\mathscr{A}(C_1) \subseteq C_2$.

Similarly, if $\mu \in C_2$, $0 \le \mu \le 1$, then $-\frac{1}{2} \le -\frac{\mu}{2} \le 0$, i.e., $\mathscr{A}\mu \in C_1$, i.e., $\mathscr{A}(C_2) \subseteq C_1$.

Observe that

$$\begin{aligned} P(\mathscr{A}\mu,\mathscr{A}\nu) &= \max\{\mathscr{A}\mu,\mathscr{A}\nu\} \\ &= \max\left\{-\frac{\mu}{2},-\frac{\nu}{2}\right\} \\ &\le \frac{1}{2}\max\{-\mu,-\nu\} \\ &\le \frac{1}{2}\max\{P(\mu,\nu),P(\mu,\mathscr{A}\mu),P(\nu,\mathscr{A}\nu),P(\mu,\mathscr{A}\nu),P(\nu,\mathscr{A}\mu)\} \end{aligned}$$

for all $\mu \in C_1$, $\nu \in C_2$. Observe that $\mathscr{A}$ is continuous and $\Xi(P,\mathscr{A},\mu_0) < \infty$. Thus all the conditions of Theorem 3.8 are entertained and so $\mu = 0 \in C_1 \cap C_2$ is exactly one fixed point of $\mathscr{A}$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Alshehri, I. Aranđelović and N. Shahzad, Symmetric spaces and fixed points of generalized contractions, *Abstract and Applied Analysis* **2014** (2014), Article ID 763547, 8 pages, DOI: 10.1155/2014/763547.

[2] A. Ammini-Harandi, Metric-like spaces, partial metric spaces and fixed points, *Fixed Point Theory and Applications* **2012** (2012), Article number: 204, DOI: 10.1186/1687-1812-2012-204.

[3] M. Asim, A. R. Khan and M. Imdad, Fixed point results in partial symmetric spaces with an application, *Axioms* **8**(1) (2019), 13, DOI: 10.3390/axioms8010013.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3** (1922), 133 – 181, URL: `http://matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3120.pdf`.

[5] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis* **1** (1993), 5 – 11, URL: `http://dml.cz/dmlcz/120469`.

[6] T. L. Hicks and B. E. Rhoades, Fixed point theory in symmetric spaces with applications to probabilistic spaces, *Nonlinear Analysis: Theory, Methods & Applications* **36** (1999), 331 – 344, DOI: 10.1016/S0362-546X(98)00002-9.

[7] M. Imdad and J. Ali, Common fixed point theorems in symmetric spaces employing a new implicit function and common property (EA), *Bulletin of the Belgian Mathematical Society, Simon Stevin* **16**(2009), 421 – 433, DOI: 10.36045/bbms/1251832369.

[8] E. Karapınar, Fixed point theory for cyclic weak ϕ contraction, *Applied Mathematics Letters* **24** (2011), 822 – 825, DOI: 10.1016/j.aml.2010.12.016.

[9] E. Karapınar, I. M. Erhan and A. Y. Ulus, Fixed point theorem for cyclic maps on partial metric spaces, *Applied Mathematics & Information Sciences* **6** (2012), 239 – 244, URL: `https://dc.naturalspublishing.com/amis/vol06/iss2/9`.

[10] W. A. Kirk, P. S. Srinavasan and P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, *Fixed Point Theory* **4** (2003), 79 – 89, URL: `http://www.math.ubbcluj.ro/~nodeacj/download.php?f=031Kirk.pdf`.

[11] P. S. Kumari and D. Panthi, Connecting various types of cyclic contractions and contractive self mappings with Hardy Rogers self mappings, *Fixed Point Theory and Applications* **2016** (2016), Article number: 15, DOI: 10.1186/s13663-016-0498-3.

[12] P. S. Kumari and D. Panthi, Cyclic compatible contractions and related fixed point theorems, *Fixed Point Theory and Applications* **2016** (2016), Article number: 28, DOI: 10.1186/s13663-016-0521-8.

[13] S. G. Matthews, Partial metric topology, *Annals of the New York Academy of Sciences* **728** (1994), 183 – 197, DOI: 10.1111/j.1749-6632.1994.tb44144.x.

[14] Z. Mustafa, J. R. Roshan, V. Parvaneh and Z. Kadelburg, Some common fixed point results in ordered partial b-metric spaces, *Journal of Inequalities and Applications* **2013** (2013), Article number: 562, DOI: 10.1186/1029-242X-2013-562.

[15] Y. Shen and M. Lu, Fixed point theorems in symmetric spaces and applications to probabilistic spaces, *Nonlinear Analysis: Theory, Methods & Applications* **72** (2010), 527 – 532, DOI: 10.1016/J.NA.2009.06.102.